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Oscillator Models and Collective Motion: Splay State Stabilization of Self-Propelled Particles

Derek A. Paley[1] and Naomi Ehrich Leonard[2]
Mechanical and Aerospace Engineering
Princeton University
Princeton, NJ 08544, USA
{dpaley}{naomi}@princeton.edu

Rodolphe Sepulchre[3]
Electrical Engineering and Computer Science
Université de Liège
Institut Montefiore B28, B-4000 Liège, Belgium
r.sepulchre@ulg.ac.be

Abstract— **This paper presents a Lyapunov design for the stabilization of collective motion in a planar kinematic model of N particles moving at constant speed. We derive a control law that achieves asymptotic stability of the splay state formation, characterized by uniform rotation of N evenly spaced particles on a circle. In designing the control law, the particle headings are treated as a system of coupled phase oscillators. The coupling function which exponentially stabilizes the splay state of particle phases is combined with a decentralized beacon control law that stabilizes circular motion of the particles.**

I. INTRODUCTION

Feedback control laws that stabilize collective motions of particle groups have a number of engineering applications including unmanned sensor networks. For example, autonomous underwater vehicles (AUVs) are used to collect oceanographic measurements in formations that maximize the information intake, see e.g. [1]. This can be achieved by matching the measurement density in space and time to the characteristic scales of the oceanographic process of interest. Coordinated, periodic trajectories such as the one studied in this paper, provide a means to collect measurements with the desired spatial and temporal separation.

In this paper, we consider a kinematic model of identical, all-to-all coupled, planar particles [2], [3]. In a sensor network application, this represents an all-to-all communication topology. The particles move at constant speed and are subject to steering controls that change their orientation. In previous work [4], [5], we observed that the norm of the average linear momentum of the group is a key control parameter: it is maximal in the case of parallel motions of the group and minimal in the case of circular motions around a fixed point. We exploited the analogy with phase models of coupled oscillators to design control laws that stabilize either parallel or circular motions.

In the present paper, we further develop this design methodology to stabilize the *splay state formation* of the group. This formation is characterized by circular motion around the (fixed) center of mass of the group, with all particles being evenly spaced on the circle. The term *splay* refers to the appearance of the particle phases when plotted on the unit circle (i.e. a phasor diagram) and is used in the coupled phase oscillator literature, see e.g. [6], [7], [8]. Our Lyapunov analysis proves asymptotic stability of the splay state formation and suggests convergence to that configuration from a large set of initial conditions. The splay state formation is relevant to the design of mobile sensor networks because it maximizes the measurement spacing of sensors on the same circular orbit. It is also illustrative of more general group formations characterized by a high level of symmetry. The applicability of the proposed design to a broader class of symmetric configurations is presented in [9].

The general philosophy of the proposed design is described in Section II. We treat the stabilization of the particle relative orientations in Section III and the stabilization of each particle position relative to the group center of mass in Section IV. Section V presents the complete control law and the construction of a composite Lyapunov function for the closed-loop dynamics.

II. PARTICLE MODEL AND CONTROL DESIGN

We consider a continuous-time kinematic model of $N \geq 2$ identical particles (of unit mass) moving in the plane at unit speed [2]:

$$\begin{aligned} \dot{r}_k &= e^{i\theta_k} \\ \dot{\theta}_k &= u_k, \end{aligned} \tag{1}$$

where $k = 1, \ldots, N$. In complex notation, the vector $r_k = x_k + iy_k \in \mathbb{C} \approx \mathbb{R}^2$ denotes the position of particle k and the angle $\theta_k \in S^1$ denotes the orientation of its (unit) velocity vector $e^{i\theta_k} = \cos\theta_k + i\sin\theta_k$. We use the variable without index to denote the corresponding N-vector, e.g. $\theta = (\theta_1, \ldots, \theta_N)$. The configuration space consists of N copies of the group $SE(2)$. In the absence of steering control ($\dot{\theta}_k = 0$), each particle moves at unit speed in a fixed direction and its motion is decoupled from the other particles.

We study the design problem of choosing feedback controls that stabilize a prescribed collective motion. The feedback controls are identical for all the particles and only depend on relative orientation and relative spacing, i.e., on the variables $\theta_{kj} = \theta_k - \theta_j$ and $r_{kj} = r_k - r_j$, $j,k = 1,\ldots,N$. Consequently, the closed-loop vector field is invariant under

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an action of the symmetry group $SE(2)$ and the closed-loop dynamics evolve on a reduced quotient manifold (shape space). Equilibria of the reduced dynamics are called relative equilibria and can be only of two types [2]: *parallel* motions, characterized by a common orientation for all the particles (with arbitrary relative spacing), and *circular* motions, characterized by circular orbits of the particles around a fixed point.

