Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Stability and optimality of distributed model predictive control

Aswin N. Venkat, James B. Rawlings and Stephen J. Wright

Abstract— **This article extends existing concepts in linear model predictive control (MPC) to a unified, theoretical framework for distributed MPC with guaranteed nominal stability and performance properties. Centralized MPC is largely viewed as impractical, inflexible and unsuitable for control of large, networked systems. Incorporation of the proposed distributed regulator provides a means of achieving optimal systemwide control performance (centralized) while essentially operating in a decentralized manner. The distributed regulators work iteratively and cooperatively towards achieving a common, systemwide control objective. An attractive attribute of the proposed MPC algorithm is that all intermediate iterates are feasible and the resulting distributed MPC controllers stabilize the nominal closed-loop system. These two features allow the practitioner to terminate the distributed control algorithm at the end of each sampling interval, even if convergence is not attained. Distributed MPC with output feedback is addressed using the well established Kalman filtering framework for state estimation.**

I. INTRODUCTION

Over the last decade, model predictive control (MPC) has established itself as a premier advanced control technology. Model predictive control subsystems have been widely implemented across the chemical industry sector, exploiting some of the latest theoretical developments in the area [1], [2], [3], [4]. In cases of tightly coupled systems, the interplay between high performance local control and subsystem-subsystem interactions leads to a deterioration in systemwide control performance. For most networked systems, the primary hurdles to centralized control are not computational but organizational. Operators of large, networked systems view centralized control as monolithic and inflexible. In many instances, different parts of the networked system are owned by different organizations making the comprehensive information sharing required by centralized control impractical. Unless these organizational impediments can be surmounted, centralized control of large, networked systems is useful primarily as the benchmark against which other control strategies can be compared and assessed.

The opportunity presented for cross-integration within the MPC framework and potential requirements and benefits of such technology has been discussed in [5], [6], [7]. A two-level decomposition coordination strategy for generalized predictive control based on the master-slave paradigm was proposed by [8]. A plantwide control strategy based on the integration of linear and nonlinear MPC coupled with a plant decomposition procedure was described by [9], [10]. While these methods have been demonstrated to work well for the cases considered, no nominal or closed-loop properties have been established. A continuous time distributed receding horizon framework for multi-vehicle formation stabilization was proposed by [11]. Unlike control of networked chemical systems, the subsystem dynamics in multi-vehicle stabilization are decoupled but the states of the system are non-separably coupled in the cost function. A distributed MPC framework for systems in which the dynamics of the subsystems are dynamically decoupled but may have coupled objectives/constraints was proposed by [12]. Global stability and feasibility issues have, however, not been addressed.

A distributed MPC formulation that considers coupling between various subsystem dynamics was proposed by [13], [14]. The authors show asymptotic stability under state feedback provided the interactions among the subsystems satisfy a stability constraint. The authors claim that their distributed MPC algorithm yields a solution that is close to the optimal, centralized solution. However, we shall show that the distributed MPC formulation described in [13], [14] is not necessarily optimal. In fact, it is shown through an example that undesirable closed-loop behavior may result. It has been established in game theory that multi-agent strategies in which the competing agents are restricted to exchange of information amongst themselves (henceforth referred to as *communication* based schemes) and are unaware of each other's cost function result in a non-cooperative equilibrium or Nash equilibrium (NE) [15]. It is also known that the NE is, in general, not Pareto optimal [16], [17]. A distributed MPC paradigm in which the effects of the interacting subsystems maybe treated as bounded uncertainties was proposed by [18]. Nominal properties of the framework have, however, not been established.

