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Motion Planning and Control of Coordinated Systems

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Abstract— **Two problems of motion planning for controlled systems which are required to attain a given target set while satisfying coordinated constraints are formulated and solved using dynamic optimization techniques. Constraint coordination arises from the fact that the state of each system is mapped onto state constraints for the other systems. The problems are formulated in terms of backward reach sets which are the sub-zero level sets of appropriate value functions for non-standard cost functions. The value functions are the solutions of Hamilton-Jacobi-Bellman type PDEs. For linear dynamics and ellipsoidal constraints the value functions are calculated through duality techniques from convex analysis.**

I. INTRODUCTION

The problem of motion planning and coordination for multiple systems has received significant attention in the literature. A significant body of this work deals with the problem of formation motion planning and control [27], [21], [23]. However, there are requirements for motion planning and coordination other than keeping a formation [3], [4]. Some of these requirements are more appropriately described by coordinated state constraints. Constraint coordination is possible when the state of each system is mapped onto state-constraints for the other systems.

Here, we address the problem of planning the motions of multiple systems to reach a certain number of targets under coordinated state constraints. The state constraints are modeled as set-valued maps mapping the state of each system onto constraints for the other systems. There is one target set for each system. The problem is solvable when the target sets are reached at some time θ within some prescribed time interval T. In this paper, we consider two versions of this problem: 1) the motion of one system is known in advance; 2) the motions of all systems are planned to take advantage of the coordinated constraints. We address these problems using backward reach set computation and dynamic optimization techniques [13], [12]. We do this for two coordinated systems. The solution methodology is directly applicable to a larger number of systems.

In [25], dynamic optimization techniques are used in an efficient algorithm to compute globally optimal trajectories for systems given by $\dot{x} = u(t), \|u\| \leq 1$ subject to simple state constraints and a traveling cost that depends only on the state. Ordered Upwind Methods have been used to solve Hamilton-Jacobi-Bellman-type equations describing path planning problems for systems modeled by an hybrid automaton with switching costs among different dynamics, [20]. Techniques from optimal control and game theory are used in [18] and [24] to design controllers for safety specifications in hybrid systems.

The paper is organized as follows. In section II, we introduce the mathematical preliminaries. In section III, we state the problems under consideration. In section IV, we use dynamic optimization techniques to characterize the solution to these problems and for controller synthesis. In section V, we find the solution for linear systems by using duality techniques from non-linear analysis. In section VI, a framework for the practical implementation of coordinated control strategies at both the planning and control levels is discussed. Finally, in section VII, we draw some conclusions.

II. PRELIMINARIES

Consider the controlled motions of a dynamic system evolving in $\mathbb{R}^n$ described as:

$$\dot{x} = f(t, x, u), \qquad u(t) \in P(t) \subset \mathbb{R}^m \tag{1}$$

with the standard conditions for uniqueness and prolongability of the solutions for $t \geq t_0$ (see for example [1]).

Definition 1: The backward reach set $W[\tau, t_f, \mathcal{X}_f]$ at time τ relative to target set $\mathcal{X}_f$ and time $t_f \geq \tau$ is the set of points $W[\tau, t_f, \mathcal{X}_f] = \bigcup \left\{ x[\tau] \middle| u(s) \in P(s),\ s \in [\tau, t_f), x[t_f] \in \mathcal{X}_f \right\}$ where $x[\tau]$ is state of the system at time τ when driven by control $u(t)$.

The definition of backward reach set for the case where the target set can be reached within some time interval $T = [t_\alpha, t_\beta]$ with $t_\alpha \geq t_0$ follows.

Definition 2: The backward reach set $W[\tau, t_\alpha, t_\beta, \mathcal{M}]$ at time $\tau \leq t_\alpha$ is the set of points $x \in \mathbb{R}^n$ such that there exists a control $u(t)$ that drives the trajectory of the system $x[t] = x(t, \tau, x)$ from state (τ, x) to the target set $\mathcal{M}$ at some time $\theta \in [t_\alpha, t_\beta]$.

