Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Multivariable practical higher order sliding mode control

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***Abstract*— This paper presents a new *practical* higher order sliding mode controller for multi input-multi output (MIMO) nonlinear systems. The advantages of the approach are multiple: first, the design is simple and the applicability is real; secondly, the conditions of existence of this class of controllers are weak ; the class of nonlinear systems for which the controllers are designed are large.**

I. Introduction

In order to overcome the classical problem of *chattering* in standard sliding modes, a new approach called "higher order sliding mode" has been recently proposed [1], [7], [15]. Instead of influencing the first time derivative of sliding manifold, the "sign" function is acting on higher time derivatives. Keeping the main advantages of the standard sliding mode control, the chattering effect is eliminated and higher order precision is provided [15]. The main results concern the second order sliding mode control (for example, [7], [15] with well-known "twisting" and "super-twisting" algorithms, [1] derived from the optimal bang-bang control and proposed for SISO nonlinear systems with uncertainties). Even if the design of r-order sliding controllers ($r \geq 3$) is difficult, first results have been proposed in [16], [17], [18] for SISO nonlinear systems and were inspired by the so-called "terminal sliding modes control" [24]. An alternative approach recently comes into view in [12] for SISO nonlinear systems and uses standard sliding mode control with linear quadratic (LQ) one converging over a finite time interval with a fixed final state [20].

The extension of these results to the MIMO nonlinear systems is an exciting challenge. Very few results on the higher order sliding mode control for MIMO nonlinear systems have been done, mainly due to the non-applicability of Lyapunov's direct method. Some existing results [2], [9], [11], [19] only concern the second order sliding mode control of systems with a "low" coupling between the inputs and the sliding variables. However, this condition is quite restrictive (for example, the Lagrangian systems do not enter in this class). For this class of systems, a solution is given by [2] but is difficult to apply and ensures only an asymptotic convergence. An other solution based on regular form has been proposed in [9]. An other solution is given in [19], but this result is based on quite restrictive conditions.

The main contribution in this current paper is the extension of [12] for MIMO nonlinear systems. Due to the control design manner, this extension is natural and quite easy. As a matter of fact, in [12], the establishment of higher order sliding mode is equivalent to establish a 1-order sliding mode on a time-varying linear manifold. Then, the generation of a higher order sliding mode is equivalent to the generation of 1-order sliding modes on p (p, output dimension) time-varying linear manifolds. Then, it is possible to use the well-known techniques of the multivariable standard sliding mode control [3], [4], [5], [6], [22], [23].

The proposed result is based on weaker conditions than [2], [9], [11], [19], [14]: in this last reference, a first attempt to generalize [12] is proposed by using the restrictive "low" coupling condition. In the present paper, this structural property is not needed. The problem of the higher order sliding mode control of MIMO minimum-phase uncertain systems is formulated in input-output terms only through the differentiation of the sliding vector s, and via a change of coordinates, is equivalent to the finite time stabilization of integrators chains with nonlinear uncertainties [12]. These latter are considered as bounded non structured parametric uncertainties: in this case, the system can be viewed as an uncertain linear system. Then, following the optimal higher order sliding mode solution for SISO uncertain nonlinear systems [12], an optimal time varying switching manifold is determined by minimizing a quadratic cost function over a finite time interval $[0,\ t_F]$ with a fixed final state. The standard sliding mode over this manifold leads to the establishment of higher order sliding mode in finite time with respect to the sliding variable vector $s = [s_1\ \dots\ s_p]^T$.

The control algorithm, which is given in the *practical* higher order sliding mode context, needs the relative degree ρ_i with respect to s_i and the bounds of uncertainties and has several advantages

- the convergence time is fixed *a priori* via parameter t_F,
- the control law can be adjusted via t_F and two constant weighting matrices P_f and Q,
- this strategy can be applied for all value of sliding mode order (greater or equal to the relative degree) under a structural condition,
- the structure of the controller is well-adapted for a practical implementation (a first version has been *experimentally* checked on electrical motors [13], [14]).

II. Problem formulation

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

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where $x \in \mathcal{X} \subset I\!R^n$ is the state variable, $u \in I\!R^p$ is the input control vector and $y \in I\!R^p$ is the output vector. $f(x)$, $g(x)$ and $h(x)$ are uncertain sufficiently smooth functions.