This work addresses the issue of distributed control of networked systems through the suitable integration of the various subsystems' MPCs. The proposed cooperation-based distributed MPC algorithm is iterative in nature. At convergence, the distributed MPC algorithm achieves optimal (centralized) control performance. In addition, the control algorithm can be terminated at any intermediate iterate without compromising feasibility or closed-loop stability of the resulting distributed controller. It is assumed that the interactions between the subsystems are stable; system re-design is recommended otherwise. The proposed method serves to equip practitioners with a low-risk plantwide (systemwide) control strategy that not only allows them to build on existing control infrastructure but also avoids the organizational impediments associated with centralized control. In cases

Aswin N. Venkat is a Doctoral candidate at the Dept. of Chemical and Biological Engineering, University of Wisconsin, Madison, WI-53706, USA venkat@bevo.che.wisc.edu

James B. Rawlings is with Faculty of Chemical and Biological Engineering, University of Wisconsin, Madison, WI-53706, USA jbraw@bevo.che.wisc.edu

Stephen J. Wright is with Faculty of Computer Sciences, University of Wisconsin, Madison, WI-53706, USA swright@cs.wisc.edu

where different subsystems of the networked process are owned/governed by different organizations, this cooperation-based strategy presents an attractive opportunity to enhance control performance.

II. Interaction modeling

a) Decentralized models: Consider a plant comprised of M subsystems. Let the decentralized (local) model for each subsystem be represented by a discrete, linear time invariant (LTI) model of the form

$$\begin{aligned} x_{ii}(k+1) &= A_{ii}x_{ii}(k) + B_{ii}u_i(k) \\ y_i(k) &= C_{ii}x_{ii}(k), \quad i = 1, 2, \ldots, M \end{aligned} \tag{1}$$

in which k is discrete time, and we assume (A_{ii}, B_{ii}, C_{ii}) is a minimal realization for each (u_i, y_i) input-output pair.

Owing to material/energy and/or information flows there exists levels of interactions between the subsystems. In the decentralized modeling framework, it is assumed that the interactions have a negligible effect on local variables. This assumption is not reliable in many situations and can lead to a deterioration in control performance.

b) Interaction models (IM): We use a framework that allows us to quantitatively assess the interactions between subsystems and retains most advantages of a decentralized approach.

Consider any subsystem $i \in \{1, M\}$ [1]. The effect of an interacting subsystem $j \neq i$ on subsystem i is represented through a discrete LTI model of the form

$$x_{ij}(k+1) = A_{ij}x_{ij}(k) + B_{ij}u_j(k) \tag{2}$$

$$y_i(k) = \sum_{l=1}^{M} C_{il}x_{il}(k), \quad i \in \{1, M\} \tag{3}$$

in which (A_{ij}, B_{ij}, C_{ij}) represents the effect of the inputs of subsystem j on subsystem i.

c) Composite models (CM): The combination of the decentralized model and the interaction models for each subsystem yields the composite model (CM). The decentralized state vector x_{ii} is augmented with states arising due to the effects of all other subsystems.

Let $x_i^T = [x_{i1}^T, \ldots, x_{ii}^T, \ldots, x_{iM}^T]$ denote the CM states for subsystem i. For notational simplicity, we represent the CM for subsystem i as

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + B_i u_i(k) + \sum_{j \neq i} W_{ij} u_j(k) \\ y_i(k) &= C_i x_i(k) \end{aligned} \tag{4}$$

in which $C_i = [C_{i1} \ldots C_{ii} \ldots C_{iM}]$ and

$$A_i = \begin{bmatrix} A_{i1} & & & & \\ & \ddots & & & \\ & & A_{ii} & & \\ & & & \ddots & \\ & & & & A_{iM} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ \vdots \\ B_{ii} \\ 0 \\ \vdots \end{bmatrix}, \quad W_{ij} = \begin{pmatrix} 0 \\ \vdots \\ B_{ij} \\ 0 \\ \vdots \end{pmatrix}$$