The relation between dynamic optimization and reachability was observed in [16]. See also [26] for a description of reach set computation using optimal control. The key observation is that the reach set is the level set of an appropriate value function, [14]. To illustrate this point consider

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the following value function

$$\left\{ \begin{array}{l} V(\tau, x) = \min_{u(.)}\{d^2(x(t_f), \mathcal{X}_f)|\, x(\tau) = x\} \\ \\ V(t_f, x) = d^2(x, \mathcal{X}_f) \end{array} \right. \tag{2}$$

where $u(\cdot)$ is an admissible control function defined on $[\tau, t_f]$ and $d(x(t_f), \mathcal{X}_f)$ is the Euclidean distance between the state of the system at time t_f and the target set $\mathcal{X}_f$ for a trajectory starting at $x(\tau) = x$. Obviously, the point $x(\tau) = x$ belongs to the backward reach set if this distance is zero. But this also means that the backward reach set is the zero level set of the value function V:

$$W[\tau, t_f, \mathcal{X}_f] = \{x|\; V(\tau, x) \leq 0\} \tag{3}$$

If the value function satisfies the principle of optimality, then it can be determined from the solution of the generalized Hamilton-Jacobi-Bellman (HJB) PDE associated with it. This is the case for V in equation (2):

$$\left\{ \begin{array}{l} V_t(t, x) + \max_{u \in P(t)}\{\langle V_x(t, x), f(t, x, u)\rangle\} = 0 \\ \\ V(t_f, x) = d^2(x, \mathcal{X}_f). \end{array} \right. \tag{4}$$

Definition 3: The ellipsoid $\mathcal{E}(a, Q)$ with center a and shape matrix $Q = Q' > 0$ is the set of points:

$$\mathcal{E}(a, Q) = \{x : \langle x - a, Q^{-1}(x - a)\rangle \leq 1\} \tag{5}$$

Its support function, [19], is

$$\begin{aligned} \rho(l|\mathcal{E}(a, Q)) &= \max\{\langle l, x\rangle | x \in \mathcal{E}(a, Q)\} \\ &= \langle l, p\rangle + \langle l, Pl\rangle^{1/2}. \end{aligned}$$

III. PROBLEM FORMULATION

Consider the motions of two controlled systems under the assumptions from section II for $t \geq t_0$

$$\dot{x}_1(t) = f_1(t, x_1, u_1), \qquad u_1(t) \in P_1(t) \tag{6}$$

$$\dot{x}_2(t) = f_2(t, x_2, u_2), \qquad u_2(t) \in P_2(t) \tag{7}$$

where $P_i(t) \in Comp_m$, $i = 1, 2$. Here, $Comp_m$ is the set of compact sets in $\mathbb{R}^m$. Moreover,

$$x_1(t_0) \in \mathcal{X}_1, \quad x_2(t_0) \in \mathcal{X}_2 \tag{8}$$

Let $\mathcal{M}_i \in Comp_n$, $i = 1, 2$ be convex target sets for the motions of systems $i = 1, 2$.

Denote by $u(.) = col\{u_1(.), u_2(.)\}$, $x = col\{x_1, x_2\}$ and $f(t, x, u) = col\{f_1(t, x_1, u_1), f_2(t, x_2, u_2)\}$, and $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$. In what follows, we will refer both to each system $i = 1, 2$ separately, and to the composed system whose state x is driven by control $u(.)$.

Consider the time interval $T = [t_\alpha, t_\beta]$ with $t_\alpha \geq t_0$. Now consider that the motions of both systems $i = 1, 2$ are coupled through the following state constraints (convex and complementary-convex as in [10]):

$$x_1(t) \in F_2(x_2(t)), \quad x_2(t) \in F_1(x_1(t)) \tag{9}$$

$$x_1(t) \notin G_2(x_2(t)), \quad x_2(t) \notin G_1(x_1(t)) \tag{10}$$

where F_1, F_2, G_1, and G_2 are continuous convex set-valued maps with values in $Comp_n$ with non-empty interior. G_1 and G_2 are avoidance sets and model safety regions to prevent the trajectories of the two systems from colliding. F_1 and F_2 are containment sets since they constrain the motions of x_2 and x_1, respectively.

