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Design of Stochastic Fault Tolerant Control for H_2 Performance

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Abstract— **In this paper, the controller synthesis problem for fault tolerant control systems (FTCS) with stochastic stability and H_2 performance is studied. The system faults of random nature are modeled by a Markov chain. Because the real system fault modes are not directly accessible in the context of FTCS, the controller is reconfigured based on the output of a Fault Detection and Identification (FDI) process, which is modeled by another Markov Chain. Then the state feedback control is developed for such systems to achieve the Mean Exponential Stability (MES) and the H_2 performance for both continuous-time and discrete-time systems. Furthermore, different types of model uncertainties are also considered in the design.**

I. INTRODUCTION

Due to the increasing demands for high reliability and survivability of the complex control systems, the fault tolerant control (FTC) has attracted extensive interests and attention from both industry and academia during the last two decades. Based on whether or not the controller needs to be reconfigured, the FTC methodologies can be classified into active and passive ones. The presence of Fault Detection and Identification (FDI) mechanism in the active FTC system make it has superior fault tolerance capability and less design constraints. However, when the separation principle does not hold under the circumstances of modeling uncertainty and unknown disturbances [1], the coupling between FDI and controller make the analysis and synthesis of active FTCS more complicated, and an integrated FTCS analysis or design is preferred for this situation.

In the previous work on integrated FTC analysis/design, if faults are modeled as external inputs of the system, then a multiple objective design approach can be taken using result from robust control theory, see [2], [3] etc. for details. On the other hand, if the random nature of faults is considered, faults/failures can be modeled by using a Markov chain, then the *open-loop* system is simply described as a Markovian jump linear system (MJLS).

Important results on stability, optimal control and robust performance of MJLS can be found in a flurry of published research papers, just to name a few, see [4]-[7]. However, these results for MJLS are not useful in the context of fault tolerant control, since practical FDI mechanisms can not always provide diagnosed results accurately and synchronously. The FDI is usually imperfect with possibilities of detection delays, false alarms and missing detections due to the model uncertainty and noises/disturbances. It does not always indicate the true operation mode of the system. For this reason, a second Markov chain was introduced to model a simple memoryless FDI decision process in [8].

This Markovian FTCS is a convenient framework for analysis and is useful for demonstrating the effects of imperfect FDI decision [9]. By using this formulation, [8], [10], [11] have studied the closed-loop FTCS stability, with or without the presence of noises. However the controller synthesis in this framework is more complicated, particularly because that the controller should only depend on the FDI process mode in practical applications. It means that the number of controllers to be designed is less than the total number of the closed loop system modes by combining those of both fault and FDI Markov chains. The design process involves searching feasible solutions of a problem where there are more constraints than the variables to be solved. Generally speaking, the synthesis problem for this stochastic FTC problem is not well solved yet. In [4], the state feedback controller for H_∞ performance was designed, which accesses not only FDI mode but also system real fault mode, just the same as in [12]. [13] relaxed this restriction by designing a controller based on cluster observation of Markov states. However a common Lyapunov function like approach is used, which means the information of FDI is at least partially neglected, conservative controllers are expected.

The authors accentuate that this FTC formulation is different from Markovian Jump Linear Systems especially in the synthesis problems. The latter is equivalent to the former only if it is assumed that the real fault mode is immediately available for the controller reconfiguration. Otherwise, the controller design for FTC system with two Markov chains is generally much more challenging. In this paper, we attempt to tackle this problem by providing a design procedure, considering not only the stability but also the H_2 performance.

The remaining of this paper is organized as follows: Section II contains the FTC systems modeling and the definition of H_2 norm for FTCS, and also the expression derived for such a definition. In section III, controller synthesis for continuous time systems has been addressed, where both polytopic and norm bounded model uncertainties have been taken into consideration. The results derived are in terms of LMIs, hence can be handled by available convex optimization tools. The design for the corresponding discrete time case is contained in Section IV, while in Section V, we include a numerical example to demonstrate the effectiveness of the algorithms, and some final conclusions are drawn in the Section VI.

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A. Notation

The notation used in the paper is fairly standard. For two homogeneous Markov Chains $r_t = i \in S_1$ and $l_t = j \in S_2$, we denote any matrix $M(r_t) = M_i$, $M(l_t) = M_j$ or $M(r_t, l_t) = M_{ij}$. $tr(.)$ denote the trace of $(.)$, $P > 0$ ($P \geq 0$) means that the matrix P is positive definite (positive semi-definite). $\mathscr{E}\{.\}$ stands for expectation. Pr means the probability, and $\mathscr{A}$ is the weak infinitesimal operator whose definition is given in Section II.

II. Modeling and Problem Formulation

A. Fault Tolerant Control Systems Modeling

The nominal system to be studied in this paper is given in the continuous-time as follows:

$$\mathscr{G}_c : \begin{cases} \dot{x}(t) = A(r_t)x(t) + B(r_t)u(t,l_t) + D(r_t)w(t) \\ y(t) = C(r_t)x(t) \end{cases} \tag{1}$$

or in discrete-time form as:

$$\mathscr{G}_d : \begin{cases} x(k+1) = A(r_k)x(k) + B(r_k)u(k,l_k) + D(r_k)w(k) \\ y(k) = C(r_k)x(k) \end{cases} \tag{2}$$

where w, x, y are external input, state and output, respectively. All the matrices have corresponding compatible dimensions. $\{r_t, t \geq 0\}$ (resp. $\{r_k, k \geq 0\}$) represents the fault process of the system, and is assumed to be a continuous/discrete time homogeneous Markov chain taking values on a finite set $S_1 = \{1,\ 2, \ldots,\ s_1\}$. Let its transition rate matrix be (α_{ij}), then it follows that:

