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$\mathscr{H}_2$ Model Reduction Using LMIs

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Abstract— **This paper considers the problem of $\mathscr{H}_2$ reduced order approximation for both continuous and discrete time MIMO systems. A heuristic algorithm is proposed that utilizes necessary and sufficient conditions expressed in terms of a set of LMIs and a matrix rank constraint, and the alternating projection method. Also a method of finding starting points is suggested. Three numerical examples are employed to show the effectiveness of the choice of starting points and the capability of the algorithm to find at least as good approximants as other methods.**

I. INTRODUCTION

There has been a significant interest in the model reduction problem, namely, the problem of approximating a high order system by a lower order, thus simpler, system. One of the most commonly used methods is the balanced truncation method [1], whose procedure is relatively simple. Another popular model reduction method is the Hankel norm approximation method [2], which also has a constructive way of finding reduced order models, or approximants. It is common practice to employ an error between the original high order system and the obtained reduced order model in some sense as an index of how good the approximant is. For both methods upper bounds of the error in the $\mathscr{H}_\infty$ sense (and also a lower bound for the Hankel norm approximation method) are explicitly expressed in terms of the Hankel singular values of the original system. These methods do not in general produce optimal approximants in the $\mathscr{H}_\infty$ sense and several methods for $\mathscr{H}_\infty$ optimal model reduction are proposed, e.g., [3].

In this paper the model reduction problem in terms of the $\mathscr{H}_2$-norm is considered. This norm has an attractive aspect in that minimizing the $\mathscr{H}_2$ error means minimizing the $\mathscr{H}_2$ error in the impulse response as well as minimizing the $\mathscr{H}_2$ error in the frequency response [4], and the $\mathscr{H}_2$ model reduction problem has attracted considerable attention over four decades.

A number of approaches, e.g., [4]–[7], just to name a few, use first order necessary conditions for optimality in one way or another and develop optimization algorithms to find solutions to resulting nonlinear equations. Most of the methods in this direction are only applicable to the single input single output (SISO) case. Furthermore it is argued [8], [9] that whether the global optimum is always achievable is unclear in the continuous time case (while it is shown to exist in the discrete time case [10]) and that, in the case of nonexistence of the optimum, these approaches can at best only find local optima which may be far from the true (global) optimum.

Even if the existence of the global optimum is guaranteed, optimization methods based on search algorithms can have difficulties [11]: There may be one or more local optima and it is difficult to guarantee that the obtained solution is close to the global optimum. Moreover there is usually no guarantee that the chosen stopping criterion for such a search algorithm is appropriate. To overcome these problems, several algorithms based on algebraic methods have been proposed that directly solve a set of nonlinear equations [11]–[13]. These approaches seem to have potential (in cases where the optima are achievable), but computation cost required for such approaches is still high and structural properties of the problem seem to require further exploitation for algorithmic development, which prevent them from becoming useful alternatives in practice at this moment.

A different type of approaches has emerged recently. In [8], [9], it is proposed to solve slightly modified problems for the continuous time case, where the global optimum is proven to exist and the use of a search algorithm makes sense. Those methods can deal with the multi input multi output (MIMO) case and thus favourable compared to many other methods in this respect. A problem of those methods may be the difficulty of measuring the conservativeness of the obtained result due to the modification of the problem.

In this paper an $\mathscr{H}_2$ model reduction method for the MIMO case based on linear matrix inequality (LMI) techniques is developed. Unlike other methods this approach allows both continuous and discrete time cases to be treated in a unified manner, as in [3] for $\mathscr{H}_\infty$ model reduction. The proposed algorithm is based on necessary and sufficient conditions for the existence of suboptimal approximants expressed by a set of LMIs and a (non-convex) matrix rank constraint, e.g., [14], and on the alternating projection method, e.g., [15]. Due to the non-convex property of the problem, the suggested method does not guarantee global convergence. In order to tackle the non-convexity, this paper proposes a method of finding good starting points. Numerical examples show that the proposed algorithm can yield approximants at least as good as those computed by other methods. It is emphasized that this method deals with the original problem rather than a modified one and thus is not affected by the potential conservativeness resulting from modification of the problem. Also the algorithm essentially solves suboptimal problems and hence avoids the issue of existence/nonexistence of the optimal solution. Moreover a search is carried out for the feasible $\mathscr{H}_2$ error by executing

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feasible tests and therefore can be terminated when a desired difference between the achieved error and the (local) optimum is reached.