Problem 1 (Motion planning): Find the set of all initial conditions $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ such that there exist controls (u_1, u_2) which starting at time t_0 steer the trajectories of both systems to reach $\mathcal{M}_1 \times \mathcal{M}_2$ at some time $\theta \in T$ under constraints (9), and (10).

The following assumptions ensure that: 1) the problem is well-posed; 2) at most two constraints are active at a time; and 3) the problem has non-empty solution set.

Assumption 1: $\forall x \in \mathbb{R}^n : G_i(x) \subset F_i(x), i = 1, 2.$

Assumption 2: $\forall x_1, x_2 \in \mathbb{R}^n : \exists y \in \mathbb{R}^n$, with $y \in G_1(x_1) \cap G_2(x_2)$, we have $G_1(x_1) \cup G_2(x_2) \subset F_i(x_i)$, $i = 1, 2$.

Assumption 3: $\exists(x_1, x_2) \in \mathcal{M}_1 \times \mathcal{M}_2 : x_1 \in F_2(x_2) \wedge x_2 \in F_1(x_1) \wedge x_1 \notin G_2(x_2) \wedge x_2 \notin G_1(x_1)$.

The solution to this problem is given in two steps.

Step 1 Find the backward reach set relative to target set $\mathcal{M}_1 \times \mathcal{M}_2$ and time interval T under state constraints (9) and (10). This is the reach-evasion set, [24].

Next we consider two versions of this problem.

Problem 2: [Given feasible motion x_2^f] Calculate the backward reach set $W_1^g[\tau, t_\alpha, t_\beta, \mathcal{M}_1]$ under coupling (9) and (10) when a feasible motion $x_2^f(\cdot)$ is known in advance.

A feasible motion of $x_2^f(\cdot)$ is a trajectory $x_2^f[t] = x_2^f(t, \tau, x_2), x_2^f(t_0) \in \mathcal{X}_2$ defined on $[t_0, t_\beta]$ such that $x_2^f(t) \in \mathcal{M}_2$ for some $t \in [t_\alpha, t_\beta]$.

Problem 3: [Coordinated controls] Calculate the backward reach set $W^c[\tau, t_\alpha, t_\beta, \mathcal{M}_1 \times \mathcal{M}_2]$ under coupling (9) and (10) and coordinated controls.

A pair of controls (u_1, u_2) is said to be coordinated when both controls are responsible for both constraints.

Step 2 The solutions to the motion planning problem (1) for the two versions of the backward reach set problem (2, 3) are given respectively by the following sets:

$$\begin{aligned} S_1^a(t_0) &= W_1^g[t_0, t_\alpha, t_\beta, \mathcal{M}_1] \cap \mathcal{X}_1 \\ S^c(t_0) &= W^c[t_0, t_\alpha, t_\beta, \mathcal{M}_1 \times \mathcal{M}_2] \cap [\mathcal{X}_1 \times \mathcal{X}_2]. \end{aligned}$$

IV. DYNAMIC PROGRAMMING APPROACH

We follow the approach described in [15] to calculate the solutions to problems (2, 3).

A. Value Functions

First, we consider problem (2). Let a feasible trajectory $x_2^f[t] = x_2^f(t, \tau, x_2)$ satisfying assumption (3) be given. Let $T_g = [t_{\alpha_g}, t_{\beta_g}]$, where $t_{\alpha_g}, t_{\beta_g}$ are the first entry and first exit times of this trajectory in $\mathcal{M}_2$. From assumption (3) and the fact that $x_2^f[t]$ is a feasible trajectory we conclude that $S = T \cap T_g \neq \emptyset$.