[1]The notation $\{1, M\}$ denotes the sequence of integers $1, 2, \ldots M$

The CM for the entire plant can be written as

$$\begin{bmatrix} x_{11} \\ \vdots \\ x_{1M} \\ \hline \vdots \\ \hline x_{M1} \\ \vdots \\ x_{MM} \end{bmatrix}(k+1) = \left[\begin{array}{ccc|c|ccc} A_{11} & & & & & & \\ & \ddots & & & & & \\ & & A_{1M} & & & & \\ \hline & & & \ddots & & & \\ \hline & & & & A_{M1} & & \\ & & & & & \ddots & \\ & & & & & & A_{MM} \end{array}\right] \begin{bmatrix} x_{11} \\ \vdots \\ x_{1M} \\ \hline \vdots \\ \hline x_{M1} \\ \vdots \\ x_{MM} \end{bmatrix}(k) + \left[\begin{array}{ccc} B_{11} & & \\ & \ddots & \\ & & B_{1M} \\ \hline & \vdots & \\ \hline B_{M1} & & \\ & \ddots & \\ & & B_{MM} \end{array}\right] \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix}(k).$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_M \end{bmatrix}(k) = \left[\begin{array}{c|c|c} C_{11} \cdots C_{1M} & & \\ & \ddots & \\ & & C_{M1} \cdots C_{MM} \end{array}\right] \begin{bmatrix} x_{11} \\ \vdots \\ x_{1M} \\ \hline \vdots \\ \hline x_{M1} \\ \vdots \\ x_{MM} \end{bmatrix}(k)$$

After identification of the significant interactions from closed-loop operating data, we expect many of the interaction terms to be zero. In the decentralized model, all of the interaction terms are zero. More details on closed-loop identification procedures for distributed MPC can be found in [19].

d) Centralized model: Finally, the full plant (centralized) model can be thought of as a minimal realization of the CM for the entire plant. The centralized model is represented as

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{5}$$

III. Distributed MPC: state feedback

Given the composite model $(A_i, B_i, C_i, \{W_{ij}\})$ for each subsystem $i \in \{1, M\}$, we consider two formulations for distributed MPC namely, communication-based MPC and cooperation-based MPC. In the sequel, the suitability of either framework for plantwide control is assessed. For both communication and cooperation-based MPC, an iteration and exchange of variables between subsystems is performed during a sample time. We may choose not to iterate to convergence. We denote this iteration number as p. The set of admissible controls for subsystem $i, \Omega_i \subseteq \mathbb{R}^{m_i}$ is assumed to be a non-empty, compact, convex set containing the origin in its interior. For convenience, we define

$$\Omega_i = \{u_i \in \mathbb{R}^{m_i} | D_i u_i \leq d_i, d_i > 0\} \tag{6}$$

The set of admissible controls for the whole plant Ω is defined to be the Cartesian product of the admissible control sets of each of the subsystems. It follows that Ω is a compact, convex set containing the origin in its interior.

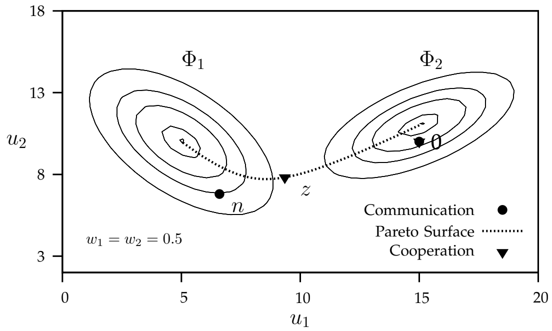

Fig. 1. Static optimization example. Location of the Nash equilibrium (n), cooperative equilibrium (z) and the Pareto optimal surface.

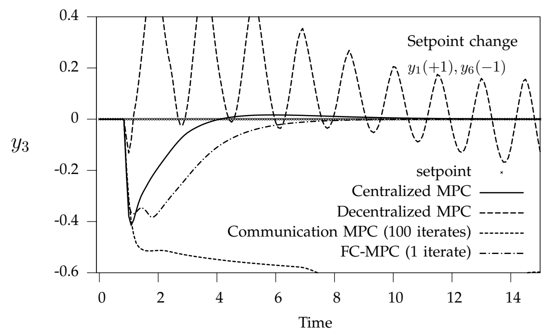

Fig. 2. Closed-loop performance of centralized, decentralized and communication-based MPC. Communication-based MPC does not converge; terminated after 100 iterates.