The structure of this paper is as follows. In Section II, the $\mathscr{H}_2$ model reduction problem is formally stated and necessary and sufficient conditions for the existence of a reduced order model within a specified error are presented for both continuous and discrete time cases. Section III reviews the alternating projection method and also presents an algorithm for the $\mathscr{H}_2$ model reduction problem which makes use of this method. Since the above algorithm is heuristic due to the non-convexity of the necessary and sufficient conditions shown in Section II, the choice of starting points has a significant effect on the practicality of the proposed algorithm. This is the topic of Section IV. In Section V, three numerical examples are employed to demonstrate the choice of starting points and the algorithm. In particular the third example shows that the obtained results are at least as good as those computed by other methods. Some concluding remarks are made in Section VI.

Notation: Given a matrix $A \in \mathbb{C}^{n\times m}$, A^* denotes the complex conjugate transpose. The trace of a square matrix is denoted by $\mathrm{tr}\{\cdot\}$. The set of all symmetric matrices in $\mathbb{C}^{n\times n}$ is denoted by $\mathscr{S}_n$. The notation $>$ (resp., $\geq$) is used to denote the positive definiteness (resp., semi-definiteness) of a square symmetric matrix, and $A > B$ (resp., $A \geq B$), $A,B \in \mathscr{S}_n$, means $A-B>0$ (resp., $A-B\geq 0$). Also, $A<0$ (resp., $A \leq 0$) means $-A>0$ (resp., $-A\geq 0$). The $\mathscr{H}_2$-norm of a stable system is denoted by $\|\cdot\|_2$. Finally, $P_{\mathscr{C}_i}$ denotes the orthogonal projection operator onto the set $\mathscr{C}_i$.

II. $\mathscr{H}_2$ Model Reduction

The $\mathscr{H}_2$ optimal model reduction problem is stated as follows: Given a stable system G of McMillan degree n with q inputs and p outputs, find a stable system $\hat{G}$ of McMillan degree $r(<n)$ with the same numbers of inputs and outputs that minimizes the $\mathscr{H}_2$-norm of the error system $E = G-\hat{G}$, i.e., minimizes the error $\|E\|_2 = \|G-\hat{G}\|_2$. Under the same set-up, the $\mathscr{H}_2$ suboptimal model reduction problem is stated as: Given $\gamma(>0)$, find, if it exists, $\hat{G}$ that achieves the $\mathscr{H}_2$ error less than γ, i.e., achieves $\|G-\hat{G}\|_2 < \gamma$. Without loss of generality, it can be assumed that both G and $\hat{G}$ are strictly proper.

A. Continuous Time Systems

In this subsection the model reduction problem for a continuous time system is considered and necessary and sufficient conditions for the existence of a reduced order model achieving a specified error are presented. Let state space realizations of $G(s)$ and $\hat{G}(s)$ be

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \tag{1}$$

$$\hat{G}(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & 0 \end{array}\right] \tag{2}$$

where $A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times q}, C\in\mathbb{R}^{p\times n}, \hat{A}\in\mathbb{R}^{r\times r}, \hat{B}\in\mathbb{R}^{r\times q}, \hat{C}\in\mathbb{R}^{p\times r}$. A state space realization of the error system is

$$\begin{aligned} E(s) &= G(s)-\hat{G}(s) \\ &= \left[\begin{array}{cc|c} A & 0 & B \\ 0 & \hat{A} & \hat{B} \\ \hline C & -\hat{C} & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_E & B_E \\ \hline C_E & 0 \end{array}\right]. \end{aligned} \tag{3}$$