Let:

$$\begin{aligned}
\varphi_0^1(x_1) &= d^2(x_1, \mathcal{M}_1) \\
\varphi_0^2(x_2) &= d^2(x_2, \mathcal{M}_2) \\
\varphi_1(t, x_1, x_2) &= d^2(x_1, F_2(x_2)) \\
\varphi_2(t, x_1, x_2) &= d^2(x_2, F_1(x_1)) \\
\varphi_3(t, x_1, x_2) &= -d^2(x_1, G_2(x_2)) \\
\varphi_4(t, x_1, x_2) &= -d^2(x_2, G_1(x_1))
\end{aligned} \quad (11)$$

The continuity and convexity of the set-valued maps G_1, G_2, F_1 and F_2, the convexity of both $\mathcal{M}_1$ and $\mathcal{M}_2$, and the fact that d is the Euclidean distance function imply the continuity of functions φ_1^0, φ_2^0 and φ_i, $i = 1, \dots, 4$.

Corresponding to this problem, we introduce the value function:

$$\begin{aligned}
V^g(\tau, x_1, S) = \min_{u_1(.)} \min_{t_f \in S} \Bigg\{ &\max \Big\{ \varphi_0^1(x_1(t_f)), \\
&\max_{i=1,2,3,4} \{ \max_t \{ \phi_i(t, x_1(t)) | t \in [\tau, t_f] \} \} \Big\} \\
&| x_1(\tau) = x_1 \Bigg\}
\end{aligned} \quad (12)$$

where

$$\phi_i(t, x_1) = \varphi_i(t, x_1, x_2^f(t)), \quad i = 1, 2, 3, 4. \quad (13)$$

The functions $\phi_i, (i = 1, \dots, 4)$ are continuous since $x_2^f(t)$ is continuous in t.

Lemma 1: The following relation is true:

$$W_1^a[\tau, t_{\alpha_g}, t_{\beta_g}, \mathcal{M}_1] = \{x_1 : V^g(\tau, x_1, S) \leq 0\}$$

Proceeding similarly for problem (3), we may write

$$\begin{aligned}
V^c(\tau, x, T) = \min_{u(.)} \min_{t_f \in T} \Bigg\{ &\max \Big\{ \varphi_0^1(x_1(t_f)), \\
&\varphi_0^2(x_2(t_f)), \\
&\max_{i=1,2,3,4} \{ \max_t \{ \varphi_i(x_1(t), x_2(t)) | t \in [\tau, t_f] \} \} \Big\} \\
&| x_1(\tau) = x_1, x_2(\tau) = x_2 \Bigg\}
\end{aligned} \quad (14)$$

Lemma 2: The following relation is true:

$$W^c[\tau, t_\alpha, t_\beta, \mathcal{M}] = \{x : V^c(\tau, x, T) \leq 0\}$$

B. Solution Approach

Here, we consider the following assumption.

Assumption 4: The functions V^c, V^g, φ_0^1, φ_0^2, $\varphi_i, (i = 1, \dots, 4)$, and $\phi_i, (i = 1, \dots, 3)$ are differentiable.

Next we describe how to calculate $V^c(\tau, x_1, x_2, T)$ (the calculation of $V^g(\tau, x_1, S)$ is identical).

First we consider the case where $t_f = t_\alpha = t_\beta$ and denote $V^c(\tau, x, T) = V^c(\tau, x, t_f) = V^c(\tau, x) = V^c(\tau, x | V^c(t_f, .))$ where

$$\begin{aligned}
V^c(t_f, x_1, x_2) \;=\; \max \Big\{ &\varphi_0^1(x_1), \varphi_0^2(x_2), \\
&\max_{i=1,2,3,4} \{ \varphi_i(x_1, x_2) \} \Big\}.
\end{aligned} \quad (15)$$

The following lemma states the Principle of Optimality for this problem.

Lemma 3: $V^c(\tau, x)$ satisfies a semi-group property, namely:

$$\begin{aligned}
&V^c(\tau, x | V^c(t_f, .)) = \\
&\quad V^c\Big(\tau, x | V^c\big(t, . | V^c(t_f, .)\big)\Big), \quad \tau \leq t \leq t_f
\end{aligned} \quad (16)$$

The proof of the lemma is based on a standard technique from [5]. Basically, this means that the value function inherits the semi-group property from the reach set. The infinitesimal form of the Principle of Optimality yields a generalized Hamilton-Jacobi-Bellman PDE for $V^c(\tau, x)$.

