Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Algebraic Structures of a Rational-in-the-State Representation after Immersion

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Abstract— **This paper discusses some algebraic structures and their geometric counterparts associated with a rational-in-the-state representation (RSR) and a polynomial-in-the-state representation (PSR) obtained via system immersion of a given nonlinear system. First, all of RSRs and PSRs obtained by an identical immersion are parameterized in terms of the relation ideal of the immersion. Second, the notions of an invariant ideal and an invariant variety of a nonlinear system over a ring are introduced, which are closely related to a differential algebraic equation. Then, it is shown that a RSR and a PSR have invariant ideals and invariant varieties associated with an immersion. In particular, an invariant variety of a RSR or a PSR is the Zariski closure of the image of the immersion, i.e., the smallest variety containing the image of the immersion.**

I. INTRODUCTION

An immersion [1–4] of a system is a mapping of the initial state from the original state space to another state space, so as to preserve the input-output map exatly, and it is usually a mapping to a higher dimensional space. The model structure of the given system may be simplifed while preserving the input-output map with an immersion. Immersions into linear [1], [5], [6], bilinear [7], rational or polynomial [4] systems have been discussed in the literature.

Although only a restricted class of nonlinear systems are immersible into linear or bilinear systems, most practical systems are immersible into rational systems and polynomial systems, as shown in [4]. More precisely, a nonlinear system is immersible into a rational-in-the-state representation (RSR), a polynomial-in-the-state representation (PSR), and a quadratic-in-the-state representation (QSR) if and only if a field generated by the observation space is finitely generated over the real number field. Moreover, it is sufficient for immersibility into those representations that all functions in the given system are differentially algebraic functions, which most practical systems are consist of. Therefore, a RSR, a PSR or a QSR can be a general model structure for a wide class of nonlinear systems.

Potential applications of immersion into a RSR or a PSR include identification, observer design, system analysis and control design, although each of them is still an open area to research. For example, if a given system is known to be immersible into a RSR or a PSR, a parameter estimation technique can be used to identify parameters in that model structure to realize the same input-output map as the given system. The immersibility is the only assumption prior to identification, and the model structure of the original system and its immersion may not be necessarily known to construct a system of the same input-output map. The unknown model structure of the original system is reflected in the set of the initial state for the identified RSR or PSR.

Immersion was used in [1], [5], [6] to design an observer for a nonlinear system that is immersible into a linear system. Similarly, immersion into a polynomial system is potentially applicable to observer design for a broad class of nonlinear systems if the general methodology of observer design is established for polynomial systems. It is also often the case in system analysis and control design to assume polynomial systems [8–11]. Those techniques may be applicable to a wider class of nonlinear systems through the use of immersion. In contrast to polynomial approximation of a nonlinear system, immersion does not raise problems due to approximation errors and can preserve the input-output map over an unbounded region in the state space, which is useful when applying theoretical results, at the expense of an increase of the dimension.

This paper discusses some algebraic structures of a RSR and a PSR obtained via immersion and their geometric counterparts in detail. Since a RSR and a PSR have particular structures of rational functions and polynomials, respectively, some properties of the original system or the immersion are reflected in additional algebraic structures in the RSR and the PSR. Although observability of a RSR and a PSR after immersion has already been discussed in [12], it is a differential geometric characterization rather than algebraic characterization in terms of a field or a ring. In this paper, algebraic structures of a RSR and a PSR is characterized in terms of rings, and their geometric counterparts are expressed naturally in terms of affine algebraic varieties rather than a manifold in differential geometry. It should be noted that algebraic structures of a RSR and a PSR are also closely related to differential algebraic equations, namely, a RSR and a PSR with additional algebraic constraints in the form of rational functions and polynomials, respectively.

II. SYSTEM IMMERSION

A. Immersion and Invariant Immersion

We treat an input-affine nonlinear system,

$$\Sigma \left\{ \begin{array}{l} \dot{x} = g_0(x) + \sum_{i=1}^{m} g_i(x) u_i \\ y = h(x) \end{array} \right. ,$$

where $x(t) \in U \subset \mathbf{R}^n$ denotes the state vector, U an open set, $u(t) = [u_1(t), \dots, u_m(t)]^{\mathrm{T}} \in \mathbf{R}^m$ the input vector,

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0-7803-9568-9/05/$20.00 ©2005 IEEE

and $y(t) \in \mathbf{R}^p$ the output vector. The system is denoted by $\Sigma(g_0, g_1, \ldots, g_m, h)$ or Σ for short hereafter. System Σ is said to be analytic on U if $g_i : U \to \mathbf{R}^n$ $(i \in I_{0,m})$ and $h : U \to \mathbf{R}^p$ are analytic functions on U. Sets of indices are denoted as $I_{i_1,i_2} = \{i \in \mathbf{Z} : i_1 \leq i \leq i_2\}$ and $I_{i_1,\infty} = \{i \in \mathbf{Z} : i \geq i_1\}$. The admissible set Ω of the input function $u : [0,\infty) \to \mathbf{R}^m$ is a set of bounded piecewise continuous functions with a common upper bound. We assume $|u_i(t)| \leq 1$ $(t \geq 0, i \in I_{1,m})$ without loss of generality. The trajectory of the state equation of system Σ starting from an initial state x_0 at $t = 0$ and driven by an input function u is denoted by $\Phi_t^{\Sigma,u} : U \to U$. That is, for a given initial state $x_0 \in U$ and an input function $u \in \Omega$, the solution of the state equation is given by $x(t) = \Phi_t^{\Sigma,u}(x_0)$. The observation space $\mathcal{O}_\Sigma$ is defined by

$$\begin{aligned}\mathcal{O}_\Sigma = \mathbf{R}\text{-}\operatorname{span}\{&L_{g_{i_1}} \ldots L_{g_{i_k}} h_j : j \in I_{1,p},\\ &(i_1, \ldots, i_k) \in I_{0,m}^k,\ k \in I_{0,\infty}\},\end{aligned}$$

where $h = [h_1, \ldots, h_p]^{\mathrm{T}}$ and $L_{g_i} h_j = (\partial h_j / \partial x) g_i$.