Then the $\mathscr{H}_2$ optimal model reduction problem can be expressed as:

$$\begin{aligned} &\text{minimize } \gamma(>0) \\ &\text{subject to } A_E P + P A_E^* + B_E B_E^* < 0 \ , \quad (4)\\ &\qquad P > 0 \ , \quad (5)\\ &\qquad \mathrm{tr}\{C_E P C_E^*\} < \gamma^2 \ . \quad (6) \end{aligned}$$

Partition P conformally with A_E and write

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}$$

where $P_{11}\in\mathscr{S}_n, P_{12}\in\mathbb{R}^{n\times r}, P_{22}\in\mathscr{S}_r$. Then, from the Schur complement formula [16, pp. 7-8], (4) is equivalent to

$$\begin{aligned} &\begin{bmatrix} A_E P + P A_E^* & B_E \\ B_E^* & -I \end{bmatrix} \\ &= \begin{bmatrix} AP_{11}+P_{11}A^* & AP_{12}+P_{12}\hat{A}^* & B \\ P_{12}^*A^*+\hat{A}P_{12} & \hat{A}P_{22}+P_{22}\hat{A}^* & \hat{B} \\ B^* & \hat{B}^* & -I \end{bmatrix} < 0 \ . \end{aligned} \tag{7}$$

Using a slack variable $W\in\mathscr{S}_n$, (5) and (6) can be expressed as

$$\mathrm{tr}\{W\} < \gamma^2 \ , \tag{8}$$

$$\begin{aligned} &\begin{bmatrix} W & C_E P \\ P C_E^* & P \end{bmatrix} \\ &= \begin{bmatrix} W & CP_{11}-\hat{C}P_{12}^* & CP_{12}-\hat{C}P_{22} \\ P_{11}C^*-P_{12}\hat{C}^* & P_{11} & P_{12} \\ P_{12}^*C^*-P_{22}\hat{C}^* & P_{12}^* & P_{22} \end{bmatrix} > 0 \ . \end{aligned} \tag{9}$$

It is observed that neither (7) nor (9) is an LMI in $P_{11}, P_{12}, P_{22}, \hat{A}, \hat{B}, \hat{C}$ since there are bilinear terms such as $\hat{A}P_{12}$.

Now those conditions are expressed with respect to two decision variables (symmetric matrices) by eliminating $\hat{A}, \hat{B}, \hat{C}$. The $\mathscr{H}_2$ model reduction problem is a special case of the $\mathscr{H}_2$ optimal controller synthesis problem and therefore the following result may readily be obtained by using some results from the literature. Indeed equivalent conditions can be found in [14]. Here the result is expressed in a form suited for the method developed later. A proof for this particular formulation can be found in [17].

Theorem 1: Consider a stable continuous time system $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ of McMillan degree n. There exists a stable continuous time system $\hat{G}(s)$ of McMillan degree at most r that satisfies $\|G(s)-\hat{G}(s)\|_2 < \gamma$ if and only if there exist $X, Z \in \mathscr{S}_n$ satisfying

$$AX + XA^* + BB^* < 0 \ , \tag{10}$$

$$A(X-Z) + (X-Z)A^* < 0 \ , \tag{11}$$

$$\mathrm{tr}\{C(X-Z)C^*\} < \gamma^2 \ , \tag{12}$$

$$Z \geq 0 \ , \tag{13}$$

$$\mathrm{rank} Z \leq r \ . \tag{14}$$

While (10)-(13) are convex constraints, the rank constraint (14) is not. An optimization problem/a feasibility problem under those constraints is a non-convex problem. Thus interior-point algorithms used for (convex) LMI feasibility/optimization problems cannot be employed and this makes the $\mathscr{H}_2$ model reduction problem a difficult task.