A. Communication-based MPC [2]

The cost function for communication-based MPC is defined over an infinite horizon and written as

$$\phi_i\left(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{j\neq i}^{p-1}, \boldsymbol{u}_{j\neq i}^{p-1}; x_i(k)\right) = \sum_{t=0}^{\infty} x_i^T(t|k) Q_i(t) x_i(t|k) + u_i^T(t|k) R_i(t) u_i(t|k) \quad (7)$$

in which $Q_i \geq 0$, $R_i > 0$ are symmetric weighting matrices with $(Q_i^{1/2}, A_i)$ detectable. For each subsystem i and iterate p, the optimal state-input trajectory $(\boldsymbol{x}_i^p, \boldsymbol{u}_i^p)$ is defined as

$$\begin{aligned}
(\boldsymbol{x}_i^p, \boldsymbol{u}_i^p) \in \ & \arg \min_{\boldsymbol{x}_i, \boldsymbol{u}_i} \ \phi_i\left(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{j\neq i}^{p-1}, \boldsymbol{u}_{j\neq i}^{p-1}; x_i(k)\right) \\
\text{s.t.} \ & x_i(t+1|k) = A_i x_i(t|k) + B_i u_i(t|k) \\
& \qquad + \sum_{j \neq i} W_{ij} u_j^{p-1}(t|k), \ k \leq t \\
& x_i(k) = \widehat{x}_i(k) \\
& u_i(t|k) \in \Omega_i, \ k \leq t \leq k+N-1 \\
& u_i(t|k) = 0, \ k+N \leq t \qquad (8) \\
\forall \ & i = 1, 2, \ldots, M
\end{aligned}$$

in which $\boldsymbol{x}_i^p = \left[x_i^p(k|k)^T, x_i^p(k+1|k)^T, \ldots\ldots\right]^T$ and $\boldsymbol{u}_i^p = \left[u_i^p(k|k)^T, u_i^p(k+1|k)^T, \ldots\ldots\right]^T$. $\widehat{x}_i(k)$ represents the current estimate of the composite model states for subsystem i. For each subsystem i at iteration p, only the subsystem input sequence $\boldsymbol{u}_i^p$ is optimized and updated. The other subsystems' inputs are not altered during this optimization; they remain at iterate $p-1$. The objective function is the one for subsystem i only.

Each communication-based MPC has no information about the cost functions of the other subsystems' MPCs. From a game theoretic perspective, such an equilibrium, if it exists, is called a non-cooperative equilibrium or Nash equilibrium [20]. The objective of each subsystem's MPC is frequently in conflict with the objectives of other interacting subsystems' MPCs. The best achievable performance is characterized by a Pareto optimal surface which represents the set of optimal trade offs among these conflicting controller objectives. Optimality of the communication-based MPC formulation implies that the Nash equilibrium (NE) lies on the Pareto optimal surface. A simple static quadratic optimization example shown in Fig. 1 is used to illustrate optimality properties of the communication-based MPC framework. The NE is at point n which does not lie on the Pareto optimal surface. Consequently, communication-based MPC is not optimal, even at convergence. To analyze convergence of the communication-based MPC formulation, we consider a multivariable 7 input–7 output plant consisting of two interacting subsystems. In this case, the communication-based MPC algorithm does not converge and is terminated after 100 iterates. The closed-loop performance of the different MPC frameworks for setpoint tracking of output y_3 is shown Fig. 2. We observe that communication-based MPC is closed-loop unstable. On the other hand, centralized MPC is stable and tracks its setpoint.