Definition 1 [1–3] An analytic system $\Sigma(g_0, \ldots, g_m, h)$ defined on an open set $U \subset \mathbf{R}^n$ is said to be *immersible* on an open set $U' \subset U$ into another $\tilde{n}$-dimensional system $\tilde{\Sigma}(\tilde{g}_0, \ldots, \tilde{g}_m, \tilde{h})$, if there exists an analytic mapping $\alpha : U' \to \mathbf{R}^{\tilde{n}}$ such that $\tilde{\Sigma}$ is analytic on an open set containing $\alpha(U')$, and for every $x_0 \in U'$ and for every $u \in \Omega$,

$$h \circ \Phi_t^{\Sigma,u}(x_0) = \tilde{h} \circ \Phi_t^{\tilde{\Sigma},u}(\alpha(x_0))$$

holds for every sufficiently small $t > 0$. Such a mapping α is called an *immersion* of Σ on U' into $\tilde{\Sigma}$. We often omit U' when it is obvious in the context or $U' = U$.

It should be noted that an immersion preserves the input-output map and, therefore, is different from a feedback transformation such as feedback linearization. An immersion is also different from a coordinate transformation because it does not necessarily preserve the dimension of the state vector. Moreover, the immersibility is a coordinate-free property because the composition of an immersion and a coordinate transformation is again an immersion.

The following proposition is useful for checking whether or not a given mapping α is an immersion.

Proposition 1 [4] *Let $U \subset \mathbf{R}^n$ be an open set, let $\alpha : U \to \mathbf{R}^{\tilde{n}}$ be an analytic mapping, let $\Sigma(g_0, \ldots, g_m, h)$ be an analytic system on U, and let $\tilde{\Sigma}(\tilde{g}_0, \ldots, \tilde{g}_m, \tilde{h})$ be an analytic system on an open set containing $\alpha(U)$. The mapping $\alpha : U \to \mathbf{R}^{\tilde{n}}$ is an immersion of Σ into $\tilde{\Sigma}$ if and only if the following holds for all $x_0 \in U$:*

$$\begin{aligned}L_{g_{i_1}} \ldots L_{g_{i_k}} h(x_0) = L_{\tilde{g}_{i_1}} \ldots L_{\tilde{g}_{i_k}} \tilde{h}(\alpha(x_0)),&\\ (i_1, \ldots, i_k) \in I_{0,m}^k,\ k \in I_{0,\infty}.&\qquad (1)\end{aligned}$$

Next, we define a particular form of an immersion.

Definition 2 [4] An analytic system $\Sigma(g_0, \ldots, g_m, h)$ defined on an open set $U \subset \mathbf{R}^n$ is said to be *invariantly immersible* on an open set $U' \subset U$ into another $\tilde{n}$-dimensional system $\tilde{\Sigma}(\tilde{g}_0, \ldots, \tilde{g}_m, \tilde{h})$, if there exists an analytic mapping $\alpha : U' \to \mathbf{R}^{\tilde{n}}$ such that $\tilde{\Sigma}$ is analytic on an open set containing $\alpha(U')$, and, for all $x \in U'$,

$$\begin{aligned}L_{g_i} \alpha(x) &= \tilde{g}_i(\alpha(x)),\ i \in I_{0,m},\\ h(x) &= \tilde{h}(\alpha(x))\end{aligned}$$

hold. Such a mapping α is called an *invariant immersion* of Σ on U' into $\tilde{\Sigma}$.

In other words, if α is an invariant immersion of $\Sigma(g_0, \ldots, g_m, h)$ into $\tilde{\Sigma}(\tilde{g}_0, \ldots, \tilde{g}_m, \tilde{h})$, each pair of corresponding vector fields g_i and $\tilde{g}_i$ are α-related, and h is the pull back of $\tilde{h}$ by α. Often, the invariant immersion is simply called immersion in the literature. However, we distinguish invariant immersion from immersion in this work because the former not only preserves the input-output map but also has additional geometric properties. That is, for every state trajectory $x(t) \in U'$ of the original system Σ, $\alpha(x(t)) \in \alpha(U')$ is a state trajectory of $\tilde{\Sigma}$.

B. Immersibility Conditions

Rational and polynomial structures with respect to the state are discussed in this paper, for which notions of fields and rings are suitable. Let $C^\omega(U)$ be the ring of all real-valued analytic functions on an open set $U \subset \mathbf{R}^n$. For a subset $A \subset C^\omega(U)$, $\mathbf{R}[A]$ denotes the ring generated by A over $\mathbf{R}$. If U is a domain (connected open set), $\mathbf{R}[A]$ is an integral domain and its fraction field $\mathbf{R}(A)$ is well defined and is called the field generated by A over $\mathbf{R}$. If $A = \{\alpha_1, \ldots, \alpha_\nu\}$ and $\alpha = [\alpha_1, \ldots, \alpha_\nu]^{\mathrm{T}}$, $\mathbf{R}[A]$ and $\mathbf{R}(A)$ are also denoted by $\mathbf{R}[\alpha]$ and $\mathbf{R}(\alpha)$, respectively. For a state vector $\tilde{x} \in \mathbf{R}^{\tilde{n}}$, its elements $\tilde{x}_1, \ldots, \tilde{x}_{\tilde{n}}$ can be regarded as analytic functions on $\mathbf{R}^{\tilde{n}}$, and $\mathbf{R}[\tilde{x}]$ and $\mathbf{R}(\tilde{x})$ denote a polynomial ring and a rational function field, respectively. We denote the subset of $\mathbf{R}[\tilde{x}]$ with the total degree less than or equal to ℓ as $\mathbf{R}[\tilde{x}]_{\leq \ell}$.