If X and Z that satisfy (10)-(14) are found, then a reduced order model that achieves an error less than γ can be obtained by firstly computing P_{11}, P_{12}, P_{22} from $X = P_{11}$ and a decomposition of $Z = P_{12}P_{22}^{-1}P_{12}^*$ and then solving the LMI feasibility problem (7), (8), (9) for $\hat{A}, \hat{B}, \hat{C}$.

B. Discrete Time Systems

Now consider the discrete time case. Suppose that state space realizations of the original system $G(z)$, the reduced order approximant $\hat{G}(z)$ and the error system $E(z)$ are given as in the right hand sides of (1), (2) and (3), respectively. The model reduction problem can be expressed identically to the continuous time case except for (4), which is to be replaced with

$$A_E P A_E^* - P + B_E B_E^* < 0 \ . \tag{15}$$

It is noted here that, by way of the Schur complement formula, (5) and (15) are equivalent to

$$\begin{bmatrix} P & -A_E P & -B_E \\ -PA_E^* & P & 0 \\ -B_E^* & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} P_{11} & P_{12} & -AP_{11} & -AP_{12} & -B \\ P_{12}^* & P_{22} & -\hat{A}P_{12}^* & -\hat{A}P_{22} & -\hat{B} \\ -P_{11}A^* & -P_{12}\hat{A}^* & P_{11} & P_{12} & 0 \\ -P_{12}^*A^* & -P_{22}\hat{A}^* & P_{12}^* & P_{22} & 0 \\ -B^* & -\hat{B}^* & 0 & 0 & I \end{bmatrix} > 0 \ . \tag{16}$$

Similar to the continuous time case, necessary and sufficient conditions with respect to two symmetric matrices can be derived [14]. Again a formulation suitable for the algorithm proposed in this paper is presented here, and a proof may be found in [17].

Theorem 2: Consider a stable discrete time system $G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ of McMillan degree n. There exists a stable discrete time system $\hat{G}(z)$ of McMillan degree at most r that satisfies $\left\|G(z) - \hat{G}(z)\right\|_2 < \gamma$ if and only if there exist $X, Z \in \mathscr{S}_n$ satisfying

$$AXA^* - X + BB^* < 0 \ , \tag{17}$$

$$A(X-Z)A^* - (X-Z) < 0 \ , \tag{18}$$

$$\mathrm{tr}\{C(X-Z)C^*\} < \gamma^2 \ , \tag{19}$$

$$Z \geq 0 \ , \tag{20}$$

$$\mathrm{rank} Z \leq r \ . \tag{21}$$

As is the case with continuous time systems, (17)-(20) are LMIs and thus convex, but the rank constraint (21) is not. It makes the problem a non-convex one.

III. $\mathscr{H}_2$ Model Reduction Algorithm Using the Alternating Projection Method

Not only the $\mathscr{H}_2$ model reduction problem but also more general controller synthesis problems with fixed controller order are formulated as LMIs with rank constraints. To tackle those control problems with controller order constraints, various heuristic algorithms have been proposed, e.g., [15], [18]–[20], though there is no guarantee for convergence of those algorithms to a solution.

In order to find reduced order models, this paper employs the alternating projection method, whose detail can be found in, e.g., [15]. The method considers a pair of sets $\mathscr{C}_1$ and $\mathscr{C}_2$ in the space $\mathscr{S}_n \times \mathscr{S}_n$ and carries out orthogonal projections onto $\mathscr{C}_1$ and $\mathscr{C}_2$ alternately in order to find an element in the intersection $\mathscr{C}_1 \cap \mathscr{C}_2$. In the case of the $\mathscr{H}_2$ model reduction problem, $\mathscr{C}_1$ can be taken as

$$\mathscr{C}_1 = \{(X,Z) | X \in \mathscr{S}_n, Z \in \mathscr{S}_n, (10), (11), (12), (13)\}$$

in the continuous time case, or

$$\mathscr{C}_1 = \{(X,Z) | X \in \mathscr{S}_n, Z \in \mathscr{S}_n, (17), (18), (19), (20)\}$$

in the discrete time case. Also,

$$\mathscr{C}_2 = \{(X,Z) | X \in \mathscr{S}_n, Z \in \mathscr{S}_n, \mathrm{rank} Z \leq r\}$$

for either case. Note that $\mathscr{C}_1$ is convex while $\mathscr{C}_2$ is not.