B. Feasible cooperation-based MPC (FC-MPC)

The unsuitability of the communication-based MPC formulation as a plantwide control strategy motivates the need for an alternate approach–one that is plantwide optimal, at least in a limiting sense. We next modify, the objective functions of the subsystems' MPCs in order to provide a means for cooperative behavior among the controllers. We replace the objective Φ_i with an objective that measures the entire system performance. Many suitable objectives are possible. Here we choose the simplest case, the overall plant objective, which is the weighted sum of all the subsystems' objectives, $\Phi = \sum_i w_i \Phi_i, \ \ w_i \geq 0, \ \ \sum_{i=1}^{M} w_i = 1$. [3]

In practical situations, the process sampling interval may be insufficient for the computation time required for convergence of the iterative algorithm. In such situations, the cooperative control formulation has to be terminated prior to convergence of the state and input trajectories (*i.e.*, when time runs out). The last calculated input trajectory is used to arrive at a suitable control law. To facilitate intermediate termination, all iterates generated by the distributed MPC algorithm have to be plantwide feasible and the resulting distributed controller must stabilize the plant in closed loop. By plantwide feasibility, we mean that the state–input sequence $\{\boldsymbol{x}_i, \boldsymbol{u}_i\}_{i=1}^M$ satisfies the model and input constraints of each subsystem $i \in \{1, M\}$. To guar-

[2]Similar strategies have been proposed by [13], [14]

[3]In this work, we choose $w_i = \frac{1}{M}, i \in \{1, M\}$. However, all results presented hold for any choice of the weight sequence $\{w_i\}_{i=1}^M$ satisfying $w_i \geq 0, \ \ \sum_{i=1}^{M} w_i = 1$

antee plantwide feasibility of the intermediate iterates, we eliminate the states $\boldsymbol{x}_i, i \in \{1, M\}$ using the composite model equation (4) and solve an optimization problem of the form

$$\boldsymbol{u}_i^{p(*)}(k) \in \ \arg(\text{FC-MPC}_i) \ \text{ where}$$

$$\begin{aligned} \text{FC-MPC}_i \triangleq \min_{\boldsymbol{u}_i} \quad & \frac{1}{M}\sum_{i=1}^{M} \Phi_i\left(\boldsymbol{u}_i, \boldsymbol{u}_{j\neq i}^{p-1}; x_i(k)\right) \\ \text{s.t.} \quad & x_i(k) = \widehat{x}_i(k) \\ & u_i(j|k) \in \Omega_i, \quad k \le j \le k+N-1 \\ & u_i(j|k) = 0, \quad k+N \le j \\ & \forall \ i = 1, 2, \ldots M \end{aligned} \tag{9}$$

For $\Phi_i(\cdot)$ quadratic and obtained by eliminating the CM states x_i from (7) using the subsystem model equation (4) for each $i \in \{1, M\}$, the cooperation-based MPC optimization problem for subsystem i is therefore

$$\begin{aligned} \min_{\boldsymbol{u}_i} \quad & \Phi\left(\boldsymbol{u}_i, \boldsymbol{u}_{j\neq i}^{p-1}; x(k)\right) = \frac{1}{2}\boldsymbol{u}_i^T(k)\mathcal{R}_i\boldsymbol{u}_i(k) \\ & + \left(r_i(k) + \sum_{j=1, j\neq i}^{M} H_{ij}\boldsymbol{u}_j^{p-1}(k)\right)^T \boldsymbol{u}_i(k) + \text{constant} \\ \text{s.t.} \quad & u_i(t|k) \in \Omega_i, \quad k \le t \le k+N-1 \\ & u_i(t|k) = 0, \quad k+N \le t \end{aligned} \tag{10}$$

in which

$$\begin{aligned} \mathcal{R}_i &= \mathbb{R}_i + E_{ii}^T\mathbb{Q}_iE_{ii} + \sum_{j\neq i}^{M} E_{ji}^T\mathbb{Q}_jE_{ji} \\ H_{ij} &= E_{ii}^T\mathbb{Q}_iE_{ij} + E_{ji}^T\mathbb{Q}_jE_{jj} \\ r_i(k) &= E_{ii}^T\mathbb{Q}_if_ix_i(k) + \sum_{j\neq i}^{M} E_{ji}^T\mathbb{Q}_jf_ix_j(k) \quad x_j(k) = \widehat{x}_j(k) \\ E_{ij} &= \begin{bmatrix} B_{ij} & 0 & \ldots & \ldots & 0 \\ A_iB_{ij} & B_{ij} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_i^{N-1}B_{ij} & \ldots & \ldots & \ldots & B_{ij} \end{bmatrix} \quad f_i = \begin{bmatrix} A_i \\ A_i^2 \\ \vdots \\ \vdots \\ A_i^N \end{bmatrix} \\ \mathbb{Q}_i &= \text{diag}\left(Q_i(1), \ldots, Q_i(N-1), \overline{Q}_i\right) \\ \mathbb{R}_i &= \text{diag}\left(R_i(0), R_i(1), \ldots, R_i(N-1)\right) \end{aligned} \tag{11}$$