Definition 1: [10] The system (1) has a *relative degree vector* $[\rho_1 \ \cdots \ \rho_p]^T$ at x_0 if $L_{g_j} L_f^k h_i(x) = 0$ for $1 \leq j \leq p$, $1 \leq i \leq p$, $k < \rho_i - 1$ and for all x in a neighborhood of x_0, and if the $p \times p$ matrix

$$\mathcal{A} = \begin{bmatrix} L_{g_1} L_f^{\rho_1 - 1} h_1(x) & \cdots & L_{g_p} L_f^{\rho_1 - 1} h_1(x) \\ L_{g_1} L_f^{\rho_2 - 1} h_2(x) & \cdots & L_{g_p} L_f^{\rho_2 - 1} h_2(x) \\ \vdots & \vdots & \vdots \\ L_{g_1} L_f^{\rho_p - 1} h_p(x) & \cdots & L_{g_p} L_f^{\rho_p - 1} h_p(x) \end{bmatrix} \tag{2}$$

is nonsingular at $x = x_0$. ■

Remark 1: If Definition 1 fulfills, preliminary feedback may be applied to system (1) [10]: this feedback is designed such that it decouples the nominal system (without uncertainties). Then, without loss of generality, in order to simplify the notations, after suitable reordering of the y_i's, the output y_i is associated to the input u_i with a relative degree ρ_i. ■

Let $y_R(t) = [y_{R1} \ y_{R2} \ \cdots \ y_{Rp}]^T$ denote the desired trajectories that outputs of (1) are forced to track through the control input u. These trajectories are supposed to be sufficiently differentiable. Let $s(x,t) = [s_1 \ s_2 \ \cdots \ s_p]^T := y - y_R(t)$ denote the "new" output of (1), called "sliding vector". These outputs are obviously the same relative as the initial outputs.

Definition 2: Consider the nonlinear system (1) and the sliding vector $s(x,t)$. The $r_1 - r_2 - \cdots - r_p$ order sliding set with respect to $s(x,t)$ is defined as

$$\begin{aligned} \mathcal{S}^r \ = \ & \{x \in \mathcal{X} \mid s_1 = \dot{s}_1 = \cdots = s_1^{(r_1 - 1)} = 0, \cdots, \\ & s_p = \dot{s}_p = \cdots = s_p^{(r_p - 1)} = 0\}. \end{aligned} \tag{3}$$

$r := [r_1 \ r_2 \ \cdots \ r_p]^T$ is called *sliding order vector*. ■

Definition 3: Consider $\mathcal{S}^r$ the not-empty r-order sliding set, and assume that it is locally an integral set in the Filippov sense [8]. Then, the behaviour of (1) on (3) is called 'r^{th}' *order sliding mode* with respect to s. ■

H1. The relative degree ρ_i $(1 \leq i \leq p)$ of each output s_i of (1) with respect to u is assumed constant and known. The sliding mode order versus each sliding variable s_i is r_i such that $r_i \geq \rho_i$.

III. Condition on existence of higher order sliding mode

Consider the *general problem of the multivariable higher order sliding mode control* for a sliding order vector $r_1 - r_2 - \cdots - r_p$ with $r_i > \rho_i$. The objective is to design a control vector u which forces, in finite time and in spite of uncertainties, the state trajectories to evolve on $\mathcal{S}^r$. Extend system (1) by a chain of integrators with length $r_i - \rho_i$ on each input u_i as

$$\begin{aligned} \dot{\overline{x}} &= \bar{f}(\overline{x}) + \bar{g}(\overline{x}) v(t) \\ y &= h(x) \end{aligned} \tag{4}$$

with $\overline{x} := [x^T \ u_1 \ \cdots \ u_1^{(r_1 - \rho_1 - 1)} \ \cdots \ u_p \ \cdots \ u_p^{(r_p - \rho_p - 1)}]^T$, *i.e.* the input variables and their time derivatives are state variables increasing the dimension of (1). $v := [u_1^{(r_1 - \rho_1)} \ \cdots \ u_p^{(r_p - \rho_p)}]^T$ is the new control input.

The r_i^{th} time derivative of each function s_i reads as

$$\begin{bmatrix} s_1^{(r_1)} \\ \vdots \\ s_p^{(r_p)} \end{bmatrix} = \underbrace{\begin{bmatrix} L_{\bar{f}}^{r_1} y_1 - y_{R1}^{(r_1)}(t) \\ \vdots \\ L_{\bar{f}}^{r_p} y_p - y_{Rp}^{(r_p)}(t) \end{bmatrix}}_{\varphi(\cdot)} + \underbrace{\begin{bmatrix} L_{\overline{g}_1} L_{\bar{f}}^{r_1 - 1} y_1 & \cdots & L_{\overline{g}_p} L_{\bar{f}}^{r_1 - 1} y_1 \\ L_{\overline{g}_1} L_{\bar{f}}^{r_2 - 1} y_2 & \cdots & L_{\overline{g}_p} L_{\bar{f}}^{r_2 - 1} y_2 \\ \vdots & \cdots & \vdots \\ L_{\overline{g}_1} L_{\bar{f}}^{r_p - 1} y_p & \cdots & L_{\overline{g}_p} L_{\bar{f}}^{r_p - 1} y_p \end{bmatrix}}_{\gamma(\cdot)} \underbrace{\begin{bmatrix} u_1^{(r_1 - \rho_1)} \\ \vdots \\ u_p^{(r_p - \rho_p)} \end{bmatrix}}_{v} \tag{5}$$

H2. $u(t) \in \mathcal{U} = \{u : |u_i| < u_M\}$ with $u_M \in I\!R^{+*}$. Furthermore, the $r_i - \rho_i - 1$ $(1 \leq i \leq p)$ first time derivatives of u_i are bounded and $v = u_i^{(r_i - \rho_i)}$ is a discontinuous function such that $|v_i| < v_M$, $v_M \in I\!R^{+*}$. The system (4) with discontinuous right-hand side admits solutions in Filippov's sense on $\mathcal{S}^r$.