- Continuous time:

$$\Pr\{r_{t+\Delta t} = j | r_t = i\} = \begin{cases} \alpha_{ij}\Delta t + o(\Delta t), & i \neq j \\ 1 + \alpha_{ii}\Delta t + o(\Delta t), & i = j \end{cases},$$

- Discrete time:

$$\Pr\{r_{k+1} = j | r_k = i\} = \alpha_{ij}$$

$\{l_t, t \geq 0\}$ ($\{l_k, k \geq 0\}$) is another independent finite state Markov Chain, which is used to model Fault Detection and Identification (FDI) mechanism of those active fault tolerant control systems, taking values on $S_2 = \{1,\ 2, \ldots,\ s_2\}$, with its one step transition probability matrix conditioned on the value of r_t (resp. r_k), e.g. for discrete time, $\Pr\{l_{k+1} = s | l_k = j, r_k = i\} = \beta^i_{js}$.

Such a formulation of fault tolerant control systems can model both system component and actuator faults/failures with random nature, using the memoryless statistical test for FDI mechanism. The fortes of this formulation lie in its capability of accounting for false alarms, missing detections and detection delay, important constraints imposed by practical FDI processes.

Given the system $\mathscr{G}_c$ (resp. $\mathscr{G}_d$), and the state feedback control law $u(t,l_t) = K(l_t)x(t)$ (resp. $u(k,l_k) = K(l_k)x(k)$ for discrete-time case), the closed-loop system model can then be written in following forms, assuming $r_t = i, l_t = j$:

$$\mathscr{G}_c : \begin{cases} \dot{x} = \tilde{A}_{ij}x + D_i w \\ y = C_i x \end{cases} \tag{3}$$

or in discrete-time form as:

$$\mathscr{G}_d : \begin{cases} x(k+1) = \tilde{A}_{ij}x(k) + D_i w(k) \\ y(k) = C_i x(k) \end{cases} \tag{4}$$

where for both cases, $\tilde{A}_{ij} = A_i + B_i K_j$.

Note that herein the controller law is solely dependant on the FDI process output l_t (or l_k in the discrete-time case). If combine both the fault modes of r_t and the FDI modes of l_t, there are totally $s_1 \times s_2$ modes; while the total number of controllers is only s_2. This fact makes the design problem more complicated. For practical systems, exact mathematical models are extremely hard or even impossible to obtain. In this paper, we consider two types of the most common used uncertainties. The first one is so-called polytopic type model uncertainty. We assume that system matrices lie within the uncertainty ploytope Ω:

$$\Omega = \{(A_i, B_i, C_i, D_i) | (A_i, B_i, C_i, D_i) = \textstyle\sum_{j=1}^m \tau_j (A_i^j, B_i^j, C_i^j, D_i^j); \tau_j \geq 0, \sum_{j=1}^m \tau_j = 1\}. \tag{5}$$

In this case, if the uncertainty is time-invariant or slowly time-varying, we can use parameter-dependent Lyapunov function approach to develop the stability conditions, which is expected to be less conservative compared with quadratic stability, where a single Lyapunov function is used.

Another type of model uncertainty adopted in this paper is norm-bounded uncertainty, which can be used to describe those time-varying uncertainties. The system matrices are assumed to have the form:$A_i = A_{0i} + A_{1i}\Delta_{1i}A_{2i}$, $B_i = B_{0i} + B_{1i}\Delta_{2i}B_{2i}$, where $\|\Delta_{1i}\| \leq 1$ and $\|\Delta_{2i}\| \leq 1$.

B. Definition of H_2 Norm for FTCS

In this subsection, first of all, the definition of H_2 norm for the stochastic Fault Tolerant Control Systems is presented. Then the expression of H_2 norm will be derived accordingly. The discussions herein are limited to continuous time systems while results for discrete-time system can be obtained similarly.

Parallel to the definition of H_2 norm for Markovian Jump Linear Systems ([14]), we define H_2 norm for stochastic FTCS $\mathscr{G}_c$ as follows:

Definition 2.1:

$$\|\mathscr{G}_c\|_2^2 = \sum_{i=1}^{s_1}\sum_{j=1}^{s_2}\sum_{m=1}^{N} \gamma_{ij}\|y_{ijm}\|_2^2 \tag{6}$$

where y_{ijm} is the output of the system with initial conditions $r(0) = i, l(0) = j$ and is disturbed by $w(t) = e_m\delta(t)$, e_m is a n dimensional vector with its ith position has 1 and all 0's at other positions, $\delta(t)$ is an impulse function and γ_{ij} is the initial probability distribution for $r(0) = i, l(0) = j$, where the norm for a stochastic signal is defined as $\|y\|_2^2 = \int_0^\infty \mathscr{E}\{y^T y\}\mathrm{d}t$. Correspondingly, for discrete-time systems, the external signal $w(0) = e_m$, and $w(t) = 0,\ t > 0$, the norm is defined as $\|y\|_2^2 = \sum_0^\infty \mathscr{E}\{y^T y\}$. □

Such a definition, when FDI process output l_t is identical to system fault mode r_t, is equivalent to the H_2 norm definition of MJLS.