Definition 3 [4] Consider a system $\tilde{\Sigma}(\tilde{g}_0, \ldots, \tilde{g}_m, \tilde{h})$ with a state vector $\tilde{x} \in \mathbf{R}^{\tilde{n}}$. System $\tilde{\Sigma}$ is said to be a *rational-in-the-state representation (RSR)* if $\tilde{g}_i(\tilde{x}) \in \mathbf{R}(\tilde{x})^{\tilde{n}}$ $(i \in I_{0,m})$ and $\tilde{h}(\tilde{x}) \in \mathbf{R}(\tilde{x})^p$. System $\tilde{\Sigma}$ is said to be a *polynomial-in-the-state representation (PSR)* if $\tilde{g}_i(\tilde{x}) \in \mathbf{R}[\tilde{x}]^{\tilde{n}}$ $(i \in I_{0,m})$ and $\tilde{h}(\tilde{x}) \in \mathbf{R}[\tilde{x}]^p$. In particular, system $\tilde{\Sigma}$ is said to be a *quadratic-in-the-state representation (QSR)* if $\tilde{g}_i(\tilde{x}) \in \mathbf{R}[\tilde{x}]_{\leq 2}^{\tilde{n}}$ $(i \in I_{0,m})$ and $\tilde{h}(\tilde{x}) \in \mathbf{R}[\tilde{x}]_{\leq 1}^p$.

It should be noted that the output equation in a QSR is at most first-order in the state. It has already been known that immersibilities are equivalent between a RSR, a PSR and a QSR.

Proposition 2 [4] *For an analytic system Σ defined on an open set U, the following three claims are equivalent:*

(i) *System Σ is immersible (resp. invariantly immersible) on U into a RSR.*
(ii) *System Σ is immersible (resp. invariantly immersible) on U into a PSR.*

(iii) *System Σ is immersible (resp. invariantly immersible) on U into a QSR.*

Since it suffices to consider a RSR consisting of rational functions, immersibility and invariant immersibility are well characterized in terms of fields.

Proposition 3 [4] *For an analytic system Σ defined on a domain U, the following four claims are equivalent:*

(i) *On an open and dense subset of U, system Σ is invariantly immersible into a RSR with an analytic mapping defined on U.*

(ii) *On an open and dense subset of U, system Σ is immersible into a RSR with an analytic mapping defined on U.*

(iii) *The observation space $\mathcal{O}_\Sigma$ is a subset of a finitely generated field over $\mathbf{R}$.*

(iv) *The field $\mathbf{R}(\mathcal{O}_\Sigma)$ is finitely generated over $\mathbf{R}$.*

Moreover, if (iv) holds, every set of analytic generators of $\mathbf{R}(\mathcal{O}_\Sigma)$ gives an invariant immersion in (i).

Many types of nonlinear systems satisfy conditions (iii) and (iv) in Proposition 3 and (invariantly) immersible into a RSR, a PSR and a QSR. Moreover, as shown in [4], it is sufficient for invariant immersibility into a RSR that all functions in a given system are differentially algebraic functions, which most practical systems are consist of.

III. Algebraic Structures after Immersion

A. Ideals Associated with an Immersion

Suppose, for a given system $\Sigma(g_0,\dots,g_m,h)$ on a domain $U \subset \mathbf{R}^n$, we have $\mathbf{R}(\mathcal{O}_\Sigma) = \mathbf{R}(\alpha)$ with an analytic mapping $\alpha : U \to \mathbf{R}^{\tilde{n}}$. Then, Proposition 3 implies that Σ is invariantly immersible into a RSR $\tilde{\Sigma}(\tilde{g}_0,\dots,\tilde{g}_m,\tilde{h})$ with α on an open and dense subset $U' \subset U$. Let the state vector of $\tilde{\Sigma}$ be $\tilde{x} \in \mathbf{R}^{\tilde{n}}$. Since all functions in $\tilde{\Sigma}$ are rational functions of $\tilde{x}$, we have $\mathcal{O}_{\tilde{\Sigma}} \subset \mathbf{R}(\tilde{x})$. Moreover, since $\tilde{\Sigma}$ is defined on an open set containing $\alpha(U')$, denominators of the rational functions in $\tilde{\Sigma}$ do not vanish identically on $\alpha(U)$, which implies a particular algebraic structure in $\tilde{\Sigma}$, as discussed below.

Definition 4 [13], [14] Given a subset $S \subset \mathbf{R}^{\tilde{n}}$, denote by

$$\mathcal{I}(S) = \{\tilde{f} \in \mathbf{R}[\tilde{x}] : \tilde{f}(\tilde{x}) = 0 \text{ for all } \tilde{x} \in S\}$$

the ideal of polynomials vanishing on S. When S is an image of a mapping $\alpha : U \to \mathbf{R}^{\tilde{n}}$, $\mathcal{I}(\alpha(U))$ is called the *relation ideal* of α.

Let $P = \mathcal{I}(\alpha(U))$ be the relation ideal of the immersion of Σ into the RSR $\tilde{\Sigma}$, and let $\alpha^* : \mathbf{R}[\tilde{x}] \to \mathbf{R}[\alpha]$ be a *substitution mapping* (or a pull back) defined by $\alpha^*(f) = f(\alpha)$ for $f \in \mathbf{R}[\tilde{x}]$. Then, α^* is a surjective ring homomorphism such that $\operatorname{Ker}\alpha^* = P$, which induces a ring isomorphism $\mathbf{R}[\alpha] \cong \mathbf{R}[\tilde{x}]/P$. Since $\mathbf{R}[\alpha]$ is an integral domain, P is a prime ideal.