In the present paper, we study the stabilization of a particular relative equilibrium characterized by a high level of symmetry: the splay state formation. In the spirit of our previous work, we decompose the design into two parts: an orientation control, aimed at stabilizing the relative orientations of the velocity vectors; and a spacing control, aimed at stabilizing the position of each particle relative to the center of mass of the group. The orientation control law is independent of the position variables r_k and is designed to stabilize the splay state of the phase variables $\theta_k \in S^1$, which corresponds to the N phases evenly distributed on the unit circle. The phase dynamics evolve in a reduced configuration space consisting of N copies of S^1 modulo the action of the symmetry group S^1 of uniform rotations.

The orientation control law, designed in Section III, achieves gradient dynamics with respect to a potential that reaches its global minimum at the splay state of the phase variables. The spacing control law stabilizes the position of each particle relative to the center of mass of the group in an identical manner to the single particle beacon control law presented in Section IV. The main result of the paper shows that the sum of the orientation control law and the spacing control law results in stabilization of the splay state formation. Stability of this relative equilibrium is proven with a composite Lyapunov function that combines the phase and spacing potentials.

III. SPLAY STATE STABILIZATION

In this section, we prove exponential stabilization of the particle phases to the splay state, originally referred to as the "antiphase" state [10] and also described as "ponies on a merry-go-round" [11]. The approach relies on including higher harmonics of the phase differences in the coupling function, as has been considered, for example, in [12]. Besides the splay state, all of the fixed points that we are able to identify are unstable, which suggests that the splay state has a large region of attraction.

Consider the system of N phases, θ_k, $k = 1,\ldots,N$, subject to control u_k, i.e. $\dot{\theta}_k = u_k$. We define the centroid of the mth harmonic of the particle phases to be

$$p_{m\theta} = \frac{1}{Nm}\sum_{j=1}^{N} e^{im\theta_j} = |p_{m\theta}|e^{i\Psi_m}, \tag{2}$$

where $m = 1,2,\ldots$, $\Psi_m \in S^1$, and $0 \le |p_{m\theta}| \le 1/m$. We refer to the phase configurations for which $m|p_{m\theta}| = 1$ ($|p_{m\theta}| = 0$) as having the mth phase harmonic synchronized (antisynchronized). Let the centroid of the $m = 1$ phase harmonic be given by $p_\theta = p_{1\theta}$.

The *synchronized state*, $|p_\theta| = 1$, occurs for $\theta_1 = \theta_2 = \ldots = \theta_N$, and, in the planar particle model (1), corresponds to parallel motion. Note that $|p_\theta| = 1$ implies that $m|p_{m\theta}| = 1$ for $m > 1$. The *antisynchronized state* (or incoherent state), $|p_\theta| = 0$, corresponds to a fixed center of mass of the group in the planar particle model (1). The *splay state* of the particle phases is characterized by $\theta_k = \frac{2\pi k}{N}$, $k = 1,\ldots,N$, i.e. the phases are evenly distributed around the unit circle. The splay state corresponds to antisynchronization of the first $N-1$ phase harmonics and synchronization of the Nth phase harmonic. We denote by $\left\lfloor \frac{N}{2} \right\rfloor$ the largest integer less than or equal to $\frac{N}{2}$. A necessary and sufficient condition for the splay state is [9]

$$|p_\theta| = |p_{2\theta}| = \cdots = |p_{\lfloor \frac{N}{2} \rfloor\theta}| = 0. \tag{3}$$

Consider a quadratic potential $U_m(\theta) = \frac{N}{2}|p_{m\theta}|^2$ for $m = 1,2,\ldots$. The potential $U_m(\theta)$ is maximal (minimal) in the synchronized (antisynchronized) state of the mth phase harmonic. To stabilize the splay state, we use the potential

$$U(\theta) = \frac{N}{2}\sum_{m=1}^{\lfloor \frac{N}{2} \rfloor} |p_{m\theta}|^2, \tag{4}$$

which is maximal for the synchronized state and minimal in the splay state (3). The kth element of the gradient of this potential, $\mathrm{grad}U(\theta)$, is the partial derivative of (4) with respect to θ_k, given by

$$\frac{\partial U}{\partial \theta_k} = \sum_{m=1}^{\lfloor \frac{N}{2} \rfloor} < p_{m\theta}, ie^{im\theta_k} >, \tag{5}$$

with the inner product $< z_1, z_2 >= \mathrm{Re}\{z_1\bar{z}_2\}$, $z_1, z_2 \in \mathbb{C}$.