with $\overline{Q}_i$ denoting an appropriately chosen terminal penalty.

1) Distributed optimization with constraints: Consider the following prototype centralized MPC optimization problem obtained by eliminating the states using the model equality constraints

$$\begin{aligned} \min_{\{\boldsymbol{u}_i\}_{i=1}^M} \quad & \Phi\left(\boldsymbol{u}_i, \boldsymbol{u}_{j\neq i}\,; x(k)\right) = \frac{1}{M}\sum_{i=1}^{M}\Phi_i\left(\boldsymbol{u}_i, \boldsymbol{u}_{j\neq i}; x_i(k)\right) \\ \text{s.t.} \quad & u_i(t|k) \in \Omega_i, \quad k \le t \le k+N-1 \\ & u_i(t|k) = 0, \quad k+N \le t, \quad i \in \{1, M\} \end{aligned} \tag{12}$$

Definition 1: The *normal cone* to a convex set Ω at a point x is denoted by $N(x, \Omega)$ and defined by

$$N(x, \Omega) = \{s \mid \langle s, y - x\rangle \le 0 \ \text{ for all } y \in \Omega\}. \tag{13}$$

Optimality is characterized by the following result (which uses convexity but does not assume that the solution is unique).

Theorem 1: $(\boldsymbol{u}_i^*, \boldsymbol{u}_{j\neq i}^*)$ is optimal for (12) if and only if

$$-\nabla_{\boldsymbol{u}_i}\Phi\left(\boldsymbol{u}_i^*, \boldsymbol{u}_{j\neq i}^*; x(k)\right) \in N(\boldsymbol{u}_i^*; \Omega_i) \tag{14}$$

for all i, $i \in \{1, M\}$.

Suppose that the following level set is bounded and closed (hence compact):

$$\mathcal{L} = \left\{(\boldsymbol{u}_i, \boldsymbol{u}_{j\neq i}) \,|\, \Phi\left(\boldsymbol{u}_i, \boldsymbol{u}_{j\neq i}\,; x(k)\right) \le \Phi\left(\boldsymbol{u}_i^0, \boldsymbol{u}_{j\neq i}^0\,; x(k)\right)\right\}$$

with $\boldsymbol{u}_i \in \Omega_i, i \in \{1, M\}$.

We have the following result concerning the limiting set of a sequence of normal cones of a closed convex set.

Lemma 1: Let $\Omega \in \mathbb{R}^m$ be closed and convex. Let $x \in \Omega$ and let $\{x_i\}$ be a sequence of points satisfying $x_i \in \Omega$ and $x_i \to x$. Let $\{v_i\}$ be any sequence satisfying $v_i \in N(x_i; \Omega)$ for all i. Then all limit points of the sequence $\{v_i\}$ belong to $N(x; \Omega)$.

2) Algorithm and properties: The state sequence generated by the input sequence $\boldsymbol{u}$ and initial state z is represented as $\boldsymbol{x}^{(\boldsymbol{u};z)}$. We have the following algorithm for cooperation-based MPC