As discussed previously, every denominator of functions in the RSR $\tilde{\Sigma}$ is not identically zero on $\alpha(U)$ or, equivalently, does not belong to the prime ideal P. Therefore, every element of $\tilde{g}_0,\dots,\tilde{g}_m$ and $\tilde{h}$ belongs to not only the rational function field $\mathbf{R}(\tilde{x})$ but also the *localization* of $\mathbf{R}[\tilde{x}]$ at P, which is given by

$$\mathbf{R}[\tilde{x}]_P = \{f/g \in \mathbf{R}(\tilde{x}) : f, g \in \mathbf{R}[\tilde{x}], \text{ and } g \notin P\}.$$

Note that $\mathbf{R}[\tilde{x}]_P$ is a *local ring* with the unique maximal ideal

$$P\mathbf{R}[\tilde{x}]_P = \{f/g \in \mathbf{R}(\tilde{x}) : f, g \in \mathbf{R}[\tilde{x}],\, g \notin P, \text{ and } f \in P\}.$$

It should also be noted that an element of $\mathbf{R}[\tilde{x}]_P$ does not necessarily define a rational function on the whole of $\alpha(U)$ because its denominator can vanish at some points. However, there is an open and dense subset $U' \subset U$ such that an element of $\mathbf{R}[\tilde{x}]_P$ is analytic on an open set containing $\alpha(U')$.

We can naturally extend the substitution mapping $\alpha^* : \mathbf{R}[\tilde{x}] \to \mathbf{R}[\alpha]$ to a mapping $\alpha^* : \mathbf{R}[\tilde{x}]_P \to \mathbf{R}(\alpha)$, whish is also a surjective ring homomorphism with $\operatorname{Ker}\alpha^* = P\mathbf{R}[\tilde{x}]_P$ and induces a ring isomorphism $\mathbf{R}(\alpha) \cong \mathbf{R}[\tilde{x}]_P/P\mathbf{R}[\tilde{x}]_P$. We use the same symbol to denote the substitution mappings for $\mathbf{R}[\tilde{x}]$, $\mathbf{R}[\tilde{x}]_P$ and their direct products, because their domains are obvious from the context.

From the definition of an invariant immersion and the fact $\operatorname{Ker}\alpha^* = P\mathbf{R}[\tilde{x}]_P$ and $\operatorname{Ker}\alpha^*|_{\mathbf{R}[\tilde{x}]} = P$, it is straightforward to show the following parameterization of all RSRs and PSRs into which a given system is invariantly immersible with the same immersion α.

Theorem 1 *Let Σ be an analytic system defined on a domain $U \subset \mathbf{R}^n$, let $\alpha : U \to \mathbf{R}^{\tilde{n}}$ be an invariant immersion of Σ on an open and dense subset of U into a RSR $\tilde{\Sigma}(\tilde{g}_0,\dots,\tilde{g}_m,\tilde{h})$, and let P be the relation ideal of α. Then all RSRs into which Σ is invariantly immersible on an open and dense subset of U with α are parameterized as $\tilde{\Sigma}'(\tilde{g}_0+r_0,\dots,\tilde{g}_m+r_m,\tilde{h}+r_{m+1})$ with $r_i \in P\mathbf{R}[\tilde{x}]_P^{\tilde{n}}$ ($i \in I_{0,m}$) and $r_{m+1} \in P\mathbf{R}[\tilde{x}]_P^p$. Moreover, if $\tilde{\Sigma}$ is a PSR, all PSRs into which Σ is invariantly immersible with α are parameterized as $\tilde{\Sigma}'(\tilde{g}_0+r_0,\dots,\tilde{g}_m+r_m,\tilde{h}+r_{m+1})$ with $r_i \in P^{\tilde{n}}$ ($i \in I_{0,m}$) and $r_{m+1} \in P^p$.*

Proof The proof is given only for the case of a RSR because the case of a PSR can be proved similarly.

First, if $\tilde{\Sigma}$, $r_i \in P\mathbf{R}[\tilde{x}]_P^{\tilde{n}}$ ($i \in I_{0,m}$) and $r_{m+1} \in P\mathbf{R}[\tilde{x}]_P^p$ are analytic on the image of an open and dense subset $U' \subset U$ by α, then it is obvious from the definition of an invariant immersion and $r_i(\alpha(x)) = 0$ ($\forall x \in U'$) that α is an invariant immersion of Σ on U' into a RSR $\tilde{\Sigma}'(\tilde{g}_0+r_0,\dots,\tilde{g}_m+r_m,\tilde{h}+r_{m+1})$.

Conversely, if α is an invariant immersion of Σ on an open and dense subset $U' \subset U$ into not only $\tilde{\Sigma}$ but also another RSR $\tilde{\Sigma}'(\tilde{g}'_0,\dots,\tilde{g}'_m,\tilde{h}')$, then all elements of $\tilde{g}'_i$ ($i \in I_{0,m}$) and $\tilde{h}'$ also belong to $\mathbf{R}[\tilde{x}]_P$, and we have, for all $x \in U'$,

$$\begin{aligned} L_{g_i}\alpha(x) &= \tilde{g}_i(\alpha(x)) = \tilde{g}'_i(\alpha(x)), \quad i \in I_{0,m}, \\ h(x) &= \tilde{h}(\alpha(x)) = \tilde{h}'(\alpha(x)). \end{aligned}$$

Therefore, for all $x \in U'$,

$$\begin{aligned}\alpha^*(\tilde{g}_i - \tilde{g}'_i)(x) &= \tilde{g}_i(\alpha(x)) - \tilde{g}'_i(\alpha(x)) = 0, \quad i \in I_{0,m}, \\ \alpha^*(\tilde{h} - \tilde{h}')(x) &= \tilde{h}(\alpha(x)) - \tilde{h}'(\alpha(x)) = 0,\end{aligned}$$

which implies $\tilde{g}_i - \tilde{g}'_i \in P\mathbf{R}[\tilde{x}]_P^{\tilde{n}}$ $(i \in I_{0,m})$ and $\tilde{h} - \tilde{h}' \in R\mathbf{R}[\tilde{x}]_P^p$. □

It is obvious that, if $\tilde{\Sigma}$ is a QSR, the parameterization of all QSRs are obtained with such additional constraints to the case of PSRs as $\tilde{g}_i + r_i \in \mathbf{R}[\tilde{x}]_{\leq 2}^{\tilde{n}}$ $(i \in I_{0,m})$ and $\tilde{h} + r_{m+1} \in \mathbf{R}[\tilde{x}]_{\leq 1}^p$.