H3. The zero dynamics of (4) are asymptotically stable.

H4. Components of vector $\varphi(\cdot)$ and matrix $\gamma(\cdot)$ are bounded functions. Furthermore, there exists positive constants s_0, $K_{ijm} \in I\!R$, $K_{ijM} \in I\!R$, $C_{0i} \in I\!R^+$ $(1 \leq i \leq p, 1 \leq j \leq p)$ such that,

$$\begin{array}{l} |s(x,t)| < s_0 \\ |\varphi_i(\cdot)| < C_{0i} \\ 0 < K_{iim} \leq |\gamma_{ii}(\cdot)| \leq K_{iiM} \\ K_{ijm} \leq |\gamma_{ij}(\cdot)| \leq K_{ijM} \quad \text{for } i \neq j. \end{array} \tag{6}$$

The problem of $r_1 - r_2 - \cdots - r_p$ order sliding mode control of (1) with respect to s is then equivalent to the stabilization in finite time, by a discontinuous control law v, of the multivariable uncertain system

$$\begin{aligned} \dot{Z}_1 &= \widehat{A}_{11} Z_1 + \widehat{A}_{12} Z_2 \\ \dot{Z}_2 &= \varphi + \gamma v \end{aligned} \tag{7}$$

with ($R = \sum_{j=1}^{p} r_j$ and $1 \leq i \leq p$)

$$
\begin{array}{rcl}
\widehat{A}_{12} & = & \mathrm{diag}\,[A_{12} \cdots A_{p2}] \in I\!R^{(R-p)\times p}, \\
\widehat{A}_{11} & = & \mathrm{diag}\,[A_{11} \cdots A_{p1}] \in I\!R^{(R-p)\times(R-p)},
\end{array}
\tag{8}
$$

$$
\begin{array}{rcl}
Z_1 & = & \left[\begin{array}{ccccccc} s_1^{(1)} & \cdots & s_1^{(r_1-2)} & \cdots & s_p & \cdots & s_p^{(r_p-2)} \end{array}\right]^T, \\
Z_2 & = & \left[\begin{array}{ccc} s_1^{(r_1-1)} & \cdots & s_p^{(r_p-1)} \end{array}\right]^T,
\end{array}
\tag{9}
$$

$$
\begin{array}{rcl}
A_{i1} & = & \left[\begin{array}{ccccc} 0 & 1 & \dots & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \dots & 1 \\ 0 & \ddots & \ddots & \ddots & 0 \end{array}\right]_{(r_i-1)\times(r_i-1)}, \\
A_{i2} & = & [\ 0\ \cdots\ 0\ 1\ ]^T_{(r_i-1)\times 1}, \\
v & = & \left[\begin{array}{ccc} u_1^{(r_1-\rho_1)} & \cdots & u_p^{(r_p-\rho_p)} \end{array}\right]^T.
\end{array}
\tag{10}
$$

The control and the stabilization of system (7) by a discontinuous control law v require the invertibility of γ. Then, system (4) has to have relative degree vector [10] $[r_1\ \cdots\ r_n]^T$ with respect to s. State the following assumption

H5. $[r_1\ \cdots\ r_p]^T$ is a relative degree vector for (4).

The first part of the next section is proposed in the context of the *ideal* higher order sliding mode, as previously presented in order to present with a sake of clarity the philosophy of the higher order sliding mode controller design. But, the establishment of an ideal sliding mode is not *really* possible for several reasons (finite frequency of control switching, singularity of control at the convergence instant, ...) . Then, the *practical* higher order sliding mode needs to be introduced [23].

Definition 4: Given the sliding variable $s(x,t)$ and a parameter ϵ , the "practical r^{th} order sliding manifold" associated to (1) is defined as

$$
\begin{array}{rcl}
\mathcal{S}^{\epsilon} & = & \{x \in \mathcal{X} \mid ||s|| < C_0(\epsilon),\ ||\dot{s}|| < C_1(\epsilon), \\
 & & \cdots, ||s^{(r-1)}|| < C_{r-1}(\epsilon)\}
\end{array}
\tag{11}
$$

with $C_i \to 0$ when $\epsilon \to 0$ ($1 \leq i \leq r-1$). ■

Definition 5: Consider the not-empty practical r^{th} order sliding set (11), and assume that it is locally an integral set in the Filippov sense, *i.e.* it consists of Filippov's trajectories of the discontinuous dynamics system. The corresponding behavior of system (1) satisfying (11) is called "practical r^{th} order sliding mode" with respect to the sliding variable $s(x,t)$. ■

IV. SYNTHESIS OF A PRACTICAL HIGHER ORDER SLIDING MODE CONTROLLER

The objective of this section is to propose a solution to the practical higher order sliding mode control for a large class of multivariable nonlinear systems. The design of the higher order sliding mode control is made in two steps

- Design of an optimal time varying switching manifold by minimizing a linear quadratic criterion over a finite time interval with a fixed final state,
- Synthesis of a discontinuous control ensuring that system trajectories evolve on the optimal time varying switching manifold in finite time and in spite of uncertainties.