The following definitions are given first that are to be used in the analysis later:

$$\begin{array}{l} Q_{ij} = \tilde{x}(t)\tilde{x}^T(t)1_{\{r(t)=i,l(t)=j\}} \\ < x,y >= \mathrm{tr}(x^* y) \\ \mathscr{T}(Q_{ij}) = \tilde{A}_{ij}Q_{ij} + Q_{ij}\tilde{A}_{ij}^T + \sum_k \alpha_{ik}Q_{kj} + \sum_k \beta_{jk}^i Q_{ik} \\ \mathscr{L}(Q_{ij}) = \tilde{A}_{ij}^T Q_{ij} + Q_{ij}\tilde{A}_{ij} + \sum_k \alpha_{ik}Q_{kj} + \sum_k \beta_{jk}^i Q_{ik} \\ \mathscr{A}V(t,r_t,l_t) \\ = \lim\limits_{\Delta\to 0}\frac{1}{\Delta}\Big(\mathscr{E}\{V(x(t+\Delta),r_t+\Delta,l_t+\Delta|x(t),r_t,l_t)\} \\ \quad -V(x(t),r_t,l_t)\Big) \end{array}$$

It is easy to verify that $\mathscr{A}(Q_{ij}) = \mathscr{T}(Q_{ij})$, where $\mathscr{A}(.)$ is the weak infinitesimal operator.

$$\begin{array}{ll} \mathscr{E}\{y^T y\} & = \sum_i \sum_j \mathscr{E}\{x^T C_i^T C_i x 1_{\{r(t)=i,l(t)=j\}}\} \\ & = \sum_i \sum_j \mathscr{E}\{\mathrm{tr}(xx^T C_i^T C_i) 1_{\{r(t)=i,l(t)=j\}}\} \\ & = \sum_i \sum_j < Q_{ij}, C_i^T C_i > \end{array}$$

Assume P_{ij} is the solution of the following coupled equations:

$$\mathscr{L}(P_{ij}) + \frac{1}{g_{ij}} C_i^T C_i = 0 \tag{7}$$

where g_{ij} is a positive scalar. Therefore

$$\begin{array}{ll} \mathscr{E}\{y^T y\} & = \sum_i \sum_j < Q_{ij}, -g_{ij}\mathscr{L}(P_{ij}) > \\ & = -\sum_i \sum_j \{g_{ij} < Q_{ij}, \mathscr{L}(P_{ij}) >\} \\ & = -\sum_i \sum_j \{g_{ij} < \mathscr{T}(Q_{ij}), P_{ij} >\} \\ & = -\sum_i \sum_j \{g_{ij} < \mathscr{A}(Q_{ij}), P_{ij} >\} \end{array}$$

$$\begin{array}{ll} & \|y_{ijm}\|_2^2|_{\{i=i_0,j=j_0,m=m_0\}} \\ = & \int_0^\infty \mathscr{E}\{y^T y\}dt \\ = & -\int_0^\infty \sum_i \sum_j \mathscr{E}\{g_{ij} < \mathscr{A}(Q_{ij}), P_{ij} >\}dt \\ = & -\sum_i \sum_j \mathscr{E}\{g_{ij} < Q_{ij}, P_{ij} >\}|_0^\infty \\ = & g_{ij} < Q_{ij}(0), P_{ij} > |_{\{i=i_0,j=j_0,m=m_0\}} \\ = & g_{i_0 j_0}\mathrm{tr}(D_{i_0} e_{m_0} e_{m_0}^T D_{i_0}^T P_{i_0 j_0}) \end{array}$$

Then we can get the expression that,

$$\begin{array}{ll} \|\mathscr{G}_c\|_2^2 & = \sum_i \sum_j \sum_m \gamma_{ij} \|y_{ijm}\|_2^2 \\ & = \sum_i \sum_j \sum_m \gamma_{ij} g_{ij} e_m^T D_i^T P_{ik} D_i e_m \\ & = \sum_i \sum_j \mathrm{tr}(D_i^T P_{ij} D_i) \end{array}$$

in the last step of derivation above, we set $g_{ij} = \frac{1}{\gamma_{ij}}$.

From above derivations, we can obtain the following expression for H_2 norm of FTCS:

$$\begin{array}{rl} & \|\mathscr{G}_c\|_2^2 = \sum_i \sum_j \mathrm{tr}(D_i^T P_{ij} D_i) \\ s.t. & \mathscr{L}(P_{ij}) + \gamma_{ij} C_i^T C_i = 0 \end{array} \tag{8}$$

Remark 1: The expression above shows that the initial distribution γ_{ij} of both system mode and FDI process will affect the H_2 norm of the system. Furthermore, we need to point out that if only the FTCS is stable, g_{ij} can take any positive value to make the above definition valid, but for simplicity, we set $g_{ij} = \frac{1}{\gamma_{ij}}$.

Using Lyapunov theorem, the formulation above can be rewritten in the convex optimization form of with all constraints are in terms of matrix inequalities:

Lemma 2.1: Controllers K are called the optimal H_2 controller of stochastic FTCS $\mathscr{G}_c$, if minimal objective value J^* is achieved as the result of the following constrained optimization:

$$\begin{array}{rl} J = & \inf\limits_K \sum_i \sum_j \gamma_i \mathrm{tr}(Z_{ij}) \\ s.t. & D_j^T P_{ij} D_j < Z_{ij} \\ & \mathscr{L}(P_{ij}) + \gamma_{ij} C_i^T C_i < 0 \\ & P_{ij} > 0 \end{array} \tag{9}$$

III. Synthesis of Continuous Time H_2 Controller

With the definition of H_2 norm of the stochastic FTCS, and matrix inequality formulation of H_2 controller design, in this section, the synthesis of continuous time controller for both polytopic and norm-bounded uncertain systems are addressed. We begin this section with some lemmas which will be used in derivation of this section and the section IV.