Observe that:

$$\begin{aligned}
V^c(\tau, x_1, x_2 | V^c(t_f, .)) &\leq \varphi_0^2(x_2), \\
V^c(\tau, x_1, x_2 | V^c(t_f, .)) &\leq \varphi_0^1(x_1), \\
V^c(\tau, x | V^c(t_f, .)) &\leq \varphi_i(x), i = 1, \dots, 4
\end{aligned} \quad (17)$$

Let:

$$\mathcal{H}(t, x, V^c, u) = V_t^c + (V_x^c(t, x), f(t, x, u)) \quad (18)$$

Following [10] we conclude that the HJB equation for $V^c(\tau, x_1, x_2)$ is

Case 1) all the inequalities in equation (17) are strict:

$$V_t^c + \min_u \langle V_x^c(t, x), f(t, x, u) \rangle = 0 \quad (19)$$

Case 2) assume there is only one equality relation in equation (17), for example $V^c(\tau, x) = \varphi_i(\tau, x)$, for some i. Consider $(x^0(t), u^0(t))$ to be an optimal solution of problem (3) that goes through point x at time t (under the usual assumptions these exist). Then,

$$\begin{aligned}
\max\{\mathcal{H}(t, x^0(t), V^c, u), \mathcal{H}(t, x^0(t), \varphi_i, u)\} &\geq \\
\mathcal{H}(t, x^0(t), V^c, u^0(t)) &= \\
\mathcal{H}(t, x^0(t), \varphi_i, u^0(t)) &= 0
\end{aligned} \quad (20)$$

Now we turn to $V^c(\tau, x, T)$.

Lemma 4: The following relation is true:

$$V^c(\tau, x, T) = \min_{t_f \in T} \{V^c(\tau, x, t_f)\}.$$

In general, value functions are not differentiable and assumption (4) does not hold. However, the above derivations are still valid if we use some generalized concept of derivative. In this case the solutions to the HJB have to treated in a generalized ("viscosity" or "min-max") sense [2], [5], [22], [17], [6].

C. Controller Synthesis

The motion planning problem (1) under coordinated controls (given feasible trajectory x_2^f) is solvable if $S^c(t_0) \neq \emptyset$ ($S_1^a(t_0) \neq \emptyset$).

Let $t_0 \in \mathbb{R}$ be such that problem (1) under coordinated controls is solvable. Consider $(x_1^0, x_2^0) \in S^c(t_0)$ and let $\theta = argmin_{t_f \in T} V^c(t_0, x_1^0, x_2^0, t_f)$. Pick the value function $V^c(t_0, x_1^0, x_2^0, \theta)$. Starting at time t_0 the control strategy which solves problem 1 under coordinated controls has a feedback form $u(t, x_1, x_2) \in \mathcal{U}(t, x_1, x_2)$, where the feasible controls $\mathcal{U}(t, x_1, x_2)$ are the minimizers in the HJB equation (19), (20) for $V^c(.,.,.,\theta)$. The same type of calculations yield the control strategy for problem 1 under a given feasible trajectory x_2^f.

It may happen that the feedback law $u(t, x_1, x_2)$ is discontinuous in the state. This requires another notion of solution for differential equations (6), (7). One possible approach is to define the solution as a "constructive" motion introduced in [9].

V. LINEAR SYSTEMS

The solution approach described above involves solving a HJB equation for the value functions V^g and V^c. This is not a trivial matter for non-linear systems and general constraints. However, for systems with linear structure and complementary convex constraints the value function can be found through techniques of convex analysis and mini-max theory [7], [8]. We illustrate these techniques to find the value function for problem 2 with linear structure and convex and complementary ellipsoidal convex constraints.

The equations of motion are

$$\dot{x}_1(t) = A(t)x_1 + B(t)u_1, \qquad u_1(t) \in \mathcal{P}_1(t) \tag{21}$$

where $A(t)$ has continuous coefficients, $\mathcal{P}_1(t) = \mathcal{E}(0, P_1(t))$, P_1 is continuous in t and $P_1 > 0$. It is assumed that the system is completely controllable.