A. Synthesis of an optimal switching manifold

Given the system (7), the synthesis of an optimal switching manifold, such that the generation of the sliding mode on this manifold allows the establishment of a $r_1 - \cdots - r_p$-order sliding mode with respect to $s(x,t)$, is based on the same approach than [12] which considers only SISO case. The synthesis of the optimal manifold is done through the minimization of a criteria over a finite time interval $[t_0, t_F]$ ($t_0 \geq 0$, $t_F < \infty$) with constraint on final state ($Z(t_F) = 0$)

$$
\begin{array}{rcl}
J & = & \displaystyle\int_{t_0}^{t_F} Z^T Q Z \ \mathrm{d}t \\
 & = & \displaystyle\int_{t_0}^{t_F} Z_1^T Q_{11} Z_1 + 2 Z_1^T Q_{12} Z_2 + Z_2^T Q_{22} Z_2 \ \mathrm{d}t
\end{array}
\tag{12}
$$

with $Z := [Z_1^T\ Z_2^T]^T$ and

$$
Q = \left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{array}\right]
\tag{13}
$$

a symmetrical definite-positive matrix, such that Q_{11}, Q_{12} and Q_{22} with respective dimensions $(R-p)\times(R-p)$, $(R-p)\times p$- and $(p\times p)$. In the first equation of (7), consider Z_1 as the state variable, and Z_2 as a fictitious control input. Then, the problem leads back to the resolution of the LQ problem (12) for the dynamics of Z_1, under the constraint $Z(t_F) = 0$. A vector Z_2 minimizing the criteria (12) under the constraint $Z(t_F) = 0$ reads as [20]

$$
Z_2 = -(Q_{22}^{-1}\widehat{A}_{12}^T P - Q_{22}^{-1}\widehat{A}_{12}^T V H^{-1} V^T + Q_{22}^{-1} Q_{12}^T) Z_1.
\tag{14}
$$

with $P(t) \in I\!R^{(R-p)\times(R-p)}$ is the unique solution of ($t_0 \leq t \leq t_F$, $P(t_F) = P_f$)

$$
\begin{array}{rcl}
-\dot{P} & = & P\tilde{A} + \tilde{A}^T P - P\widehat{A}_{12} Q_{22}^{-1} \widehat{A}_{12}^T P \\
 & & +(Q_{11} - Q_{12} Q_{22}^{-1} Q_{12}^T),
\end{array}
\tag{15}
$$

with $\tilde{A} := \widehat{A}_{11} - \widehat{A}_{12} Q_{22}^{-1} Q_{12}^T$ and $V \in I\!R^{(R-p)\times(R-p)}$ et $H \in I\!R^{(R-p)\times(R-p)}$ respectively solutions of ($t_0 \leq t \leq t_F$, $V(t_F) = I$, $H(t_F) = 0$)

$$
\begin{array}{rcl}
-\dot{V} & = & (\widehat{A}_{11} - \widehat{A}_{12} Q_{22}^{-1} Q_{12}^T - \widehat{A}_{12} Q_{22}^{-1} \widehat{A}_{12}^T P)^T V, \\
\dot{H} & = & V^T \widehat{A}_{12} Q_{22}^{-1} \widehat{A}_{12}^T V.
\end{array}
\tag{16}
$$

From (14), one gets the optimal sliding variables vector $S_{opt}(\cdot)$ defined as

$$\begin{array}{rcl} S_{opt}(\cdot) & = & \left[S_{1_{opt}} \ \cdots \ S_{p_{opt}}\right]^T \\ & = & Z_2 + (Q_{22}^{-1}\widehat{A}_{12}^T P - Q_{22}^{-1}\widehat{A}_{12}^T V H^{-1} V^T \\ & & +Q_{22}^{-1} Q_{12}^T) Z_1 \end{array} \tag{17}$$