Lemma 3.1: (Reciprocal Projection Lemma, [15]): Let P be any given positive-definite matrix, the following statements are equivalent:

1) $\Psi + S + S^T < 0$
2) the LMI

$$\begin{bmatrix} \Psi + P - (W + W^T) & S^T + W^T \\ S + W & -P \end{bmatrix} < 0 \tag{10}$$

is feasible with respect to W.

Lemma 3.2: The following conditions are equivalent ($f(P) > 0$ is a matrix function of P):

1) There exists a symmetric matrix $P > 0$ such that

$$A^T PA - f(P) < 0 \tag{11}$$

2) There exist a symmetric matrix P and a matrix G such that

$$\begin{bmatrix} f(P) & A^T G^T \\ GA & G + G^T - P \end{bmatrix} > 0 \tag{12}$$

Proof: The prototype of this lemma is shown in [16], and the detailed proof is omitted here. □

A. Controller Synthesis for Polytopic Uncertain Systems

Under the assumption that all states are accessible, the state feedback control strategy can be adopted.

the inequality $\mathscr{L}(P_{ij}) + \gamma_{ij} C_i^T C_i < 0$ in Eq. (9) takes the form:

$$\begin{array}{r} A_i^T P_{ij} + P_{ij} A_i + K_j^T B_i^T P_{ij} + P_{ij} B_i K_j \\ + \sum_k \alpha_{ik} P_{kj} + \sum_k \beta_{jk}^i P_{ik} + \gamma_{ij} C_i^T C_i < 0 \end{array} \tag{13}$$

Using Reciprocal Projection Lemma to expand this matrix inequality into:

$$\begin{bmatrix} \sum_k \alpha_{ik} P_{kj} + \sum_k \beta_{jk}^i P_{ik} + \gamma_{ij} C_i^T C_i + \bar{P}_{ij} - W_{ij} - W_{ij}^T & P_{ij} A_i + P_{ij} B_i K_j + W_{ij} \\ A_i^T P_{ij} + K_j^T B_i^T P_{ij} + W_{ij}^T & -\bar{P}_{ij} \end{bmatrix} < 0 \tag{14}$$

Define: $X_{ij} = P_{ij}^{-1}$, $\bar{W}_{ij} = X_{ij} W_{ij}$, and pre-, post-multiply the left side of the inequality by diag$\{X_{ij}, I\}$, we obtain:

$$\begin{bmatrix} X_{ij}(\sum_k \alpha_{ik} P_{kj} + \sum_k \beta_{jk}^i P_{ik} + \gamma_{ij} C_i^T C_i + \bar{P}_{ij}) X_{ij} - \bar{W}_{ij} X_{ij} - X_{ij} \bar{W}_{ij}^T & A_i + B_i K_j + \bar{W}_{ij} \\ A_i^T + K_j^T B_i^T + \bar{W}_{ij}^T & -\bar{P}_{ij} \end{bmatrix} < 0 \tag{15}$$

According to Reciprocal Projection Lemma, $\bar{P}_{ij}$ can be any positive matrix, we choose $\bar{P}_{ij} = \lambda_{ij} I - \gamma_{ij} C_i^T C_i$, where λ_{ij} is a scaler large enough to guarantee the positivity of $\bar{P}_{ij}$.

Then (15) changes into

$$\begin{bmatrix} X_{ij}(\sum_k \alpha_{ik} P_{kj} + \sum_k \beta^i_{jk} P_{ik}) X_{ij} + \lambda_{ij} X_{ij} X_{ij} - \bar{W}_{ij} X_{ij} - X_{ij} \bar{W}_{ij}^T & A_i + B_i K_j + \bar{W}_{ij} \\ A_i^T + K_j^T B_i^T + \bar{W}_{ij}^T & -\lambda_{ij} I + \gamma_{ij} C_i^T C_i \end{bmatrix} < 0 \tag{16}$$

For the inequality above, use Schur complement and the inequality $-\lambda_{ij}^{-1} \bar{W}_{ij} \bar{W}_{ij}^T \leq -\bar{W}_{ij} - \bar{W}_{ij}^T + \lambda_{ij} I$ to get a sufficient condition. Then we obtain:

$$\begin{bmatrix} (\alpha_{ii} + \beta^i_{jj}) X_{ij} - \bar{W}_{ij} - \bar{W}_{ij}^T + \lambda_{ij} I & \lambda_{ij} X_{ij} - \bar{W}_{ij} & H_{1ij} & A_i + B_i K_j + \bar{W}_{ij} & 0 \\ \lambda_{ij} X_{ij} - \bar{W}_{ij}^T & -\lambda_{ij} I & 0 & 0 & 0 \\ H_{1ij}^T & 0 & H_{2ij} & 0 & 0 \\ A_i^T + K_j^T B_i^T + \bar{W}_{ij}^T & 0 & 0 & -\lambda_{ij} I & \sqrt{\gamma_{ij}} C_i^T \\ 0 & 0 & 0 & \sqrt{\gamma_{ij}} C_i & -I \end{bmatrix} < 0 \tag{17}$$

where

$$\begin{aligned} H_{1ij} = \; & X_{ij} \left[\sqrt{\alpha_{i1}} I, \cdots, \sqrt{\alpha_{i,i-1}} I, \sqrt{\alpha_{i,i+1}} I, \cdots, \right. \\ & \left. \sqrt{\beta^i_{j1}} I, \cdots, \sqrt{\beta^i_{j,j-1}} I, \sqrt{\beta^i_{j,j+1}} I, \cdots \right] \\ H_{2ij} = \; & -\mathrm{diag}\{X_{1j}, \cdots, X_{i-1,j}, X_{i+1,j}, \cdots, X_{i1}, \cdots, \\ & X_{i,j-1}, X_{i,j+1}, \cdots\} \end{aligned}$$