The ellipsoidal and the complementary ellipsoidal convex constraints are given by the set valued-maps F_2 and G_2 which map points to ellipsoids in $Comp_n$ with non-empty interior. For example $x_1 \in F_2(x_2^f)$ is given by $\langle (x_1 - x_2^f), F_2^e(x_1 - x_2^f)\rangle \leq 1)$. The target sets are also non-degenerate ellipsoids ($M_1 > 0, M_2 > 0$) $\mathcal{M}_1 = \mathcal{E}(m_1, M_1)$, and $\mathcal{M}_2 = \mathcal{E}(m_2, M_2)$.

In order to calculate the backward reach set $W_1^g[t_0, t_\alpha, t_\beta, \mathcal{M}_1]$ through $V^g(\tau, x_1, S)$ we need to consider a constraint qualification from [15]:

Assumption 5: There exists a control $u_1(t) \in \mathcal{P}_1$, $t \in [t_0, t_{\beta_g}]$, a point $x_1^0 \in \mathcal{X}_1$, and a number $\epsilon > 0$ such that the trajectory $x_1[t] = x_1(t, t_0, x_1^0 | u_1(.))$ generated by $u_1(t)$ produces a tube

$$x_1(t, t_0, x_1^0) + \epsilon \mathcal{B}_n(0) \subseteq F_2(x_2^f(t)), \quad t \in [t_0, t_{\beta_g}]$$

where $\mathcal{B}_n$ is the unit ball in $\mathbb{R}^n$.

As in [7] we find a solvability condition for $V^g(\tau, x_1, t_f)$ of the system of inequalities

$$\begin{aligned} (x_1[t] - x_2^f[t]), F_2^e(t)(x_1[t] - x_2^f[t]) &\leq \mu^2 \\ (x_1[t_f] - m_1), M_1(t_f)(x_1[t_f] - m_1) &\leq \mu^2 \end{aligned} \tag{22}$$

and find the smallest μ that ensures solvability.

Furthermore, we consider that assumption 3 holds.

Now let $s[t]$ be a row-vector solution to the adjoint equation

$$ds = -sAdt - q'(t)d\Lambda(t), s(t_f) = l' \tag{23}$$

where $q(t)$ is continuous and Λ is a nondecreasing, finite variation function of bounded variation, then

Theorem 1: $V^g(\tau, x_1, t_f)$ is given by the formula

$$\begin{aligned} V^g(\tau, x_1, t_f) &= \max_{q(.)} \max_{\Lambda(.)} \max_{l} \{ (s[\tau], x_1) \\ &\quad + \int_\tau^{t_f} (s[t]B(t)P_1(t)B'(t)s'[t])^{1/2} dt \} \\ &= \mu^0(\tau, x_1) \end{aligned} \tag{24}$$

where the maximums are taken over all functions $(q(t), N^{-1}q(t)) \leq 1, t \in [\tau, t_f], N = F_2^e$ and all elements $(l, M_1^{-1} l^{1/2}) + \int_\tau^{t_f} d\Lambda(t) \leq 1$.

From this theorem we obtain as a corollary that the backward reach set is convex and compact.

VI. IMPLEMENTATION

In this section, we present a framework for the implementation of feedback coordinated control of the motion of two vehicles based on the results presented in this article.

This framework encompasses both the motion planning and the motion execution levels and provides a joint feedback controller.

Let us cast the coordinated motion control problem in somewhat simpler terms having in mind the clarity of the exposition.

Let us assume that the dynamics of the two vehicles are given in a differential inclusion form in a certain fixed time interval $[t_0, \theta]$, their initial state constraint and target sets, are, for $i = 1, 2$, as follows:

$$\dot{x}_i(t) \in \tilde{F}_i(t, x_i(t)),\ t \in [t_0, \theta] \tag{25}$$

$$x_i(t_0) \in \mathcal{X}_i, \tag{26}$$

$$x_i(\theta) \in \mathcal{M}_i. \tag{27}$$

Besides the constraints above, the coordinated control strategy has also to satisfy the following joint state constraints:

$$x_1 \in H_2(x_2), \quad x_2 \in H_1(x_1) \tag{28}$$

for given set-valued maps $H_i : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, $i = 1, 2$.