Theorem 1: Gradient dynamics of the system $\dot{\theta} = u$ with respect to the potential (4) are obtained by choosing $u = -K\mathrm{grad}U(\theta)$, i.e.

$$u_k = \frac{K}{N}\sum_{j=1}^{N}\sum_{m=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{m}\sin m\theta_{kj} \tag{6}$$

where $K > 0$ is a scalar gain. All trajectories asymptotically converge to the critical points of the potential $U(\theta)$. In particular, the splay state is a stable equilibrium and a global minimum of the potential $U(\theta)$.

Proof: The time derivative of the potential $U(\theta)$ along the trajectories of the particle phases, θ_k, is

$$\dot{U}(\theta) = -K\|\mathrm{grad}U(\theta)\|^2 \le 0. \tag{7}$$

Therefore, all trajectories converge to the largest invariant set for which $\dot{U}(\theta) = 0$, i.e. the critical points of the potential $U(\theta)$ defined by $\mathrm{grad}U(\theta) = 0$, which includes the splay state. The splay state (3) is the global minimum of the potential since $U(\theta) = 0$ for $|p_{m\theta}| = 0$, $m = 1,\ldots,\left\lfloor \frac{N}{2} \right\rfloor$. ■

Theorem 1 proves asymptotic convergence of (6) to the set of critical points of $U(\theta)$. Next, we address the stability of the critical points that we have identified. Let M be the number of clusters of synchronized phases in an arbitrary phase configuration. A *symmetric M-pattern* has M equally spaced clusters of N/M phases; an *asymmetric M-pattern* has M equally spaced clusters with a different number of phases in each cluster.

Theorem 2: The splay state is the only exponentially stable symmetric pattern of the orientation control law (6). All other symmetric patterns of $M<N$ phase clusters are unstable equilibria of (6). In addition, the set of asymmetric $M=2$ patterns for which $\theta_{kj}=0$ or π for $j,k=1,\dots,N$ are unstable equilibria of (6).

Proof: Let the control (6) be defined in terms of the coupling function, $\Gamma(\theta_{kj})$, i.e.

$$\dot{\theta}_k = \frac{1}{N}\sum_{j=1}^{N}\Gamma(\theta_{kj}) \tag{8}$$

where

$$\Gamma(\theta_{kj}) = K\sum_{m=1}^{\lfloor \frac{N}{2}\rfloor}\frac{1}{m}\sin m\theta_{kj}. \tag{9}$$

Let $\Gamma'(\theta_{kj})$ be the derivative of $\Gamma(\theta_{kj})$ with respect to θ_{kj}, given by

$$\Gamma'(\theta_{kj}) = K\sum_{m=1}^{\lfloor \frac{N}{2}\rfloor}\cos m\theta_{kj}. \tag{10}$$

As shown in [12], the linearization of coupling functions of this form about symmetric patterns of $M\le N$ phase clusters has N eigenvalues that can be described as the sum of two sets. The first set consists of the eigenvalue $\tilde{\lambda}^{(M)}$ with multiplicity $N-M$. These eigenvalues are associated with intra-cluster fluctuation. The second set consists of M eigenvalues $\lambda_p^{(M)}$, $p=0,\dots,M-1$. These eigenvalues are associated with inter-cluster fluctuation. Both sets of eigenvalues can be expressed as functions of the Fourier coefficients of $\Gamma'(\theta_{kj})$. For a general coupling function, the Fourier expansion of $\Gamma'(\theta_{kj})$ is

$$\Gamma'(\theta_{kj}) = \sum_{l=1}^{\infty}\left(a'_l\cos l\theta_{kj} + b'_l\sin l\theta_{kj}\right).$$

The formulas for calculating the (real part of) the eigenvalues are as follows [12]:

$$\tilde{\lambda}^{(M)} = \sum_{l=1}^{\infty} a'_{Ml} \tag{11}$$

$$\mathrm{Re}\{\lambda_p^{(M)}\} = \sum_{l=1}^{\infty}\left(a'_{Ml} - \frac{a'_{M(l-1)+p}+a'_{Ml-p}}{2}\right). \tag{12}$$

Note that only the a'_l coefficients determine stability and that $\mathrm{Re}\{\lambda_p^{(M)}\} = \mathrm{Re}\{\lambda_{M-p}^{(M)}\}$.