To summarize the formulation above, we have the following theorem:

Theorem 3.1: For stochastic FTCS $\mathscr{G}_c$ with polytopic uncertainty, the H_2 controllers exist, if for preset positive scalars λ_{ij}, substitute into $A_i^k, B_i^k, C_i^k, D_i^k$, and feasible solution for $X_{ij}^k, W_{ij}, Z_{ij}, \lambda_{ij}$, where $i \in S_1$, $j \in S_2$, $k = 1, 2, \cdots, m$, for all i, j, k, LMIs list below are feasible:

$$\begin{array}{l} \sum_i \sum_j \mathrm{tr}(Z_{ij}) < \mu \\ \begin{bmatrix} -Z_{ij} & D_i^T \\ D_i & -X_{ij} \end{bmatrix} < 0 \\ Eq.(17) \\ X_{ij} > 0 \end{array} \tag{18}$$

B. Design for Norm-bounded Uncertain Systems

To accommodate model uncertainties in the system, we need the following lemma.

Lemma 3.3: For all admissible norm-bounded model uncertainty, there exists $\varepsilon_{ij}, \delta_{ij} > 0$, following two inequalities hold:

$$\begin{aligned} & P_{ij} A_{1i} \Delta_{1i} A_{2i} + A_{2i}^T \Delta_{1i}^T A_{1i}^T P_{ij} \leq \varepsilon_{ij}^{-1} A_{2i}^T A_{2i} + \varepsilon_{ij} P_{ij} A_{1i} A_{1i}^T P_{ij} \\ & P_{ij} B_{1i} \Delta_{2i} B_{2i} K_j + K_j^T B_{2i}^T \Delta_{2i}^T B_{1i}^T P_{ij} \leq \delta_{ij} P_{ij} B_{1i} B_{1i}^T P_{ij} \\ & \qquad + \delta_{ij}^{-1} K_j^T B_{2i}^T B_{2i} K_j \end{aligned} \tag{19}$$

Using this lemma, and substitute the result into (13) to obtain:

$$\begin{aligned} & A_{0i}^T P_{ij} + P_{ij} A_{0i} + K_j^T B_{0i}^T P_{ij} + P_{ij} B_{0i} K_j + \sum_k \alpha_{ik} P_{kj} \\ & + \sum_k \beta^i_{jk} P_{ik} + \gamma_{ij} C_i^T C_i + \varepsilon_{ij}^{-1} A_{2i}^T A_{2i} + \varepsilon_{ij} P_{ij} A_{1i} A_{1i}^T P_{ij} \\ & + \delta_{ij} P_{ij} B_{1i} B_{1i}^T P_{ij} + \delta_{ij}^{-1} K_j^T B_{2i}^T B_{2i} K_j < 0 \end{aligned} \tag{20}$$

Still use Reciprocal Projection Lemma to expand the inequality as:

$$\begin{bmatrix} G_{ij} & P_{ij} A_{0i} + P_{ij} B_{0i} K_j + W_{ij} & \varepsilon_{ij} P_{ij} A_{1i} & \delta_{ij} P_{ij} B_{1i} \\ A_{0i}^T P_{ij} + K_j^T B_{0i}^T P_{ij} + W_{ij}^T & -\bar{P}_{ij} & 0 & 0 \\ \varepsilon_{ij} A_{1i}^T P_{ij} & 0 & -\varepsilon_{ij} I & 0 \\ \delta_{ij} B_{1i}^T P_{ij} & 0 & 0 & -\delta_{ij} I \end{bmatrix} < 0 \tag{21}$$

where $G_{ij} = \sum_k \alpha_{ik} P_{kj} + \sum_k \beta^i_{jk} P_{ik} + \gamma_{ij} C_i^T C_i + \varepsilon_{ij}^{-1} A_{2i}^T A_{2i} + \delta_{ij}^{-1} K_j^T B_{2i}^T B_{2i} K_j + \bar{P}_{ij} - W_{ij} - W_{ij}^T$, and $\bar{P}_{ij}, W_{ij}$ are slack variables introduced.

define $X_{ij} = P_{ij}^{-1}$, $\bar{W}_{ij} = X_{ij} W_{ij}$, and $\bar{P}_{ij} = \lambda_{ij} I - \gamma_{ij} C_i^T C_i - \delta_{ij}^{-1} K_j^T B_{2i}^T B_{2i} K_j$, where λ_{ij} is a positive scalar large enough to guarantee that $\bar{P}_{ij} > 0$ is satisfied. Using the similar technique as for polytopic type uncertainty case, we obtain the following inequality. Due to the page limit, the detailed derivation procedure is omitted here.

$$\begin{bmatrix} (\alpha_{ii} + \beta^i_{jj}) bar X_{ij} - \bar{W}_{ij} - \bar{W}_{ij}^T + \lambda_{ij} I & X_{ij} A_{2i}^T & \varepsilon_{ij} A_{1i} & \delta_{ij} B_{1i} & H_{1ij} & \lambda_{ij} X_{ij} - \bar{W}_{ij} & 0 & A_{0i} + B_{0i} K_j + \bar{W}_{ij} & 0 \\ A_{2i} X_{ij} & -\varepsilon_{ij} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{ij} A_{1i}^T & 0 & -\varepsilon_{ij} I & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_{ij} B_{1i}^T & 0 & 0 & -\delta_{ij} I & 0 & 0 & 0 & 0 & 0 \\ H_{1ij}^T & 0 & 0 & 0 & H_{2ij} & 0 & 0 & 0 & 0 \\ \lambda_{ij} X_{ij} - \bar{W}_{ij}^T & 0 & 0 & 0 & 0 & -\lambda_{ij} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I & \sqrt{\gamma_{ij}} C_i & 0 \\ A_{0i}^T + K_j^T B_{0i}^T + \bar{W}_{ij}^T & 0 & 0 & 0 & 0 & 0 & \sqrt{\gamma_{ij}} C_i^T & -\lambda_{ij} I & K_j^T B_{2i}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{2i} K_j & -\delta_{ij} I \end{bmatrix} < 0 \tag{22}$$

where all variables are defined as the same as in Eq. (17).