By equipping the space $\mathscr{S}_n \times \mathscr{S}_n$ with the inner product

$$\langle (X_1, Z_1), (X_2, Z_2) \rangle = \mathrm{tr}\{X_1 X_2\} + \mathrm{tr}\{Z_1 Z_2\} \ ,$$

the orthogonal projection of $(X_0, Z_0) \in \mathscr{S}_n \times \mathscr{S}_n$ onto $\mathscr{C}_1$ can be found by solving a convex optimization problem [15]. As for the projection $P_{\mathscr{C}_2}(X_0, Z_0)$, since $\mathscr{C}_2$ is not convex, there may be more than one matrix pair that minimize the distance from (X_0, Z_0). Let $Z_0 = U \Sigma V^*$ be the singular value decomposition of Z_0. Then a projection of (X_0, Z_0) onto $\mathscr{C}_2$ is given by

$$P_{\mathscr{C}_2}(X_0, Z_0) = (X_0, U \Sigma_r V^*)$$

where Σ_r is the diagonal matrix obtained from Σ by replacing the $(n-r)$ smallest diagonal elements of Σ by zero [21, Section 7.4].

Now the following algorithm is naturally suggested for $\mathscr{H}_2$ model reduction.

1) Find $X, Z \in \mathscr{S}_n$ and an $\mathscr{H}_2$-norm bound γ that satisfy (10)-(14) in the continuous time case (resp., (17)-(21) in the discrete time case).
2) Reduce γ. Find $X, Z \in \mathscr{S}_n$ that satisfy (10)-(14) (resp., (17)-(21)) using the alternating projection method, taking (X, Z) from the previous step as a starting point.
3) If successful, go back to Step 2. Otherwise, compute an approximant from the best (X, Z) available by solving the feasibility problem (7) (resp., (16)), (8) and (9).

It is also possible to use a bisection method with respect to γ.

It is emphasized that the presented algorithm is heuristic since $\mathscr{C}_2$ is non-convex and thus the alternating projection method becomes heuristic. Hence this method may not provide a suboptimal approximant whose achieved error is as close to the optimal error as desired. It is of great significance to find a nice starting point in Step 1. This is because it can determine whether an approximant which is close to the global optimum will be obtained. Also, Step 2 is not in general an inexpensive task and it is desired to have initial γ close to the optimal γ, which may be achieved by having (X,Z) close to the optimum (or a suboptimum achieving practically the optimal error in the case where the optimum cannot be achieved). The choice of the starting point is the topic of the next section.

IV. Choice of Starting Points

A. Choosing from the Balanced Realization

From now on, it is supposed that the given state space realization is balanced. This is a sensible assumption from the practical point of view since the use of balanced realizations in general improves the reliability of numerical computation and thus is common practice. In such a case the Gramian of the form

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

satisfies

$$A\Sigma + \Sigma A^* + BB^* = 0 \ , \qquad A^*\Sigma + \Sigma A + C^*C = 0$$

in the continuous time case, or

$$A\Sigma A^* - \Sigma + BB^* = 0 \ , \qquad A^*\Sigma A - \Sigma + C^*C = 0$$

in the discrete time case. The diagonal elements of Σ are called Hankel singular values and ordered as

$$\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n \geq 0 \ .$$

To make the left hand side of (12) (or (19)) small, a straightforward idea is to make $X - Z$ "small". From (10) (or (17)), it is observed that $X > \Sigma$ [22]. Inequality (11) (or (18)) implies that $X > Z$, and, if X is taken to be very close to Σ, the i-th diagonal element of Z is smaller than σ_i. Write $C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$ where c_i are column vectors and observe that

$$\mathrm{tr}\{C\Sigma C^*\} = \mathrm{tr}\{C^*C\Sigma\} = \sum_i (c_i^* c_i)\sigma_i \ . \qquad (22)$$