The splay state of particle phases has $M=N$ evenly spaced clusters of one phase each. In this case, $\tilde{\lambda}^{(N)}$ has multiplicity zero so all N eigenvalues are in the set $\lambda_p^{(N)}$. Also, since (8) is a gradient system, the Jacobian is symmetric and all the eigenvalues are real. The a'_l coefficients are given by integrating

$$a'_l = \frac{1}{\pi}\int_{-\pi}^{\pi}\Gamma'(\theta_{kj})\cos l\theta_{kj}\,d\theta_{kj} \tag{13}$$

which gives

$$a'_l = K,\ l=1,\dots,\left\lfloor\frac{N}{2}\right\rfloor \text{ and } a'_l=0,\ l=0,\ l>\left\lfloor\frac{N}{2}\right\rfloor. \tag{14}$$

As a result, $a'_{\lfloor \frac{N}{2}\rfloor l}=0$ for $l>1$, which, using (12), yields $\lambda_p^{(N)} = -K/2<0$ for $p=1,\dots,N-1$ and $\lambda_0^{(N)}=0$. The only exception is for even N and $p=\frac{N}{2}$, in which case $\lambda_p^{(N)} = -K<0$. The zero eigenvalue corresponds to rigid rotation of all N phases [12]. Therefore, the splay state is exponentially stable because the remaining $N-1$ eigenvalues are negative definite.

Next, we show that the other critical points of $U(\theta)$ that we have identified are unstable. Symmetric patterns of $M<N$ equally spaced clusters of N/M phases are fixed points of (8) [12]. In this case, the eigenvalue $\tilde{\lambda}^{(M)}$ has multiplicity $N-M>0$ and, using (11) and (14), is positive definite. Therefore, all symmetric patterns other than the splay state are unstable equilibria of the coupling function (9), which includes the synchronized state ($M=1$) and the symmetric $M=2$ pattern.

Finally, we show that the asymmetric $M=2$ patterns for which $\theta_{kj}=0$ or π for $j,k=1,\dots,N$ (and $j\neq k$) are unstable equilibria of (8). Note that that this configuration exists only for $N\ge 3$. In an asymmetric $M=2$ pattern, all of the even phase harmonics, i.e. $p_{m\theta}$ with m even, are synchronized. The odd phase harmonics, i.e. $p_{m\theta}$ with m odd, satisfy $m|p_{m\theta}|=\alpha$, where $\alpha\in(0,1)$. If $\bar{\theta}=(\bar{\theta}_1,\dots,\bar{\theta}_N)$ is an asymmetric $M=2$ pattern, then there exists at least one pair k and l in $\{1,\dots,N\}$ such that $\bar{\theta}_k=\bar{\theta}_l$, $k\neq l$. Define the variation $\delta\theta=(\delta\theta_1,\dots,\delta\theta_N)$. In the vicinity of the critical point $\bar{\theta}$, $U(\theta)$ can be expanded as

$$U(\bar{\theta}+\delta\theta) = U(\bar{\theta}) + \delta\theta^T H\delta\theta + \mathscr{O}(|\delta\theta|^3), \tag{15}$$

where H is the Hessian of $U(\theta)$ evaluated at $\bar{\theta}$. Using (2), equation (5) can be written

$$\frac{\partial U}{\partial\theta_k} = \frac{1}{N}\sum_{m=1}^{\lfloor \frac{N}{2}\rfloor}\frac{1}{m} < \sum_{j=1,j\neq k}^{N} e^{im\theta_j}, ie^{im\theta_k} >$$

so that the diagonal terms of H are

$$\frac{\partial^2 U}{\partial\theta_k^2} = -\frac{1}{N}\sum_{m=1}^{\lfloor \frac{N}{2}\rfloor} < \sum_{j=1,j\neq k}^{N} e^{im\theta_j}, e^{im\theta_k} > \tag{16}$$

and the off-diagonal terms of H are

$$\frac{\partial^2 U}{\partial\theta_l\partial\theta_k} = \frac{1}{N}\sum_{m=1}^{\lfloor \frac{N}{2}\rfloor} < e^{im\theta_l}, e^{im\theta_k} >, \tag{17}$$

for $k\neq l$.