The problem that we address here consists in computing control strategies for both dynamic systems so that the corresponding trajectories satisfy the respective endpoint constraints as well as the weak invariance property with respect to the joint state constraint (28).

Notice that, formally, the maps H_i can be easily related with the ones in the problem formulation in section III by putting $H_i(x) = F_i(x) \cap G_i(x)^c$, for all $x \in \mathbb{R}^n$. Here, A^c denotes the complement of the set A.

Obviously mappings $\tilde{F}_i$ can be easily related with the dynamics defined earlier as follows $\tilde{F}_i(t, x) = f_i(t, x, P(t))$ and the required assumptions are naturally inherited.

In what concerns the coordination constraints consider the set-valued maps $H_i(\cdot)$ satisfying following properties:

A1 For each $x \in \mathbb{R}^n$, $H_i(x)$, $i = 1, 2$, are compact, convex, and have nonempty interior in $\mathbf{R}^n$.

A2 The set-valued maps $H_i(\cdot)$, $i = 1, 2$, are Hausdorff Lipschitz continuous, i.e., $\exists K_i > 0$, s.t., $d_H(H_i(y), H_i(x)) \leq K_i\|x - y\|$. The Hausdorff distance between sets A and B is defined as follows

$$d_H(A, B) := \max\Big\{\max_{x\in A} \rho(x, B), \max_{x\in B} \rho(x, A)\Big\}$$

where $\rho(c, C) := \inf_{c'\in C} \|c - c'\|$.

Moreover, we also require several additional hypotheses ensuring the consistency among the different constraints intervening in the problem in order to ensure the existence of a coordinating control strategy for both vehicles. Therefore besides the existence of a feasible trajectory for each one of the dynamic systems in isolation, i.e., without considering the coordinating constraints, we also need the following assumptions

C1 $\text{int}\big[\mathcal{M}_1 \cap H_2(\mathcal{M}_2)\big] \neq \emptyset$ and $\text{int}\big[\mathcal{M}_2 \cap H_1(\mathcal{M}_1)\big] \neq \emptyset$.

C2 $\text{int}\big[\mathcal{X}_1 \cap H_2(\mathcal{X}_2)\big] \neq \emptyset$ and $\text{int}\big[\mathcal{X}_2 \cap H_1(\mathcal{X}_1)\big] \neq \emptyset$.

C3 There is a viable interior joint trajectory. That is, a pair $(x_1(t), x_2(t))$, $t \in [t_0, \theta]$ such that, for $i = 1, 2$, we have, besides the differential constraints, $x_i(t_0) \in \text{int}\mathcal{X}_i$, $x_i(\theta) \in \text{int}\mathcal{M}_i$, and

$$x_1(t) \in \text{int}F_2(x_2(t)), \text{ and } x_2(t) \in \text{int}F_1(x_1(t)).$$

Here and in what follows, int$A$ denotes the interior of the set A relatively to the space in which it was originally introduced. We also denote the unit ball by B.

Now, we are in position to address the proposed implementation framework which involves two stages.

A first one - the planning stage - where the finite sequences

$$\{t^i\}_{i=0}^N, \quad \{\varepsilon^i\}_{i=0}^N, \quad \{(x_1^i, x_2^i 9\}_{i=0}^N,$$

are computed a priori with the following properties:

- The $\varepsilon^k > 0$, and $t^k < t^{k+1}$, $k = 0, \ldots, N-1$, with $t^N = \theta$ and $t^0 = t_0$, are such that, for $i = 1, 2$,
$$x_i(t) \in x_i^k + \varepsilon^k B, \quad t \in [t^{k-1}, t^k].$$
- The pair (x_1^k, x_2^k) satisfies, for $k = 1, \ldots, N$,
$$x_1^k + \varepsilon^k B \subset H_2(x), \forall x \in x_2^k + \varepsilon^k B, \quad \text{and}$$
$$x_2^k + \varepsilon^k B \subset H_1(x), \forall x \in x_1^k + \varepsilon^k B.$$
- For $i = 1, 2$, $x_i(\theta) = x_i^N \in \mathcal{M}_i$, $x_i(t_0) = x_i^0 \in \mathcal{X}_i$, and $x_i(t^k) = x_i^k$, $k = 1, \ldots, N-1$.