B. System over a Ring

Now, we can characterize some algebraic structures and their geometric counterparts of the RSR after immersion in terms of such a *subring* of the rational function field $\mathbf{R}(\tilde{x})$ as $\mathbf{R}[\tilde{x}]$ and $\mathbf{R}[\tilde{x}]_P$ rather than $\mathbf{R}(\tilde{x})$ itself. To this end, we prepare some ring theoretic notions of nonlinear system theory in place of usual differential geometric settings.

Let R be a partial differential subring of $\mathbf{R}(\tilde{x})$, such as $\mathbf{R}[\tilde{x}]$ and $\mathbf{R}[\tilde{x}]_P$, satisfying $(\partial/\partial\tilde{x}_i)R \subset R$ for all $i \in I_{1,\tilde{n}}$. Then, a PSR or a RSR $\tilde{\Sigma}(\tilde{g}_0, \ldots, \tilde{g}_m, \tilde{h})$ is regarded as a *system over a ring* such that $\tilde{g}_0, \ldots, \tilde{g}_m \in R^{\tilde{n}}$ and $\tilde{h} \in R^p$ for an appropriate ring R. Note that the state $\tilde{x}$ belongs to Euclidean space as usual in the present notion of a system over a ring.

The vector fields $\tilde{g}_0, \ldots, \tilde{g}_m$ can be viewed as elements of a free R-module $R^{\tilde{n}}$ rather than sections of a tangent bundle. Moreover, the Lie derivative $L_{\tilde{g}_i}$ is regarded as a mapping $L_{\tilde{g}_i} : R \to R$, which is not a ring endomorphism of R in general but a derivation of R regarded as an $\mathbf{R}$-module. An important algebraic structure in a system over a ring is the invariance of an ideal under the Lie derivative.

Definition 5 An ideal $I \subset R$ is said to be an *invariant ideal* of a system $\tilde{\Sigma}(\tilde{g}_0, \ldots, \tilde{g}_m, \tilde{h})$, if

$$L_{\tilde{g}_i} I \subset I, \quad i \in I_{0,m}$$

holds.

If the ring R is Noetherian, every ideal is finitely generated. For example, $\mathbf{R}[\tilde{x}]$ is Noetherian by Hilbert's Basis Theorem, and its localization $\mathbf{R}[\tilde{x}]_P$ is also Noetherian [15]. If an invariant ideal I is finitely generated and is represented as $I = (\tilde{f}_1, \ldots, \tilde{f}_s)$, which is equivalent to $R\tilde{f}_1 + \cdots + R\tilde{f}_s$, with $\tilde{f}_i \in R$, it is meaningful to consider a differential algebraic equation (DAE) over R:

$$\tilde{\Sigma}_I \begin{cases} \dot{\tilde{x}} = \tilde{g}_0(x) + \displaystyle\sum_{i=1}^{m} \tilde{g}_i(\tilde{x}) u_i \\ 0 = \tilde{f}(\tilde{x}) \end{cases}, \tag{2}$$

where $\tilde{f}(\tilde{x}) = [\tilde{f}_1(\tilde{x}), \ldots, \tilde{f}_s(\tilde{x})]^{\mathrm{T}} \in R^s$. We call, in this paper, $\tilde{x}_0 \in \mathbf{R}^{\tilde{n}}$ a *regular point* of DAE $\tilde{\Sigma}_I$, if all denominators in $\tilde{g}_i$ and $\tilde{f}_i$ are nonzero at x_0. Since $I = (\tilde{f}_1, \ldots, \tilde{f}_s)$ is an invariant ideal, we have

$$L_{g_i} \tilde{f}_j \in (\tilde{f}_1, \ldots, \tilde{f}_s), \quad i \in I_{0,m},\ j \in I_{1,s},$$

which implies that, if a regular point x_0 satisfies $\tilde{f}(x_0) = 0$, the trajectory starting from x_0 satisfies $\tilde{f}(\Phi_t^{\Sigma,u}(x_0)) = 0$ for every admissible input $u \in \Omega$, as long as the trajectory $\Phi_t^{\Sigma,u}(x_0)$ is a regular point. If $\tilde{\Sigma}$ is a PSR, $R = \mathbf{R}[\tilde{x}]$ and every point in $\mathbf{R}^{\tilde{n}}$ is a regular point of the DAE $\tilde{\Sigma}_I$.

A geometric object related to an invariant ideal is an invariant variety.

Definition 6 [14] Let I be an ideal of $\mathbf{R}[\tilde{x}]$. An *(affine algebraic) variety* (or an algebraic set) defined by I is a subset of $\mathbf{R}^{\tilde{n}}$ given by

$$\mathcal{V}(I) = \{\tilde{x} \in \mathbf{R}^{\tilde{n}} : \tilde{f}(\tilde{x}) = 0 \text{ for all } \tilde{f} \in I\}.$$

When $I = (\tilde{f}_1, \ldots, \tilde{f}_s)$, $\mathcal{V}(I)$ is also denoted by $\mathcal{V}(\tilde{f}_1, \ldots, \tilde{f}_s)$.

Definition 7 Let R be a partial differential ring such that $\mathbf{R}[\tilde{x}] \subset R \subset \mathbf{R}(\tilde{x})$, and let P be an ideal of $\mathbf{R}[\tilde{x}]$. $\mathcal{V}(P)$ is called an *invariant variety* of a system over R, if RP is an invariant ideal of the system.