Assume, without loss of generality, that $\bar{\theta}_1=\bar{\theta}_2=\Psi_m=0$, i.e. we choose two phases from the larger cluster and this cluster is aligned with the positive real axis. Consider a variation with $\delta\theta_1\delta\theta_2\neq 0$ and $\delta\theta_k=0$ for $k=3,\dots,N$. Evaluating the Hessian at $\theta=\bar{\theta}$, the diagonal terms (16)

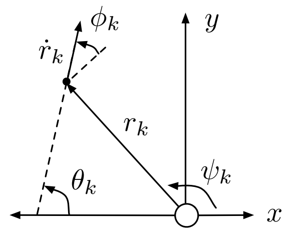

Fig. 1. The coordinates used in Section IV that describe the position and velocity of the kth planar particle with respect to the beacon at the origin.

become

$$\begin{aligned}\frac{\partial^2 U}{\partial \theta_k^2} &= -\frac{1}{N}\sum_{m=1}^{\lfloor \frac{N}{2} \rfloor} < Nmp_{m\theta} - e^{im\theta_k}, e^{im\theta_k} > \\ &= -P_{even} - \alpha P_{odd} + \frac{1}{N}\left\lfloor \frac{N}{2} \right\rfloor,\end{aligned}$$

for $k = 1,2$ where P_{even} (P_{odd}) is the cardinality of the positive even (odd) integers less than or equal to $\lfloor \frac{N}{2} \rfloor$. The off-diagonal terms (17) become

$$\frac{\partial^2 U}{\partial \theta_l \partial \theta_k} = \frac{1}{N}\left\lfloor \frac{N}{2} \right\rfloor,$$

for $k,l = 1,2$, $k \neq l$. The upper left 2×2 block of the Hessian is

$$H_{12} = \begin{bmatrix} -P_{even} - \alpha P_{odd} + \frac{1}{N}\lfloor \frac{N}{2} \rfloor & \frac{1}{N}\lfloor \frac{N}{2} \rfloor \\ \frac{1}{N}\lfloor \frac{N}{2} \rfloor & -P_{even} - \alpha P_{odd} + \frac{1}{N}\lfloor \frac{N}{2} \rfloor \end{bmatrix}$$

This matrix is negative definite for $N \geq 4$ since

$$P_{even} > \frac{2}{N}\left\lfloor \frac{N}{2} \right\rfloor \tag{18}$$

For $N = 3$, we obtain $\delta\theta^T H \delta\theta < 0$ by choosing the variation $\delta\theta_1 = -\delta\theta_2$. Consequently, $\bar{\theta}$ is not a minimum of $U(\theta)$ since, using (15), $U(\bar{\theta} + \delta\theta) < U(\bar{\theta})$. Therefore, all asymmetric $M = 2$ patterns are unstable equilibria of (8). ■

Since the splay state is exponentially stable and all other critical points of $U(\theta)$ that we have identified are unstable, Theorems 1 and 2 suggest a large region of attraction of the dynamics (6) to the splay state.

IV. BEACON CONTROL LAW

In this section, we set aside the splay state control term of Section III and derive the spacing control term for stabilization of the splay state formation. This is identical to the control of a single particle circling a beacon at a fixed radius, ρ_0. Consider the kinematic model (1) for N self-propelled particles in the plane subject to steering control. The position, $r_k = \rho_k e^{i\psi_k}$, and heading, θ_k, of the kth particle, respectively, are shown in Figure 1. We consider a control law that is the composition of Hamiltonian and dissipative terms. A constant control such as $u_k = -\omega_0 < 0$ drives the kth particle in a clockwise circular motion with radius $\rho_0 = \omega_0^{-1} > 0$ about an arbitrary fixed center.

To stabilize clockwise circular motion with radius ρ_0 about a fixed beacon at the origin of the inertial coordinate system, we add dissipation to the constant control, so the spacing control becomes

$$u_k = -\omega_0 - \kappa\omega_0 < r_k, \dot{r}_k >, \tag{19}$$

where $\kappa > 0$ is a scalar gain. The potential, $S_k(r_k, \theta_k)$, given by

$$S_k(r_k, \theta_k) = \frac{1}{2}|r_k - i\rho_0 e^{i\theta_k}|^2 \tag{20}$$

is nonincreasing along solution trajectories because

$$\dot{S}_k(r_k, \theta_k) = -\kappa < r_k, \dot{r}_k >^2 \leq 0.$$

The only invariant set for which $\dot{S}_k(r_k, \theta_k) = 0$ is a circle of radius ρ_0 centered at the origin, on which the kth particle travels clockwise at constant angular speed $\omega_0 = \rho_0^{-1}$.