Equation $S_{opt}(Z,t) = 0$ described the desired dynamics which satisfy the finite time stabilization of $Z = [s_1 \ \dot{s}_1 \ \cdots \ s_1^{(r_1-1)} \ \cdots \ s_p \ \dot{s}_p \ \cdots \ s_p^{(r_p-1)}]^T$ to zero and minimize the criteria (12). In short, one gets a sliding surface with time-varying coefficients, these latter being computed from a finite-time convergence LQ control approach so that $S_{opt}(Z,t)$ and each component of Z equal zero in a finite time t_F. Thus, the optimal time varying switching manifold reads as

$$\mathcal{S}_{opt} \ = \ \bigcap_{i=1}^{p} \mathcal{S}_{i_{opt}} = \bigcap_{i=1}^{p} \{x \in \mathcal{X} \mid S_{i_{opt}}(\cdot) = 0\} \tag{18}$$

B. Controller design

Once the optimal switching manifold designed, the second step of the synthesis of the control law is the design of the control input vector v composed by p discontinuous components and ensuring the establishment of a 1-order sliding mode with respect to each component $S_{i_{opt}}$ of S_{opt}, which induces the establishment of a r_i-order sliding mode with respect to each component s_i of s, *in spite of uncertainties*. The multivariable context makes this synthesis more difficult than in monovariable one, because the objective is to force the system trajectories to slide on the intersection of p surfaces, even if there exists interaction (coupling) between the outputs. This interaction is due to the structure of uncertain matrix γ. As a matter of fact, one has

$$\dot{S}_{opt} \ = \ \Theta + \varphi + \gamma \cdot v \tag{19}$$

with

$$\begin{array}{rcl} \Theta & = & \Psi \cdot [\widehat{A}_{11} Z_1 + \widehat{A}_{12} Z_2] + \Delta \cdot Z_1^T, \\ \Psi & = & Q_{22}^{-1}\hat{A}_{12}^T P - Q_{22}^{-1}\hat{A}_{12}^T V H^{-1} V^T + Q_{22}^{-1} Q_{12}^T, \\ \Delta & = & Q_{22}^{-1}\hat{A}_{12}^T \cdot (\dot{P} - \dot{V} H^{-1} V^T - V(\dot{H^{-1}}) V^T \\ & & -V H^{-1} (\dot{V^T})). \end{array} \tag{20}$$

In order to determine a control law ensuring the establishment of a sliding mode on S_{opt}, one uses the classical methods for stability analysis, *i.e.* Lyapunov's direct method. Consider the following function, which is a Lyapunov's function candidate $\mathcal{V} = \dfrac{1}{2} S_{opt}^T S_{opt}$. The first time derivative of $\mathcal{V}$ reads as

$$\dot{\mathcal{V}} = S_{opt}^T (\Theta + \varphi) + S_{opt}^T \gamma v \tag{21}$$

Consider v a discontinuous control law defined as ($\alpha \in I\!R^{+*}$)

$$v := -\alpha \ \mathrm{sign}\left[S_{opt}^*\right] = -\alpha \begin{bmatrix} \mathrm{sign}(S_{1_{opt}}^*) \\ \vdots \\ \mathrm{sign}(S_{p_{opt}}^*) \end{bmatrix} \tag{22}$$

with

$$S_{opt}^* = D \ S_{opt}, \tag{23}$$

D an invertible matrix such that the matrix $L := (D^{-1})^T \cdot \gamma$ is a dominant diagonal matrix.

Note that the transformation (23) is non singular [23]: the establishment of a sliding mode behaviour on S_{opt}^* implies the establishment of sliding mode behaviour on S_{opt} [23], *i.e.* $S_{opt}^* = 0$ implies $S_{opt} = 0$. Equation (21) reads as

$$\begin{array}{rcl} \dot{\mathcal{V}} & = & (S_{opt}^*)^T (D^{-1})^T \ (\Theta + \varphi) \\ & & -\alpha (S_{opt}^*)^T L \ \left[\mathrm{sign}(S_{1_{opt}}^*) \ \cdots \ \mathrm{sign}(S_{p_{opt}}^*)\right]^T \\ & = & \displaystyle\sum_{i=1}^{p} \Bigg[S_{i_{opt}}^* \Pi_i - \alpha \Bigg(|S_{i_{opt}}^*| l_{ii} \\ & & \displaystyle + \sum_{j=1, j\neq i}^{p} l_{ij} S_{i_{opt}}^* \ \mathrm{sign}(S_{j_{opt}}^*) \Bigg) \Bigg] \end{array} \tag{24}$$

where Π_i ($1 \leq i \leq p$) are the components of the vector $(D^{-1})^T \ (\Theta + \varphi)$. If The time derivative of the Lyapunov function candidate is negative, then a sliding mode exists on S_{opt}^* after a finite time. One gets (for $1 \leq i \leq p$)

$$S_{i_{opt}}^* \Pi_i - \alpha \left(|S_{i_{opt}}^*| l_{ii} + \sum_{j=1, j\neq i}^{p} l_{ij} S_{i_{opt}}^* \mathrm{sign}(S_{j_{opt}}^*) \right) < 0 \tag{25}$$

Then, one has

$$\Pi_i - \alpha \left(l_{ii} + \sum_{j=1, j\neq i}^{p} l_{ij} \ \mathrm{sign}(S_{j_{opt}}^*) \right) < 0 \tag{26}$$