Note that if $R = \mathbf{R}[\tilde{x}]$ then $RP = P$, and if $R = \mathbf{R}[\tilde{x}]_P$ then $RP = P\mathbf{R}[\tilde{x}]_P$. Since $\mathbf{R}[\tilde{x}]$ is Noetherian, $\mathcal{V}(P)$ can always be expressed as a set of common zeros of a finite number of polynomials. By the invariance of the ideal RP, those polynomials are identically zero along a trajectory starting from a regular point $\tilde{x}_0 \in \mathcal{V}(P)$ of $\tilde{\Sigma}$. That is, the trajectory stays in $\mathcal{V}(P)$ as long as it is defined, which motivates the notion of the invariant variety.

The notions of an invariant ideal and an invariant variety can be regarded as a generalization of an *algebraic particular integral* and *invariant algebraic surface* [16], [17] of a polynomial vector field. It should be noted that an invariant variety may have a singular point as a variety and is not necessarily a manifold globally. In particular, the existence of an invariant variety does not necessarily imply the existence of a foliation of manifolds.

C. Invariance in a System after Immersion

Through the use of algebraic and geometric notions defined above, we can characterize an invariance in a system after immersion.

Theorem 2 *Let α be an invariant immersion of system Σ defined on a domain $U \subset \mathbf{R}^n$ into a RSR $\tilde{\Sigma}$ and let P be the relation ideal of α. Then, $P\mathbf{R}[\tilde{x}]_P$ is an invariant ideal of $\tilde{\Sigma}$. Moreover, if $\tilde{\Sigma}$ is a PSR, P is also an invariant ideal.*

Proof For any $r \in P\mathbf{R}[\tilde{x}]_P$, $L_{\tilde{g}_i} r \in \mathbf{R}[\tilde{x}]_P$ because $\tilde{g}_i \in \mathbf{R}[\tilde{x}]_P^{\tilde{n}}$. Moreover, there is an open and dense subset $U' \subset U$ such that r is analytic on an open set containing $\alpha(U')$ and $r(\alpha(x)) = 0$ for all $x \in U'$. Therefore, we have $L_{g_i} r(\alpha(x)) = 0$ $(i \in I_{0,m})$ on U'. Meanwhile, from the definition of an invariant immersion, we have

$$\begin{aligned}L_{g_i} r(\alpha(x)) &= \frac{\partial r(\alpha(x))}{\partial \tilde{x}} \frac{\partial \alpha(x)}{\partial x} g_i(x) \\ &= \frac{\partial r(\alpha(x))}{\partial \tilde{x}} \tilde{g}_i(\alpha(x)) = (L_{\tilde{g}_i} r)(\alpha(x)),\end{aligned}$$

for all $x \in U'$. In summary, $(L_{\tilde{g}_i} r)(\tilde{x}) = 0$ for all $\tilde{x} \in \alpha(U')$, which means $L_{\tilde{g}_i} r \in P\mathbf{R}[\tilde{x}]_P$. When $\tilde{\Sigma}$ is a PSR, $L_{\tilde{g}_i}$ maps $\mathbf{R}[\tilde{x}]$ into $\mathbf{R}[\tilde{x}]$ and $L_{\tilde{g}_i} P \subset P$ can be shown similarly. □

Definition 8 [14] The *Zariski closure* of a subset $S \subset \mathbf{R}^{\tilde{n}}$ is the smallest algebraic variety containing S.

Proposition 4 [14] *The Zariski closure of $S \subset \mathbf{R}^{\tilde{n}}$ is given by $\mathcal{V}(\mathcal{I}(S))$.*

Theorem 3 *Let α, $\tilde{\Sigma}$ and P be the same as in Theorem 2. Then, $\mathcal{V}(P)$ is the Zariski closure of $\alpha(U)$ and, moreover, an invariant variety of $\tilde{\Sigma}$.*

Proof Since P is the relation ideal of α, i.e., $P = \mathcal{I}(\alpha(U))$, Proposition 4 implies that $\mathcal{V}(P)$ is the Zariski closure of $\alpha(U)$. Moreover, since $P\mathbf{R}[\tilde{x}]_P$ is an invariant ideal of $\tilde{\Sigma}$ by Theorem 2, $\mathcal{V}(P)$ is also an invariant algebraic variety. □

For an invariant immersion $\alpha : U \to \mathbf{R}^{\tilde{n}}$ of a given system Σ on an open and dense subset $U' \subset U$ into a RSR $\tilde{\Sigma}(\tilde{g}_0, \ldots, \tilde{g}_m, \tilde{h})$, its relation ideal $P = \mathcal{I}(\alpha(U)) \subset \mathbf{R}[\tilde{x}]$ is always finitely generated, and its generators are also generators of the maximal ideal $P\mathbf{R}[\tilde{x}]_P$ of the local ring $\mathbf{R}[\tilde{x}]_P$. If $P = (\tilde{f}_1, \ldots, \tilde{f}_s)$, an image of any trajectory of the original system by the immersion, $\alpha(\Phi_t^{\Sigma,u}(x_0))$ $(x_0 \in U')$, is always a solution of the DAE given in (2). That is, $\alpha(\Phi_t^{\Sigma,u}(x_0))$ $(x_0 \in U')$ belongs to $\mathcal{V}(\tilde{f}_1, \ldots, \tilde{f}_s)$. Moreover, Theorem 3 says that not only the image of a trajectory of the original system but also any trajectory starting from a point on $\mathcal{V}(\tilde{f}_1, \ldots, \tilde{f}_s)$ stays on $\mathcal{V}(\tilde{f}_1, \ldots, \tilde{f}_s)$, as long as it is defined.