If Z is assumed to be a diagonal matrix whose r (diagonal) elements are only nonzero, and the only constraint is $\Sigma > Z$, then the optimal choice of Z is obtained by identifying the $n-r$ smallest terms of $(c_i^* c_i)\sigma_i, i = 1, \ldots, n$, and letting the corresponding diagonal elements of Σ be zero.

Unfortunately such Z does not in general satisfy (11) (or (18)). An option may be to restrict the structure of Z and minimize γ under LMIs (10)-(13) (or (17)-(20)). Another method is to use the balanced realization. This has an effect similar to having structured Z. As is mentioned above, a balanced realization is often obtained beforehand, so it can be used without extra cost. However, instead of the ordinary balanced truncation, modes whose contributions to (22) are small are truncated. Indeed this idea can be used to find upper bounds of $\mathscr{H}_2$ error [9], [23].

Once a reduced order model and the $\mathscr{H}_2$-norm of the error system, which will serve as initial γ, are obtained in this way, the controllability Gramian of the error system is computed, from which initial X and Z are obtained. Note that, if the modes are chosen such that the retained modes and the truncated modes do not share the same Hankel singular values, then the reduced order system is stable [24], [25, Theorem 21.29] and (10) and (11) (or (17) and (18)) are automatically satisfied.

In fact it is observed that a good starting point is not necessarily obtained from the modes of the r largest contributions. In the worst case, $\binom{n}{r}$ combinations of modes are to be examined. Nevertheless, in practice, combinations containing modes with small contributions may be ignored and the number of combinations to be tested can be greatly reduced.

B. Improving Starting Points

It is observed that, once an initial approximant is obtained from the balanced realization as in the previous subsection, a better approximant may be obtained by solving a feasibility problem with LMIs. The set of LMIs (7) (or (16)), (8) and (9) is to be solved for $\hat{A}, \hat{B}, \hat{C}$ where P is replaced with the controllability Gramian of the error system and γ in the right hand side of (8) is replaced with the $\mathscr{H}_2$-norm of the error system. Notice that, once P is fixed, the set of inequalities are LMIs with respect to $\hat{A}, \hat{B}, \hat{C}$.

The above method usually gives a reduced order model that achieves a smaller error. The effect is sometimes trivial. (If the model obtained from the balanced realization is close to the (local) minimum, there is little room for improvement.) However the required computation cost is relatively small compared to the alternating projection method, so it is worth carrying out.

V. Numerical Examples

In this section three numerical examples are presented. The first two examples demonstrate the effectiveness of the choice of starting points in Section IV, and the third example shows that the algorithm presented in Section III works at least as well as other methods. The error achieved by an approximant is shown in the relative error, i.e.,

$$\mathscr{J}(G, \hat{G}) = \frac{\|G - \hat{G}\|_2}{\|G\|_2}.$$

Example 1. This second order continuous time SISO example is taken from [4, Example 2]:

$$G(s) = \frac{10000s + 5000}{s^2 + 5000s + 25}.$$

A first order approximant is sought. Its Hankel singular values $\sigma_i, i = 1, 2$, are

$$99.000,\ 0.99990,$$

but, $(c_i^* c_i)\sigma_i, i = 1, 2$, are

$$102.05,\ 9997.9.$$

It is seen that the contribution of the mode corresponding to the second (i.e., the smaller) Hankel singular value to (22) is the larger. Indeed,

$$\mathscr{J}(G(s), \hat{G}_1(s)) = 0.99494,$$
$$\mathscr{J}(G(s), \hat{G}_2(s)) = 0.0985088$$

where $\hat{G}_i(s)$ is the approximant obtained by retaining the i-th mode.