Since clockwise circular motion of particle k is a relative equilibrium of (1) with control (19), exponential stability is established in the shape coordinates, (ρ_k, ϕ_k), shown in Figure 1. Differentiating with respect to time $r_k = \rho_k e^{i\psi_k}$ and $\phi_k = \theta_k - \psi_k + \pi/2$ and using (1) gives

$$\dot{\rho}_k e^{i\psi_k} + \rho_k i\dot{\psi}_k e^{i\psi_k} = e^{i\theta_k}$$

and

$$\dot{\phi}_k = \dot{\theta}_k - \dot{\psi}_k.$$

In the coordinates (ρ_k, ϕ_k, ψ_k), the system (1) with control (19) becomes

$$\begin{aligned}\dot{\rho}_k &= \sin\phi_k \\ \dot{\phi}_k &= -\omega_0 - \kappa\omega_0\rho_k \sin\phi_k + \rho_k^{-1}\cos\phi_k,\end{aligned} \tag{21}$$

and

$$\dot{\psi}_k = -\rho_k^{-1}\cos\phi_k.$$

Note that the equations of motion of the shape coordinates, ρ_k and ϕ_k, are independent of ψ_k, which reflects the rotational symmetry of the system.

Theorem 3: For particle k, the relative equilibrium corresponding to clockwise circular motion with radius ρ_0 about the origin is the exponentially stable fixed point of (21) given by $(\rho_k, \phi_k) = (\rho_0, 0)$ with $\dot{\psi}_k = -\omega_0$. Furthermore, this fixed point is globally asymptotically stable.

Proof: The Jacobian of the system (21), evaluated at the unique fixed point $(\rho_k, \phi_k) = (\rho_0, 0)$, has eigenvalues, $\lambda = (-\kappa \pm \sqrt{\kappa^2 - 4\omega_0^2})/2$, with strictly negative real part. Global attractivity of this fixed point is proved using the Lyapunov function (20) in the (ρ_k, ϕ_k) coordinates which is radially unbounded in the coordinate ρ_k. ■

V. COMPOSITE LYAPUNOV FUNCTION

In this section, we construct a composite Lyapunov function to prove stabilization of the splay state formation: i.e. uniform clockwise rotation of N evenly spaced particles on a circle of prescribed radius. The control law combines the orientation control from Section III with the spacing control of Section IV. Numerical simulation results of stabilizing the splay state formation are included in Figure 3.

Define the center of mass of the particles to be $R = \frac{1}{N}\sum_{k=1}^{N} r_k$. Note that the average linear momentum, $\dot{R}$, is

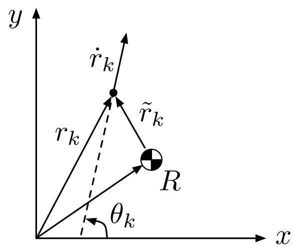

Fig. 2. The coordinates used in Section V that describe the position and velocity of the kth planar particle with respect to the center of mass, R.

equivalent to the centroid, p_θ, of the particle headings on the unit circle, which is defined in (2) for $m=1$. The vector from the center of mass to particle k is $\tilde{r}_k = r_k - R = \frac{1}{N}\sum_{j=1}^{N} r_{kj}$, as shown in Figure 2. Define the distance from the center of mass to the kth particle $\rho_k = |\tilde{r}_k|$ and let $\rho_0 = \omega_0^{-1} > 0$ be the desired equilibrium radius.

Consider a composite Lyapunov function, $V(r,\theta)$, which combines the splay potential (4) with a modified beacon potential (20), given by

$$V(r,\theta) = KU(\theta) + \kappa S(r,\theta), \tag{22}$$

where $K > 0$ and $\kappa > 0$ are scalar gains as before. The potential $S(r,\theta)$ is given by

$$S(r,\theta) = \frac{1}{2}\sum_{k=1}^{N} |\tilde{r}_k - i\rho_0 e^{i\theta_k}|^2. \tag{23}$$

The potential $V(r,\theta)$ is positive definite and is minimum (zero) for clockwise circular motion with radius ρ_0 in the splay state formation.

The time derivative of $U(\theta)$ is

$$\dot{U}(\theta) = <\mathrm{grad}U(\theta), \dot{\theta}>, \tag{24}$$

where the kth element of $\mathrm{grad}U(\theta)$ is given by (5). Let $\mathbf{1} = (1,\ldots,1) \in \mathbb{R}^N$. Then, using (5), we observe that

$$<\mathrm{grad}U(\theta), \mathbf{1}> = \sum_{m=1}^{\lfloor \frac{N}{2} \rfloor} Nm <p_{m\theta}, ip_{m\theta}> = 0.$$

Consequently, (24) can be written,

$$\dot{U}(\theta) = <\omega_0 \mathrm{grad}U(\theta), \mathbf{1} + \rho_0\dot{\theta}>. \tag{25}$$