Theorem 3.2: The H_2 controller exists, if we can find feasible solution $\bar{X}_{ij}, \bar{W}_{ij}, \lambda_{ij}, \bar{\varepsilon}_{ij}, \bar{\delta}_{ij}$ for the following LMIs:

$$\begin{array}{l} \sum_i \sum_j \mathrm{tr}(Z_{ij}) < \mu \\ \begin{bmatrix} -Z_{ij} & D_i^T \\ D_i & -X_{ij} \end{bmatrix} < 0 \\ Eq.(22) \\ X_{ij} > 0, \varepsilon_{ij} > 0, \delta_{ij} > 0 \end{array} \tag{23}$$

IV. Discrete Time Design

Similar to continuous time situation, the expression of H_2 norm for FTCS is given as follows:

$$\begin{aligned} & \|\mathscr{G}_d\|_2^2 = \sum_i \sum_j \mathrm{tr}(D_i^T P_{ij} D_i) \\ s.t. \quad & \mathscr{R}(P_{ij}) + \gamma_{ij} C_i^T C_i = 0 \end{aligned} \tag{24}$$

where

$$\mathscr{R}(P_{ij}) = \tilde{A}_{ij}^T (\sum_k \alpha_{ik} P_{kj} + \sum_k \beta^i_{jk} P_{ik}) \tilde{A}_{ij} - P_{ij} \tag{25}$$

So the optimal H_2 control problem of the system $\mathscr{G}_d$ can be expressed in a convex optimization problem as:

$$\begin{array}{ll} & \inf_K \sum_i \sum_j \operatorname{tr}(Z_{ij}) \\ s.t. & D_j^T P_{ij} D_j < Z_{ij} \\ & \mathscr{R}(P_{ij}) + \gamma_{ij} C_i^T C_i < 0 \\ & P_{ij} > 0, \end{array} \tag{26}$$

Like in continuous-time case, here we still consider guaranteed cost control, i.e. pres-set the upper bound μ for H_2 norm to be achieved.

A. Design for Polytopic Uncertain Systems

The inequality $\mathscr{R}(P_{ij}) + \gamma_{ij} C_i^T C_i < 0$ takes the form:

$$\begin{array}{l} (A_i + B_i K_j)^T (\sum_k \alpha_{ik} P_{kj} + \sum_k \beta^i_{jk} P_{ik})(A_i + B_i K_j) \\ -P_{ij} + \gamma_{ij} C_i^T C_i < 0 \end{array} \tag{27}$$

using the lemma 3.2, we know that the feasibility of of the matrix inequality above is equivalent the feasibility of

$$\begin{bmatrix} \gamma_{ij} C_i^T C_i - P_{ij} & (A_i + B_i K_j)^T G_{ij}^T \\ G_{ij}(A_i + B_i K_j) & (\sum_k \alpha_{ik} P_{kj} + \sum_k \beta^i_{jk} P_{ik}) - G_{ij} - G_{ij}^T \end{bmatrix} < 0 \tag{28}$$

notice that $(\sum_k \alpha_{ik} P_{kj} + \sum_k \beta^i_{jk} P_{ik}) - G_{ij} - G_{ij}^T < 0$, hence $G_{ij} + G_{ij}^T > 0$, so G_{ij} is nonsingular. Therefore we can define $\bar{G}_{ij} = G_{ij}^{-1}$, pre- and post-multiply by $\operatorname{diag}\{I, \bar{G}_{ij}\}$ and its transpose,

$$\begin{bmatrix} \bar{G}_{ij}(\sum_k \alpha_{ik} P_{kj} + \sum_k \beta^i_{jk} P_{ik})\bar{G}_{ij}^T - \bar{G}_{ij} - \bar{G}_{ij}^T & A_i + B_i K_j \\ A_i^T + K_j^T B_i^T & \gamma_{ij} C_i^T C_i - P_{ij} \end{bmatrix} < 0 \tag{29}$$

Expand the inequality using Schur complement to obtain:

$$\begin{bmatrix} -\bar{G}_{ij} - \bar{G}_{ij}^T & H_{3ij} & A_i + B_i K_j \\ H_{3ij}^T & H_{4ij} & 0 \\ A_i^T + K_j^T B_i^T & 0 & \gamma_{ij} C_i^T C_i - P_{ij} \end{bmatrix} < 0 \tag{30}$$

where

$$\begin{array}{l} H_{3ij} = \bar{G}_{ij}\left[\sqrt{\alpha_{i1}} I, \sqrt{\alpha_{i2}} I, \cdots, \sqrt{\beta^i_{j1}} I, \cdots\right] \\ H_{4ij} = -\operatorname{diag}\{X_{1j}, X_{2j}, \cdots, X_{i1}, \cdots\} \end{array}$$

We notice that in Eq. (30), both P_{ij},X_{ij} appear and we have the non-convex constraint $P_{ij} X_{ij} = I$. The same situation appears in *static output-feedback* (SOF) stabilization problem. A number of numerical algorithms have been proposed for solving this problem. The LMI-based algorithms include alternating projection, min-max algorithm, the XY-centering algorithm, the cone complementarity linearization (CCL) algorithm. In [17], numerical experiments have been made to compare the performance and convergence of some algorithms. Here, we adopt the *sequential linear programming matrix method* (SLPMM) proposed in [18], which improved the CCL algorithm to guarantee the convergence. In the following, we will briefly state the algorithm.