IV. Example

A. Rational-in-the-State Representation

Consider a one-dimensional analytic system on $\mathbf{R}$:

$$\Sigma \left\{ \begin{array}{l} \dot{x} = \dfrac{\sin x}{x} \\ y = x \end{array} \right. .$$

The observation space of this system is given by

$$\mathcal{O}_\Sigma = \mathbf{R}\text{-}\operatorname{span}\left\{ x, \frac{\sin x}{x}, \frac{\cos x \cdot x - \sin x}{x^2} \cdot \frac{\sin x}{x}, \ldots \right\}.$$

We have $\mathbf{R}(\mathcal{O}_\Sigma) = \mathbf{R}(x, \sin x, \cos x)$ and Σ is immersible into a RSR with $\alpha(x) = [x, \sin x, \cos x]^{\mathrm{T}}$. In fact, a RSR $\tilde{\Sigma}(\tilde{g}_0, \tilde{h})$ can readily be constructed from $(\partial\alpha/\partial x)\dot{x} = [\sin x/x, \sin x \cdot \cos x/x, -\sin^2 x/x]^{\mathrm{T}}$ as follows:

$$\tilde{\Sigma} \left\{ \begin{array}{l} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2/\tilde{x}_1 \\ \tilde{x}_2\tilde{x}_3/\tilde{x}_1 \\ -\tilde{x}_2^2/\tilde{x}_1 \end{bmatrix} \\ y = \tilde{x}_1 \end{array} \right. .$$

The RSR $\tilde{\Sigma}$ is analytic on an open set $\tilde{U} = \{\tilde{x} \in \mathbf{R}^3 : \tilde{x}_1 \neq 0\} \supset \alpha(\mathbf{R} \setminus \{0\})$. Therefore, Σ is immersible into the RSR $\tilde{\Sigma}$ on $\mathbf{R} \setminus \{0\}$. The immersion α itself is analytic on $\mathbf{R}$ and its image is a helix along the $\tilde{x}_1$ axis.

From the algebraic relation between the trigonometric functions, $(\sin x)^2 + (\cos x)^2 - 1 = 0$, the relation ideal of α is $P = (\tilde{x}_2^2 + \tilde{x}_3^2 - 1)$. The maximal ideal of the local ring $\mathbf{R}[\tilde{x}]_P$ has the form

$$P\mathbf{R}[\tilde{x}]_P = \left\{ \frac{n(\tilde{x})}{d(\tilde{x})}(\tilde{x}_2^2 + \tilde{x}_3^2 - 1) : n, d \in \mathbf{R}[\tilde{x}], \text{ and } d \notin P \right\}.$$

Note that any element of $P\mathbf{R}[\tilde{x}]_P$ vanishes when $\tilde{x} = \alpha(x)$ is substituted. Theorem 1 says that all RSRs into which Σ is invariantly immersible with α are parameterized as

$$\tilde{\Sigma}' \left\{ \begin{array}{l} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2/\tilde{x}_1 + r_{01}(\tilde{x}) \\ \tilde{x}_2\tilde{x}_3/\tilde{x}_1 + r_{02}(\tilde{x}) \\ -\tilde{x}_2^2/\tilde{x}_1 + r_{03}(\tilde{x}) \end{bmatrix} \\ y = \tilde{x}_1 + r_1(\tilde{x}) \end{array} \right. , \tag{3}$$

where r_{01}, r_{02}, r_{03} and r_1 belong to $P\mathbf{R}[\tilde{x}]_P$.

Theorem 2 says that $P\mathbf{R}[\tilde{x}]_P$ is an invariant ideal of $\tilde{\Sigma}$. In fact, for any $n(\tilde{x})(\tilde{x}_2^2 + \tilde{x}_3^2 - 1)/d(\tilde{x}) \in P\mathbf{R}[\tilde{x}]_P$, we have

$$\begin{aligned} & L_{\tilde{g}_0}\left[\frac{n(\tilde{x})}{d(\tilde{x})}(\tilde{x}_2^2 + \tilde{x}_3^2 - 1)\right] \\ &= \left[L_{\tilde{g}_0}\frac{n(\tilde{x})}{d(\tilde{x})}\right](\tilde{x}_2^2 + \tilde{x}_3^2 - 1) + \frac{n(\tilde{x})}{d(\tilde{x})} L_{\tilde{g}_0}(\tilde{x}_2^2 + \tilde{x}_3^2 - 1) \\ &= \left[L_{\tilde{g}_0}\frac{n(\tilde{x})}{d(\tilde{x})}\right](\tilde{x}_2^2 + \tilde{x}_3^2 - 1) \\ &\quad + \frac{n(\tilde{x})}{d(\tilde{x})}\left(2\tilde{x}_2 \cdot \frac{\tilde{x}_2\tilde{x}_3}{\tilde{x}_1} + 2\tilde{x}_3 \cdot \frac{-\tilde{x}_2^2}{\tilde{x}_1}\right) \\ &= \left[L_{\tilde{g}_0}\frac{n(\tilde{x})}{d(\tilde{x})}\right](\tilde{x}_2^2 + \tilde{x}_3^2 - 1). \end{aligned}$$

Note that the Lie derivative of $n(\tilde{x})(\tilde{x}_2^2 + \tilde{x}_3^2 - 1)/d(\tilde{x}) \in P\mathbf{R}[\tilde{x}]_P$ vanishes even when $\tilde{x}$ does not necessarily belong to the image of α. It is also straightforward to check that $P\mathbf{R}[\tilde{x}]_P$ is an invariant ideal of every system $\tilde{\Sigma}'$ in the form of (3).

Finally, Theorem 3 claims that a variety $\mathcal{V}(\tilde{x}_2^2 + \tilde{x}_3^2 - 1)$, a cylinder along the $\tilde{x}_1$ axis, is the Zariski closure of $\alpha(\mathbf{R})$, a helix, and an invariant algebraic variety of $\tilde{\Sigma}$. Then, for any solution $x(t)$ of Σ, its image $\alpha(x(t))$ satisfies the DAE:

$$\tilde{\Sigma}_P \left\{ \begin{array}{l} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2/\tilde{x}_1 \\ \tilde{x}_2\tilde{x}_3/\tilde{x}_1 \\ -\tilde{x}_2^2/\tilde{x}_1 \end{bmatrix} \\ 0 = \tilde{x}_2^2 + \tilde{x}_3^2 - 1 \end{array} \right. .$$

Moreover, any solution of the state equation of $\tilde{\Sigma}$ starting from a point on $\mathcal{V}(\tilde{x}_2^2 + \tilde{x}_3^2 - 1)$ always satisfy the DAE, as long as it is defined (Fig. 1).