The time derivative of $S(r,\theta)$ is

$$\begin{aligned} \dot{S}(r,\theta) &= \sum_{k=1}^{N} <\tilde{r}_k - i\rho_0 e^{i\theta_k}, \dot{r}_k - \dot{R} + \rho_0 e^{i\theta_k}\dot{\theta}_k> \\ &= \sum_{k=1}^{N} (1+\rho_0\dot{\theta}_k) <\tilde{r}_k, e^{i\theta_k}>. \end{aligned} \tag{26}$$

Combining (25) and (26) gives

$$\dot{V}(r,\theta) = \sum_{k=1}^{N} <K\omega_0\frac{\partial U}{\partial\theta_k} + \kappa <\tilde{r}_k, e^{i\theta_k}>, 1 + \rho_0\dot{\theta}_k>.$$

Choosing the control $\dot{\theta} = u$, such that

$$u_k = -\omega_0(1 + K\omega_0\frac{\partial U}{\partial\theta_k} + \kappa <\tilde{r}_k, e^{i\theta_k}>) \tag{27}$$

results in

$$\dot{V}(r,\theta) = -\sum_{k=1}^{N}\left(K\omega_0\frac{\partial U}{\partial\theta_k} + \kappa <\tilde{r}_k, e^{i\theta_k}>\right)^2 \leq 0. \tag{28}$$

The control (27) is the composition of the orientation control (6) with the spacing control (19), where the coordinate ρ_k (previously distance to the beacon) is now defined with respect to the center of mass. The control law (27) can be written

$$u_k = -\omega_0 - \omega_0\frac{\kappa}{N} <\sum_{j=1}^{N} r_{kj}, \dot{r}_k> + \omega_0^2\frac{K}{N}\sum_{j=1}^{N}\sum_{m=1}^{\lfloor \frac{N}{2} \rfloor}\frac{1}{m}\sin m\theta_{kj}$$

Note that choosing $K=1$ and $\kappa = \omega_0$ weights the orientation and spacing controls equally with the constant gain ω_0^2.

Theorem 4: The system (1) with control (27) asymptotically stabilizes all particles to clockwise circular motion with radius ρ_0 about a fixed center and with relative phases determined by the critical points of the potential (4). In particular, rotation in a splay state formation is a stable relative equilibrium which minimizes the Lyapunov function $V(r,\theta)$. The fixed center of rotation is the center of mass of the group.

Proof: The Lyapunov $V(r,\theta)$ is nonincreasing along the solutions and, by the LaSalle Invariance principle, solutions converge to the largest invariant set Λ where

$$\dot{\theta}_k = -\omega_0, \tag{29}$$

for $k = 1,\ldots,N$. In this set, each particle orbits a fixed circle of radius ρ_0. We want to show that all centers coincide. Differentiating (2) along the trajectories of (29) gives

$$\dot{p}_{m\theta} = -\frac{i\omega_0}{N}\sum_{k=1}^{N} e^{im\theta_k} = -im\omega_0 p_{m\theta}. \tag{30}$$

For $m=1$, this implies that the center of mass R satisfies the differential equation,

$$\ddot{R} = -i\omega_0\dot{R}. \tag{31}$$

Another consequence of (30) is that $\mathrm{grad}U(\theta)$ is constant in Λ since, using (5), (29), and (30),

$$\begin{aligned} \frac{d}{dt}\frac{\partial U}{\partial\theta_k} &= \sum_{m=1}^{\lfloor \frac{N}{2} \rfloor} <\dot{p}_{m\theta}, ie^{im\theta_k}> + <p_{m\theta}, -me^{im\theta_k}\dot{\theta}_k> \\ &= \sum_{m=1}^{\lfloor \frac{N}{2} \rfloor} <-im\omega_0 p_{m\theta}, ie^{im\theta_k}> + <p_{m\theta}, m\omega_0 e^{im\theta_k}> = 0 \end{aligned}$$

Combining this result with (27) and (29) yields

$$\frac{d}{dt} <\tilde{r}_k, \dot{r}_k> = 0 \tag{32}$$

for every solution in the invariant set Λ.