One gets $\alpha > \dfrac{|\Pi_i|}{|l_{ii}| - \sum_{j=1, j\neq i}^{p} |l_{ij}|}$. This relation can be also found in the case $S_{i_{opt}}^* < 0$. As $L = (D^{-1})^T \gamma$ and by denoting d_{ij}^{-1} the (i,j) component of matrix D^{-1} , the time derivative of the Lyapunov function candidate is negative, *i.e.* a sliding mode exists on S_{opt}^* if $\alpha > \mathrm{Max}_{1 \leq i \leq p} \left[\frac{|\Pi_i|}{\chi_i}\right]$ with

$$\begin{array}{rcl} \chi_i & = & |d_{ii}^{-1} K_{iim} + \sum_{j=1, j\neq i}^{m} d_{ji}^{-1} K_{jim}| \\ & & -| \sum_{j=1, j\neq i}^{m} d_{ii}^{-1} K_{ijM} + \sum_{j=1, j\neq i}^{m} d_{ji}^{-1} K_{jjM}| \end{array} \tag{27}$$

Now, let suppose that

H6. At $t = t_0$, $S_{opt}(Z,t) = 0$.

The function $S_{opt}(Z,\cdot)$ is a switching variable with time-varying coefficients depending on $P(\cdot)$, $V(\cdot)$ and $H(\cdot)$. These coefficients do not depend on state variables and then, can be computed *off-line* and stored from resolution of (15)-(16) for each time between t_0 and t_F. Then, when the controller is implemented, these coefficients are *fully known*. However, at t_F, the function S_{opt} can not be evaluated because it is undetermined. As a matter of fact, it depends on the inverse of $H(\cdot)$ (with $H(t_F) = 0$) which is multiplied by Z_1 (with $Z_1(t_F) = 0$). From [21], it is known that H^{-1} exists for $t \in [t_0, t_F - \epsilon]$, with ϵ an arbitrarily small constant. Then, via a discontinuous control u, the final control objective consists in forcing, the trajectories of (1) to slide on

$$\mathcal{S}_{opt}^{\epsilon} = \{Z \mid S_{opt}(Z,t) = 0,\ t \in [t_0, t_F - \epsilon],\ 0 < \epsilon << t_F\}$$

in finite time. The design of a switching control function u, which allows the sliding on $\mathcal{S}_{opt}^{\epsilon}$, follows the conventional path [23]; the variable structure control u can be selected to satisfy the sliding mode condition $S_{opt} \cdot \dot{S}_{opt} < -\eta |S_{opt}|$ where $\eta > 0$ is a positive real number.

Theorem 1: Consider the nonlinear system (1) with a relative degree vector $[\rho_1 \ \cdots \ \rho_n]^T$ with respect to the sliding vector $s(x,t)$ and a sliding order vector $[r_1 \ \cdots \ r_n]^T$. Suppose the assumptions H1, H2, H3, H4, H5 and H6 fulfilled. Let S_{opt} denote the following vector (with $t_0 \leq t \leq t_F - \epsilon$, $t_F > 0$ and $0 < \epsilon << t_F$)

$$\begin{aligned} S_{opt} &= \left[S_{1_{opt}} \ \cdots \ S_{p_{opt}}\right]^T \\ &:= Z_2 + \left[Q_{22}^{-1} A_{12}^T P(t) \right. \\ &\quad \left. -Q_{22}^{-1} A_{12}^T V(t) H(t)^{-1} V(t)^T + Q_{22}^{-1} Q_{12}^T\right] Z_1 \end{aligned}$$

where the matrix A_{12} is defined by (8), $P(\cdot)$ the unique solution of (15) with $P(t_F) = P_f$ positive definite matrix, $V(\cdot)$ and $H(\cdot)$ the solutions of (16) with $V(t_F) = I$ and $H(t_F) = 0$), and $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$ a weighting matrix which is symmetric definite positive. Consider the vector S_{opt}^* defined by (23) from S_{opt}. Then, the control input vector $v = [v_1 \ \cdots \ v_p]^T$ defined by (with $1 \leq i \leq p$)

$$v_i = -\alpha \left[\mathrm{sign}(S_{i_{opt}}^*)\right]$$

with

$$\alpha > \mathrm{Max}_{1 \leq i \leq p} \left[\frac{|\Pi_i|}{\chi_i}\right], \tag{28}$$

with χ_i defined by (27), allows the establishment of a practical $r_1 - r_2 - \cdots - r_p$-order sliding mode with respect to the sliding vector $s(x,t)$. ∎

Implementation in practice/Algorithm. The matrix $S_{opt}(Z,t)$ is a switching variable with time-varying coefficients, which depend on $P(t)$, $V(t)$ and $H(t)$. These coefficients do not depend on state variables and then, can be computed *off-line* and stored from resolution of (15)-(16) for each time between $t = t_0$ and $t = t_0 + t_F$. Then, when the controller is implemented, these coefficients are *fully known*. Assumption H6 in Theorem 1 can be relaxed through the following algorithm:

Stage 1. $t \in [0,\ t_0[$. The goal of this stage is force the system trajectories of (1) to reach the surface $S_{opt}^{*0} = DS_{opt}^0 = D(Z_2(t) + \lambda_0 Z_1(t)) = 0$ (note that, as D is invertible, $S_{opt}^{*0} = 0$ implies $S_{opt}^0 = 0$), with λ_0 issued from the *off-line* computations/resolutions of (15)-(16) at $t = t_0$, *i.e.*

$$\begin{aligned} \lambda_0 &= Q_{22}^{-1} A_{12}^T P(t = t_0) - Q_{22}^{-1} A_{12}^T V(t = t_0) \\ &\quad H(t = t_0)^{-1} V(t = t_0)^T + Q_{22}^{-1} Q_{12}^T, \end{aligned}$$

by applying the control law $u = -\alpha \ \mathrm{sign}\ \left(S_{opt}^{0*}\right)$ with

$$S_{opt}^{0*} := D(Z_2(t) + \lambda_0 Z_1(t)).$$

Of course, during all this stage, the coefficients vector λ_0 is constant. The time $t = t_0$ is defined such that $S_{opt}^{0*} = D(Z_2(t_0) + \lambda_0 Z_1(t_0)) = 0$.

Stage 2. $t \in [t_0,\ t_0 + t_F - \epsilon]$. The control law $u = -\alpha \ \mathrm{sign}\ (S_{opt}^*) = -\alpha \ \mathrm{sign}\ (D\left[Z_2(t) + \lambda Z_1(t)\right])$, with λ issued from the *off-line* computations/resolutions of (15)-(16) for $t \in [t_0,\ t_0 + t_F - \epsilon]$, *i.e.*

$$\lambda = Q_{22}^{-1} A_{12}^T P(t) - Q_{22}^{-1} A_{12}^T V(t) H(t)^{-1} V(t)^T + Q_{22}^{-1} Q_{12}^T,$$

maintains $S_{opt}^* = 0$ (and $S_{opt} = 0$). Consequently, the equality (14) minimizing (12) under the constraint $Z(t_F) = 0$, holds. Then, practical higher order sliding mode occurs.

Stage 3. $t \in]t_0 + t_F - \epsilon,\ \infty[$. The control task consists in maintaining the system trajectories at the origin. This objective is fulfilled by the control law $u = -\alpha \ \mathrm{sign}\ (S_{opt}^{*f}) = -\alpha \ \mathrm{sign}\ (D\left[Z_2(t) + \lambda_f Z_1(t)\right])$, with λ_f issued from the *off-line* computations/resolutions of (15)-(16) at $t = t_F - \epsilon$, *i.e.*

$$\begin{aligned} \lambda_f &= Q_{22}^{-1} A_{12}^T P(t = t_F - \epsilon) - Q_{22}^{-1} A_{12}^T V(t = t_F - \epsilon) \\ &\quad H(t = t_F - \epsilon)^{-1} V(t = t_F - \epsilon)^T + Q_{22}^{-1} Q_{12}^T \end{aligned}$$

which allows the continuation of the sliding on $S_{opt}^{*f} = 0$ (and $S_{opt}^f = 0$).

Remark 2: Equations (15) and (16) are three differential equations which do not depend on the state trajectories. Since only their final conditions are available, these equations have to be integrated backward from a *a priori* final time t_F over a time interval $\tau \in [t_F, 0]$ in order to find the initial conditions of P, V and H at $\tau = 0$ (which corresponds to $t = t_0$). ∎

V. EXAMPLE

The academic example is taken from [19], in which a second order sliding mode controller is designed, and reads as (with $x = [x_1 \ x_2 \ x_3]^T = y$ the state vector and the output and $u = [u_1 \ u_2 \ u_3]^T$ the input vector)

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -x_2 x_3 \\ x_1 x_3 \\ -\frac{1}{3} x_1 x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \Delta f_1 \\ \Delta f_2 \\ \Delta f_3 \end{bmatrix}}_{\Delta f} + \underbrace{\begin{bmatrix} \Delta g_1 \\ \Delta g_2 \\ \Delta g_3 \end{bmatrix}}_{\Delta g} \\ &\quad + \begin{bmatrix} 1 & 1.2 & 1.5 \\ 1.5 & 1 & 1.2 \\ 1.2 & 1.5 & 1 \end{bmatrix} u \end{aligned} \tag{29}$$

with Δf and Δg are considered as bounded uncertainties defined as

$$\Delta f = \begin{bmatrix} \cos(t)(1 + 0.05\sin(4t) + 0.1\cos(t)) \\ \sin(t)\cos(t)(1 + 0.05\sin(4t) + 0.1\cos(t)) \\ \sin^2(t)(1 + 0.05\sin(4t) + 0.1\cos(t)) \end{bmatrix}$$

$$\Delta g = \begin{bmatrix} 0.01\sin(t + 2.1)(u_1 - 0.5u_3) \\ 0.01\cos(t)(-0.2u_2 + 0.8u_3) \\ 0.01\cos(t + 1.3)(-0.2u_1 - u_2 + 0.7u_3 \end{bmatrix} \tag{30}$$