First we give out the definition of two sets:

$$\Sigma_1(Z_{ij}, \bar{G}_{ij}, P_{ij}, X_{ij}) : \begin{array}{l} \sum_i \sum_j \operatorname{tr}(Z_{ij}) < \mu \\ \begin{bmatrix} -Z_{ij} & D_i^T \\ D_i & -X_{ij} \end{bmatrix} < 0 \\ Eq.(30) \\ X_{ij} > 0 \end{array} \tag{31}$$

$$\Sigma_2(P_{ij}, X_{ij}) : \begin{bmatrix} P_{ij} & I \\ I & X_{ij} \end{bmatrix} > 0 \tag{32}$$

Algorithm 1:

1) Determine $(P_{ij}^0, X_{ij}^0, Z_{ij}^0, \bar{G}_{ij}^0) \in \Sigma_1 \cap \Sigma_2$
For $k = 0, 1, 2, \cdots, do$
2)
$$\begin{array}{rl} (\tilde{P}_{ij}^k, \tilde{Q}_{ij}^k, \tilde{Z}_{ij}^k, \tilde{G}_{ij}^k) = & \min \sum_i \sum_j \operatorname{tr}(P_{ij} X_{ij}^k + P_{ij}^k X_{ij}) \\ & s.t. Z_{ij}, \bar{G}_{ij}, P_{ij}, X_{ij} \in \Sigma_1 \cap \Sigma_2 \end{array} \tag{33}$$
3) If $\operatorname{tr}(\tilde{P}_{ij}^k X_{ij}^k + P_{ij}^k \tilde{X}_{ij}^k) = 2\operatorname{tr}(P_{ij}^k Q_{ij}^k)$, $\longrightarrow Stop$
4)
$$c^* = \min_{c \in [0,1]} \sum_i \sum_j \operatorname{tr}((P_{ij}^k + c(\tilde{P}_{ij}^k - P_{ij}^k))(P_{ij}^k + c(\tilde{P}_{ij}^k - P_{ij}^k))) \tag{34}$$
5) Set $P_{ij}^{k+1} = (1-c^*)P_{ij}^k + c^* \tilde{P}_{ij}^k$, $X_{ij}^{k+1} = (1-c^*)X_{ij}^k + c^* \tilde{X}_{ij}^k$, $Z_{ij}^{k+1} = (1-c^*)Z_{ij}^k + c^* \tilde{Z}_{ij}^k$,$\bar{G}_{ij}^{k+1} = (1-c^*)\bar{G}_{ij}^k + c^* \tilde{G}_{ij}^k$.

For polytopic uncertain systems, matrices corresponding to all vertex of the polytope should submit to solve the solution, as like continuous time case.

B. Discrete Time Synthesis for Norm-bounded Uncertain Systems

Within this situation, we have

$$\begin{bmatrix} P_{ij} - \gamma_{ij} C_i^T C_i & (A_i + B_i K_j)^T G_{ij}^T \\ G_{ij}(A_i + B_i K_j) & G_{ij} + G_{ij}^T - (\sum_k \alpha_{ik} P_{kj} + \sum_k \beta^i_{jk} P_{ik}) \end{bmatrix} > 0 \tag{35}$$

that is:

$$\begin{array}{l} \begin{bmatrix} P_{ij} - \gamma_{ij} C_i^T C_i & (A_{0i} + B_{0i} K_j)^T G_{ij}^T \\ G_{ij}(A_{0i} + B_{0i} K_j) & G_{ij} + G_{ij}^T - (\sum_k \alpha_{ik} P_{kj} + \sum_k \beta^i_{jk} P_{ik}) \end{bmatrix} \\ + \begin{bmatrix} A_{2i}^T & K_j^T B_{2i}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{1i}^T & 0 \\ 0 & \Delta_{2i}^T \end{bmatrix} \begin{bmatrix} 0 & A_{1i}^T G_{ij}^T \\ 0 & B_{1i}^T G_{ij}^T \end{bmatrix} \\ + \begin{bmatrix} 0 & 0 \\ G_{ij} A_{1i} & G_{ij} B_{1i} \end{bmatrix} \begin{bmatrix} \Delta_{1i} & 0 \\ 0 & \Delta_{2i} \end{bmatrix} \begin{bmatrix} A_{2i} & 0 \\ B_{2i} K_j & 0 \end{bmatrix} > 0 \end{array}$$

Considering all model uncertainties, we have that the inequality holds for all admissive model uncertainties, if and only if we can find positive scalars $\varepsilon_{ij} > 0$ and $\delta_{ij} > 0$ such that the following inequality holds

$$\begin{bmatrix} H_{5ij} & (A_{0i} + B_{0i} K_j)^T G_{ij}^T \\ G_{ij}(A_{0i} + B_{0i} K_j) & H_{6ij} \end{bmatrix} < 0 \tag{36}$$

where,

$$\begin{array}{rl} H_{5ij} = & \gamma_{ij} C_i^T C_i - P_{ij} + \varepsilon_{ij}^{-1} A_{2i}^T A_{2i} + \delta_{ij}^{-1} K_j^T B_{2i}^T B_{2i} K_j \\ H_{6ij} = & -G_{ij} - G_{ij}^T + (\sum_k \alpha_{ik} P_{kj} + \sum_k \beta^i_{jk} P_{ik}) \\ & + G_{ij}(\varepsilon_{ij} A_{1i} A_{1i}^T + \delta_{ij} B_{1i} B_{1i}^T) G_{ij}^T \end{array}$$