An improved starting point is computed from $\hat{G}_2(s)$ using the method described in Subsection IV-B. However the improvement is practically zero and the error the new approximant $\hat{G}_2'(s)$ achieves is

$$\mathscr{J}(G(s), \hat{G}_2'(s)) = 0.0985086.$$

The algorithm in Section III hardly improves this, but this is natural since the optimal relative error is 0.0985 [4].

Example 2. This discrete time system is taken from [13]:

$$G(z) = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{-3}{4} \\ 0 & 0 & 1 & 0 & \frac{383}{2080} & \frac{279}{1040} \\ 0 & 0 & 0 & 1 & \frac{1839}{8320} & \frac{-1317}{4160} \\ 0 & \frac{-1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1419}{33280} & \frac{99}{1280} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

This is a 4th order MIMO system, with 2 inputs and 2 outputs. A 2nd order approximant is to be found. Its Hankel singular values σ_i, $i = 1, \ldots, 4$, are

$$1.6752,\ 0.90114,\ 0.16511,\ 0.083916,$$

and, $(c_i^* c_i)\sigma_i, i = 1, \ldots, 4$, are

$$2.0191,\ 0.18054,\ 0.0045092,\ 0.00082210.$$

In this example the order of the contributions is identical to that of the Hankel singular values. Indeed the approximant $\hat{G}_{12}(z)$, constructed by retaining the 1st and 2nd modes and discarding the 3rd and 4th modes, achieves the smallest error among $\binom{4}{2} = 6$ possible approximants of order 2:

$$\mathscr{J}(G(z), \hat{G}_{12}(z)) = 0.089277.$$

The method in Subsection IV-B is carried out to find an improved starting point, but again the improvement is not significant:

$$\mathscr{J}(G(z), \hat{G}_{12}'(z)) = 0.089119.$$

Also the algorithm described in Section III hardly improves this since $\hat{G}_{12}'(z)$ is nearly optimal; The optimal approximant given in [13] achieves

$$\mathscr{J}(G(z), \hat{G}_{\mathrm{opt}}(z)) = 0.089039.$$

Notice that

$$\frac{\mathscr{J}(G(z), \hat{G}_{12}'(z))}{\mathscr{J}(G(z), \hat{G}_{\mathrm{opt}}(z))} = 1.00089675.$$

Example 3. A more realistic example, the System AUTM in [26], is examined. The system to be approximated is a 12th order continuous time MIMO system, with 2 inputs and 2 outputs. The Hankel singular values σ_i, $i = 1, 2, \ldots, 12$, are

$$7.1833,\ 1.4904,\ 0.92791,\ 0.58756,\ 0.46331,\ 0.23683,$$
$$0.16132,\ 0.093582,\ 0.56596 \times 10^{-3},\ 0.20608 \times 10^{-4},$$
$$0.14124 \times 10^{-5},\ 0.34341 \times 10^{-7},$$

and, $(c_i^* c_i)\sigma_i, i = 1, 2, \ldots, 12$, are

$$17.001,\ 3.6844,\ 0.23558,\ 0.076721,\ 2.5247,\ 0.30031,$$
$$0.065789,\ 0.20424,\ 4.0153 \times 10^{-6},\ 1.1683 \times 10^{-8},$$
$$9.8808 \times 10^{-13},\ 2.1223 \times 10^{-15}.$$

It is observed that modes corresponding to large Hankel singular values in general have large contributions in (22), but the orders are slightly different.