Using (31), we note that solutions in Λ also satisfy

$$\ddot{\tilde{r}}_k = \ddot{r}_k - \ddot{R} = -i\omega_0\dot{\tilde{r}}_k,$$

which, integrated twice, provides the explicit solution

$$\tilde{r}_k(t) = \tilde{r}_k(0) + i\rho_0\dot{\tilde{r}}_k(0)(e^{-i\omega_0 t} - 1). \tag{33}$$

Similarly, integrating (31) twice yields

$$R(t) = R(0) + i\rho_0 \dot{R}(0)(e^{-i\omega_0 t} - 1). \quad (34)$$

Substituting (33) in (32) results in

$$\frac{d}{dt} < \tilde{r}_k(0) + i\rho_0 \dot{\tilde{r}}_k(0)(e^{-i\omega_0 t} - 1), \dot{r}_k(0)e^{-i\omega_0 t} >= 0,$$

which can be rewritten as

$$\frac{d}{dt} < \tilde{r}_k(0) - i\rho_0 \dot{\tilde{r}}_k(0), \dot{r}_k(0)e^{-i\omega_0 t} >= 0 \quad (35)$$

since $< i\dot{\tilde{r}}_k(0)e^{-i\omega_0 t}, \dot{r}_k(0)e^{-i\omega_0 t} >=< i\dot{\tilde{r}}_k(0), \dot{r}_k(0) >$ is a constant. But (35) can be satisfied only if

$$\tilde{r}_k(0) = i\rho_0 \dot{\tilde{r}}_k(0) \quad (36)$$

for each $k = 1, \ldots, N$.

Substituting (36) in (33) shows that solutions in Λ satisfy

$$r_k(t) = R(t) + i\rho_0 \dot{\tilde{r}}_k(0)e^{-i\omega_0 t}. \quad (37)$$

Using (34) in (37), we thus arrive at the explicit solution

$$r_k(t) = R(0) - i\rho_0 \dot{R}(0) + i\rho_0 e^{i\theta_k}$$

which shows that all solutions in Λ circle with radius ρ_0 around the same fixed point $R(0) - i\rho_0\dot{R}(0)$. Because $S(r,\theta) = \frac{N}{2}\rho_0^2|\dot{R}(0)|^2$ is constant along these solutions, $U(\theta)$ must be constant in Λ and the relative phases must correspond to a critical point of $U(\theta)$. Rotation in the splay state formation is a stable relative equilibrium by Theorem 1 since it minimizes $U(\theta)$. Furthermore, since $p_\theta = \dot{R} = 0$ in the splay state formation, the fixed center of rotation is the center of mass of the group. ■

We include a simulation for the splay state formation in Figure 3. Simulations suggest a large region of attraction of the splay state formation using the control (27).

VI. CONCLUSIONS

In this paper we provide a control law that stabilizes the splay state formation in a kinematic model of N particles moving at constant velocity. The control law is the sum of an orientation control and a spacing control. The orientation control is independent of the position variables and assigns gradient dynamics for the phase variables with respect to a potential that reaches its minimum in the splay state configuration, that is, when the phase variables are evenly spaced on the unit circle. The spacing control stabilizes the position of each particle relative to the center of mass of the particle system. The sum of the two controls is shown to stabilize the splay state formation by means of Lyapunov analysis. We show in [9] a generalization of this result that stabilizes all symmetric patterns of N particles in a circular formation. One can also break the translational symmetry of the control in order to stabilize the splay state formation about a fixed beacon [9]. Similar studies on connected but not complete coupling networks suggest that the splay state formation is can be stabilized by topologies that are not all-to-all [13], [14].

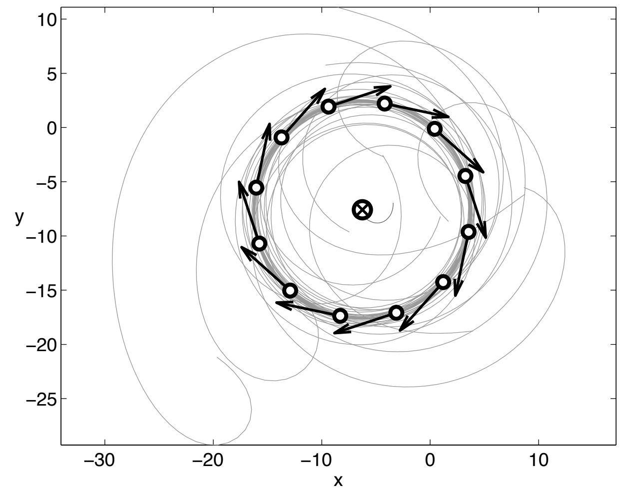

Fig. 3. The result of a numerical simulation of stabilizing the splay state formation using control (27) with $N = 12$, $\rho_0 = 10$, $K = 1$, $\kappa = \omega_0 = 0.1$, and random initial conditions. The particle trajectories are shown in grey and their final positions are black circles. The center of mass is depicted by the black crossed circle.

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