The positive integer N should be chosen in order to ensure a robust viability of the computed sequences.

The procedure to compute the above sequences is recursive and it is initialized by finding the numbers ε^N, and (x_1^N, x_2^N) such that

$$x_1^N + \varepsilon^N B \subset \text{int}\Big[\mathcal{M}_1 \bigcap \Big\{\bigcap_{x\in x_2^N+\varepsilon^N B} H_2(x)\Big\} \bigcap W_1^f[\theta; t_0, \mathcal{X}_1]\Big],$$

$$x_2^N + \varepsilon^N B \subset \text{int}\Big[\mathcal{M}_2 \bigcap \Big\{\bigcap_{x\in x_1^N+\varepsilon^N B} H_1(x)\Big\} \bigcap W_2^f[\theta; t_0, \mathcal{X}_2]\Big].$$

Here, $W_i^f[\theta; t_0, \mathcal{X}_i]$, $i = 1, 2$, is the set reachable at time θ from the set $\mathcal{X}_i$ at time t_0.

Then, for $k = N, N-1, \ldots, 1$, the above sequences are computed so that

$$x_1^{k-1} + \varepsilon^{k-1} B \subset \text{int}\Big[W_1^b(t^{k-1}; t^k, x_1^k) \bigcap \Big\{\bigcap_{x\in W_2^b(t^{k-1}; t^k, x_2^k)} H_2(x)\Big\} \bigcap W_1^f[t^{k-1}; t_0, \mathcal{X}_1]\Big],$$

$$x_2^{k-1} + \varepsilon^{k-1} B \subset \text{int}\Big[W_2^b(t^{k-1}; t^k, x_2^k) \bigcap \Big\{\bigcap_{x\in W_1^b(t^{k-1}; t^k, x_1^k)} H_1(x)\Big\} \bigcap W_2^f[t^{k-1}; t_0, \mathcal{X}_2]\Big],$$

where $W_i^b(t^i; t^f, x_i^f)$ is the backward reach set at time t_i when the system reaches x_i^f at time t_f.

Each one of these sets is the level set of a certain value function which can be computed by solving the corresponding Hamilton-Jacobi equation. We should remark that there are several degrees of freedom which can be used in order to choose the best trade-off between complexity and robustness: the finer the time partition (i.e., the greater the number of points with smaller time subintervals $[t^{k-1}, t^k)$), the higher the complexity but also the higher the feasible values of ε^k, and hence the more robust is the obtained solution.

The second stage is the on-line computation of the control that drives the i^{th} dynamic control system, $i = 1, 2$, between the point x_i^{k-1} at time t^{k-1} to the point x_i^k at time t^k, while keeping it within the cylinder $x_1^k + \varepsilon^k B$, $k = 1, \ldots, N$. Notice that, because of the way these points are produced, there exists always such a control strategy.

On the other hand, since this control synthesis process involves each system individually, i.e., decoupled from the other, this is a conventional problem for which there are standard results. Therefore, we will not dwell on it here.

A final remark concerns the fact that this scheme lends itself to replanning. In other words, after the motion has already been initiated, there is always the possibility of recomputing new waypoints for the part of the motion that remains to be executed. This possibility can be used in order to optimize some pertinent performance criterion.

VII. CONCLUSIONS

We have described motion planning problems under coordinated constraints and used dynamic programming techniques to characterize the solution and to synthesize controllers. The solution approach involves solving a HJB equation. This is not a trivial matter. However, for systems with linear structure and ellipsoidal constraints we can use the techniques from [11] to obtain numerical solutions to the HJB equation. We have not yet explored the geometry of coordinated constraints so as to obtain a better characterization of the solution properties. An outline of a practical implementation scheme is also discussed.

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