Reduced order models of McMillan degrees 4, 5 and 6 are sought. First a 4th order approximant is found. In this case the best initial approximant is obtained by discarding all the modes except for those corresponding to the 1st, 2nd, 3rd and 5th Hankel singular values, or the 1st, 2nd, 3rd and 5th largest $(c_i^* c_i)\sigma_i$:

$$\mathscr{J}(G(z), \hat{G}_{1235}(s)) = 0.15231.$$

It is pointed out that the approximant constructed from the modes corresponding to the 4 largest $(c_i^* c_i)\sigma_i$ yields a much worse error:

$$\mathscr{J}(G(z), \hat{G}_{1256}(s)) = 0.26578.$$

By means of the method in Subsection IV-B, a slight improvement of the starting point is made:

$$\mathscr{J}(G(z), \hat{G}_{1235}'(s)) = 0.15171.$$

Finally the $\mathscr{H}_2$ model reduction algorithm in Section III is invoked and an approximant $\hat{G}_{\mathrm{APM}}^4(s)$ that achieves the following error is obtained:

$$\mathscr{J}(G(z), \hat{G}_{\mathrm{APM}}^4(s)) = 0.13494.$$

The data of the approximant (and also that of the approximants obtained later) is provided in [17]. It is pointed out that this is a slight improvement over the approximant $\hat{G}_{\mathrm{YL}}^4(s)$ reported in [9]:

$$\mathscr{J}(G(z), \hat{G}_{\mathrm{YL}}^4(s)) = 0.1354.$$

The same procedure is executed for $r = 5$. By retaining the modes corresponding to the 1st, 2nd, 3rd, 5th and 8th Hankel singular values, or the 1st, 2nd, 3rd, 5th and 6th largest $(c_i^* c_i)\sigma_i$, the best initial approximant is obtained:

$$\mathscr{J}(G(z), \hat{G}_{12358}(s)) = 0.10831.$$

This is improved a little by the method described in Subsection IV-B:

$$\mathscr{J}(G(z), \hat{G}'_{12358}(s)) = 0.10634.$$

Finally the model reduction algorithm finds an approximant achieving

$$\mathscr{J}(G(z), \hat{G}^5_{\mathrm{APM}}(s)) = 0.078078.$$

Again a slight improvement over the result in [9] is observed:

$$\mathscr{J}(G(z), \hat{G}^5_{\mathrm{YL}}(s)) = 0.0795.$$

For $r = 6$, the best initial approximant is constructed by keeping the modes corresponding to the 1st, 2nd, 3rd, 4th, 5th and 8th Hankel singular values, or the 1st, 2nd, 3rd, 5th, 6th and 7th largest $(c_i^* c_i)\sigma_i$. It achieves the error

$$\mathscr{J}(G(z), \hat{G}_{123458}(s)) = 0.078882.$$

The method in Subsection IV-B yields a slightly better approximant:

$$\mathscr{J}(G(z), \hat{G}'_{123458}(s)) = 0.076574.$$

Finally the model reduction algorithm finds a reduced order model whose error is

$$\mathscr{J}(G(z), \hat{G}^6_{\mathrm{APM}}(s)) = 0.052709.$$

This approximant is better than the one reported in [9]:

$$\mathscr{J}(G(z), \hat{G}^6_{\mathrm{YL}}(s)) = 0.0541.$$

VI. Conclusion

This paper has considered the $\mathscr{H}_2$ model reduction problem for both continuous and discrete time MIMO systems. The proposed algorithm employs necessary and sufficient conditions for the existence of an $\mathscr{H}_2$ suboptimal reduced order model obtained by means of LMI techniques, and the alternating projection method. The algorithm is a heuristic one due to a non-convex constraint. However, along with the suggested method for choosing starting points, it can find suboptimal approximants which are as good as those computed by previously proposed methods, which is demonstrated by numerical examples.

This algorithm is believed to have several advantages. It relies on off-the-shelf routines and requires rather simple programming. It covers both continuous and discrete time systems; The difference in the programs is trivial. The conditions share the same structure—several LMIs and a matrix rank constraint—as various controller synthesis problems with fixed controller order. Observe in particular some similarity of the conditions in [3] for $\mathscr{H}_\infty$ model reduction and those in [27] for model reduction in the ν-gap metric. Research on the solution of such problems is one of areas where intensive studies are carried out and the potential for the development of efficient, numerically reliable algorithms to tackle those problems including this approach to $\mathscr{H}_2$ model reduction can be large.

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