B. Polynomial-in-the-State Representation

According to Proposition 2, Σ is invariantly immersible on $\mathbf{R} \setminus \{0\}$ into not only the RSR but also a PSR. In fact, an invariant immersion into a PSR is readily obtained by augmenting $\alpha(x)$ with $1/x$ $(= 1/\tilde{x}_1)$ as $\beta(x) = [x, \sin x, \cos x, 1/x]^{\mathrm{T}}$, and a PSR

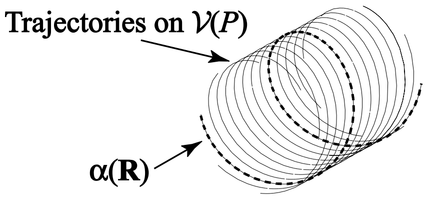

Fig. 1. Trajectories on the invariant algebraic variety.

$\bar{\Sigma}(\bar{g}_0, \bar{h})$ is obtained from $(\partial\beta/\partial x)\dot{x} = [\sin x/x, \sin x \cdot \cos x/x, -\sin^2 x/x, -\sin x/x^3]^{\mathrm{T}}$ as follows:

$$\bar{\Sigma}\left\{\begin{array}{l} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} \bar{x}_2\bar{x}_4 \\ \bar{x}_2\bar{x}_3\bar{x}_4 \\ -\bar{x}_2^2\bar{x}_4 \\ -\bar{x}_2\bar{x}_4^3 \end{bmatrix} \\ y = \bar{x}_1 \end{array}\right. .$$

The PSR $\bar{\Sigma}$ is analytic on $\mathbf{R}^4 \supset \beta(\mathbf{R} \setminus \{0\})$.

From an additional algebraic relation $x \cdot (1/x) - 1 = 0$, the relation ideal of β is $Q = (\bar{x}_2^2 + \bar{x}_3^2 - 1, \bar{x}_1\bar{x}_4 - 1)$. Then, Theorem 1 says that all PSRs into which Σ is invariantly immersible with β are parameterized as

$$\bar{\Sigma}'\left\{\begin{array}{l} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} \bar{x}_2\bar{x}_4 + q_{01}(\bar{x}) \\ \bar{x}_2\bar{x}_3\bar{x}_4 + q_{02}(\bar{x}) \\ -\bar{x}_2^2\bar{x}_4 + q_{03}(\bar{x}) \\ -\bar{x}_2\bar{x}_4^3 + q_{04}(\bar{x}) \end{bmatrix} \\ y = \bar{x}_1 + q_1(\bar{x}) \end{array}\right. ,$$

where q_{01}, q_{02}, q_{03}, q_{04} and q_1 belong to Q.

Theorem 2 says that Q is an invariant ideal of $\bar{\Sigma}$. In fact, it is readily confirmed that $L_{\bar{g}_0}Q \subset Q$. For example, we have, for any $p \in \mathbf{R}[\bar{x}]$,

$$\begin{aligned} & L_{\bar{g}_0}[p(\bar{x})(\bar{x}_1\bar{x}_4 - 1)] \\ & \quad = [L_{\bar{g}_0}p(\bar{x})](\bar{x}_1\bar{x}_4 - 1) \\ & \qquad + p(\bar{x})[\bar{x}_4 \cdot \bar{x}_2\bar{x}_4 + \bar{x}_1 \cdot (-\bar{x}_2\bar{x}_4)] \\ & \quad = [L_{\bar{g}_0}p(\bar{x}) - p(\bar{x})\bar{x}_2\bar{x}_4^2](\bar{x}_1\bar{x}_4 - 1) \in Q. \end{aligned}$$

Finally, Theorem 3 states that $\mathcal{V}(\bar{x}_2^2 + \bar{x}_3^2 - 1, \bar{x}_1\bar{x}_4 - 1)$ is the Zariski closure of $\beta(\mathbf{R} \setminus \{0\})$ and an invariant algebraic variety of $\bar{\Sigma}$. Then, for any solution $x(t)$ of Σ, its image $\beta(x(t))$ satisfies a DAE:

$$\bar{\Sigma}_Q\left\{\begin{array}{l} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} \bar{x}_2\bar{x}_4 \\ \bar{x}_2\bar{x}_3\bar{x}_4 \\ -\bar{x}_2^2\bar{x}_4 \\ -\bar{x}_2\bar{x}_4^3 \end{bmatrix} \\ 0 = \bar{x}_2^2 + \bar{x}_3^2 - 1 \\ 0 = \bar{x}_1\bar{x}_4 - 1 \end{array}\right. .$$

Moreover, any solution of $\bar{\Sigma}$ starting from a point on $\mathcal{V}(\bar{x}_2^2 + \bar{x}_3^2 - 1, \bar{x}_1\bar{x}_4 - 1)$ satisfies the DAE, as long as it is defined.

V. Conclusion

Some algebraic and geometric structures associated with a RSR and a PSR obtained via system immersion have been discussed in this paper. First, it has been shown that all of RSRs or PSRs into which a given system is invariantly immersible with an identical immersion are parameterized in terms of the relation ideal of the immersion. In particular, the algebraic structures of a RSR after immersion are well described in terms of the localization of a polynomial ring at the relation ideal rather than the rational function field.

Second, the notions of an invariant ideal and an invariant variety of a nonlinear system over a ring have been introduced, which are closely related to a DAE. Then, it has been shown that the maximal ideal of a local ring associated with a RSR is an invariant ideal and, in particular, the relation ideal of an immersion is an invariant ideal of a PSR. As a geometric counterpart of the algebraic structures, it has also been shown that a variety defined by the relation ideal of an immersion is the Zariski closure of the image of the immersion and also an invariant variety. Therefore, any trajectory starting from a point on the variety stays in that variety, as long as it is defined, even when it is not necessarily the image of the trajectory of the original system.

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