The objective is to force the state vector $x = [x_1\ x_2\ x_3]^T$ to track in finite time the desired trajectories defined as $x_1^{ref}(t) = 1 - \sin(0.5t)$, $x_2^{ref}(t) = 0.5\cos(0.5t)\cos(t)$ and $x_3^{ref}(t) = 0.5\cos(0.5t)\sin(t)$. Then, the sliding variable vector (new output) is defined as $s(x,t) = [x_1 - x_1^{ref}(t)\ x_2 - x_2^{ref}(t)\ x_2 - x_2^{ref}(t)]^T$. The relative degree vector of (29) with respect to $s(x,t)$ are $[1\ 1\ 1]^T$. As an other objective is to design a robust sliding mode controller without chattering, a second order sliding mode is then designed. Then, one gets (with $s = [s_1\ s_2\ s_3]^T$) $\ddot{s} = \varphi(x,t,u) + \gamma(t)\dot{u}$ with φ and γ bounded matrix. By denoting $Z_1 := [s_1\ s_2\ s_3]^T$, $Z_2 := [\dot{s}_1\ \dot{s}_2\ \dot{s}_3]^T$ and $v = [\dot{u}_1\ \dot{u}_2\ \dot{u}_3]^T$, one gets

$$\begin{array}{rcl} \dot{Z}_1 & = & \hat{A}_{11}Z_1 + \hat{A}_{12}Z_2 \\ \dot{Z}_2 & = & \varphi(x,t,u) + \gamma(t)v \end{array} \qquad (31)$$

The matrices Q_{11}, Q_{22} and Q_{12} have been stated as $Q_{11} = I_{3\times3}$, $Q_{22} = I_{3\times3}$ and $Q_{12} = 0_{3\times3}$. The matrix P_f and the convergence time t_F are $P_f = 0_{3\times3}$, $t_F = 1.5$ s and $\epsilon = 1$ ms. The matrix D is stated as

$$D = \begin{bmatrix} 0.0005 & 0.0015 & 0.0015 \\ 0.0012 & 0.0005 & 0.0020 \\ 0.0020 & 0.0015 & 0.0005 \end{bmatrix}$$

and ensures that $L = (D^{-1})^T\gamma$ is a diagonal dominant matrix in spite of uncertainties. The gain α is tuned to 2000. Figures 2 and 3 displays the outputs and the trajectories tracking of state variables. The convergence to zero for the three sliding variables appears to be effective at $t = 1.5$ sec, as stated via the parameter t_F, in spite of uncertainties.

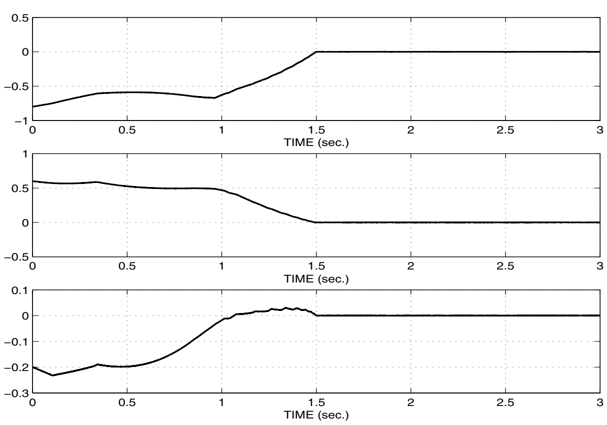

Fig. 1. Outputs s_1 (top), s_2 (middle) and s_3 (bottom) versus time ($sec.$)

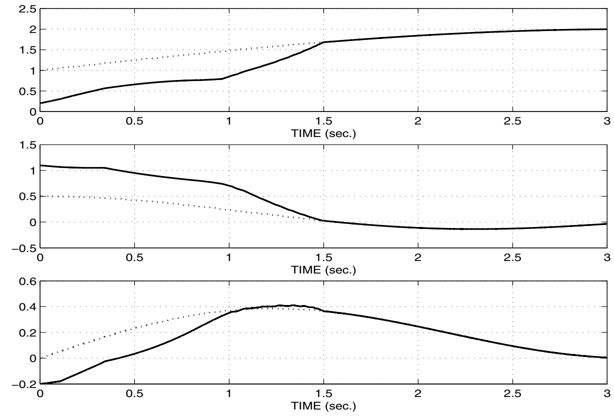

Fig. 2. State variables (solid line) x_1 (top), x_2 (middle) and x_3 (bottom), and their respective references (dotted line) versus time ($sec.$)

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