Use the similar technique we applied for previous case, we can obtain:

$$\begin{bmatrix} -\bar{G}_{ij} - \bar{G}_{ij}^T + \varepsilon_{ij} A_{1i} A_{1i}^T + \delta_{ij} B_{1i} B_{1i}^T & H_{3ij} & 0 & A_{0i} + B_{0i} K_j & 0 \\ H_{3ij}^T & H_{4ij} & -\delta_{ij} I & B_{2i} K_j & 0 \\ A_{0i}^T + K_j^T B_{0i}^T & 0 & K_j^T B_{2i}^T & \gamma_{ij} C_i^T C_i - P_{ij} & A_{2i}^T \\ 0 & 0 & 0 & A_{2i} & -\varepsilon_{ij} I \end{bmatrix} < 0 \tag{37}$$

All the definitions are the same as in previous section, and similarly, SLPMM algorithm can be used to calculate $P_{ij}, X_{ij}, \bar{G}_{ij}, Z_{ij}, \varepsilon_{ij}, \delta_{ij}$.

V. Numerical Example

In the simulation, the controllers are designed for continuous-time systems with polytopic and norm-bounded uncertainties. The systems to considered have two mode, subscript 1 stands for normal operation mode while subscript 2 stands for faulty systems.

Case 1: Polytopic Type Uncertainty

$$A_1^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix}, A_1^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.9 \end{bmatrix},$$
$$A_2^1 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix}, A_2^2 = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.9 \end{bmatrix}$$
$$B_1^1 = \begin{bmatrix} 0 & 1 \\ -0.25 & 0.25 \end{bmatrix}, B_1^2 = \begin{bmatrix} 0 & 0.9 \\ -0.25 & 0.20 \end{bmatrix},$$
$$B_2^1 = \begin{bmatrix} 0 & 0.2 \\ -0.25 & 0.05 \end{bmatrix}, B_2^2 = \begin{bmatrix} 0 & 0.18 \\ -0.25 & 0.04 \end{bmatrix}$$
$$C_1^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, C_1^2 = \begin{bmatrix} 0.9 & 0 \end{bmatrix}, C_2^1 = \begin{bmatrix} 1.5 & 0 \end{bmatrix}, C_2^2 = \begin{bmatrix} 1.7 & 0 \end{bmatrix}$$
$$D_1^1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, D_1^2 = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}, D_2^1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, D_2^2 = \begin{bmatrix} 0.4 \\ 1 \end{bmatrix}$$

Transition rate matrices are $\alpha = \begin{bmatrix} -0.5 & 0.5 \\ 1 & -1 \end{bmatrix}$, $\beta^1 = \begin{bmatrix} -0.1 & 0.1 \\ 0.15 & -0.15 \end{bmatrix}$, $\beta^1 = \begin{bmatrix} -0.2 & 0.2 \\ 0.1 & -0.1 \end{bmatrix}$ and in the simulation, $\lambda_{ij} = 10$, $\mu = 10$, $\gamma = \begin{bmatrix} 1/3 & 1/3 & 1/6 & 1/6 \end{bmatrix}$ is set according to the stationary distribution probability of the Markov Chain r_t.

The simulation Results are as follows:

$$K_1 = \begin{bmatrix} -1.2805 & 9.1339 \\ -9.3639 & -1.224 \end{bmatrix}, K_2 = \begin{bmatrix} -1.2739 & 9.1386 \\ -9.3772 & -1.2155 \end{bmatrix}$$
$$Z_{11} = 1.6249,\ Z_{12} = 1.625,\ Z_{21} = 3.5232,\ Z_{22} = 3.5198$$

Case 2: Norm-bounded Type Uncertainty

$$A_{01} = A_{02} = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix}, A_{11} = A_{12} = A_{21} = A_{22} = 0.2 * I_2$$
$$B_{01} = \begin{bmatrix} 0 & 1 \\ -0.25 & 0.25 \end{bmatrix}, B_{02} = \begin{bmatrix} 0 & 0.2 \\ -0.25 & 0.05 \end{bmatrix},$$
$$B_{11} = B_{12} = B_{21} = B_{22} = 0.2 * I_2,$$
$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, D_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

And other parameter settings are the same as previous case. Simulation results are as follows:

$$K_1 = \begin{bmatrix} -1.2662 & 8.1203 \\ -7.8225 & -1.2159 \end{bmatrix}, K_2 = \begin{bmatrix} -1.263 & 8.1282 \\ -7.8332 & -1.214 \end{bmatrix}$$
$$Z_{11} = 3.1678,\ Z_{12} = 3.168,\ Z_{21} = 5.4573,\ Z_{22} = 5.4567$$

VI. Conclusion

In this paper, we study H_2 controller synthesis for the uncertain stochastic fault tolerant systems. For both polytopic type and norm-bounded state space parameter uncertainties, design algorithms are derived for both continuous-time and discrete-time cases. In the discrete-time case, an SLPMM iterative LMI algorithm is adopted to solve nonconvex optimization. Compared with the ordinary MJLS state feedback design, the design problem of FTCS is more involved due to the imperfect FDI scheme. A Numerical example is given to demonstrate the effectiveness of the design.

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