Algorithm 1: Given $\left(\boldsymbol{u}_i^0, x_i(k)\right)$ $\mathbb{Q}_i \ge 0, \mathbb{R}_i \ge 0, i \in \{1, M\}$ $p_{\max}(k) \ge 0$ and $\epsilon > 0$
$p \leftarrow 1, \rho_i \leftarrow \Gamma\epsilon, \Gamma \gg 1$
while $\rho_i > \epsilon$ for some $i \in \{1, M\}$ and $p \le p_{\max}$
do $\forall \ i \in \{1, M\}$
$\boldsymbol{u}_i^{p(*)} \in \ \arg(\text{FC-MPC}_i)$, (see (9), (10))
end (do)
for each $i \in \{1, M\}$
$\boldsymbol{x}_i^{p(*)} \leftarrow \boldsymbol{x}_i^{(\boldsymbol{u}_i^{p(*)}, \boldsymbol{u}_{j\neq i}^{p(*)}; x(k))}$
$(\boldsymbol{x}_i^p, \boldsymbol{u}_i^p) = \frac{1}{M}(\boldsymbol{x}_i^{p(*)}, \boldsymbol{u}_i^{p(*)}) + \left(1 - \frac{1}{M}\right)(\boldsymbol{x}_i^{p-1}, \boldsymbol{u}_i^{p-1})$
$\rho_i = \|(\boldsymbol{x}_i^p, \boldsymbol{u}_i^p) - (\boldsymbol{x}_i^{p-1}, \boldsymbol{u}_i^{p-1})\|$
end (for)
$p \leftarrow p + 1$
end (while)

For the distributed MPC formulation (9), (10) with Algorithm 1, the following properties can be established

Lemma 2: Given the distributed MPC formulation $\text{FC-MPC}_i \ \forall \ i \in \{1, M\}$, the sequence of cost functions $\left\{\Phi(\boldsymbol{u}_i^p, \boldsymbol{u}_{j\neq i}^p; x(k))\right\}$ generated by Algorithm 1 is a nonincreasing function of the iteration number p.

Lemma 3: All limit points of Algorithm 1 are optimal.

Revisiting the static optimization example presented in Section III-A, we note that the cooperation-based optimization scheme converges after 7 iterates. The cooperative equilibrium (Point z in Fig. 1) lies on the Pareto surface (hence optimal).

3) Closed-loop properties: Let $\mathcal{X}$ represent the constrained stabilizable set for the plant under the set of input constraints $\{\Omega_i\}_{i=1}^M$. At time k, let the FC-MPC scheme be terminated after $p(k) = l$ iterates. Let

$$\boldsymbol{u}_i^l(x(k)) = \left[u_i^l(x(k), k)^T, \ldots, u_i^l(x(k), k+N-1)^T\right]^T \tag{15}$$

$\forall \ i \in \{1, M\}$ represent the solution to the FC-MPC algorithm (Algorithm 1) after l iterates. The distributed

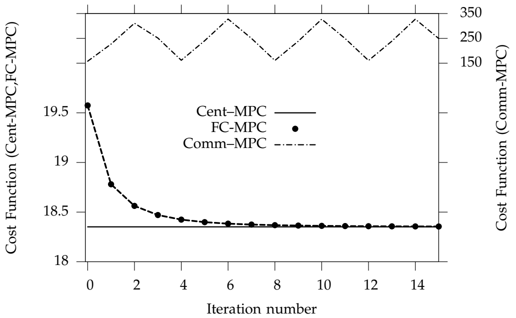

Fig. 3. Behavior of the communication and FC-MPC cost function with iteration number at time=1.75.

MPC control law is obtained through a receding horizon implementation of optimal control whereby the input applied to subsystem i is $u_i(k) = u_i^l(k|k) \equiv u_i^l(x(k), k)$. Lemmas 2 and 3 lead to the following theorems on closed-loop stability.

Theorem 2 (Stable decentralized modes): Given Algorithm 1 and the state feedback distributed MPC formulation (9), (10) with $N \geq 1$. If all interaction models are stable and if $\forall \ i \in \{1, M\}$

- A_{ii} stable.
- $Q_i(0) = Q_i(1) = \ldots = Q_i(N-1) = Q_i > 0$.
- $R_i(0) = R_i(1) = \cdots = R_i(N-1) = R_i > 0$.
- $\mathbb{Q}_i = \text{diag}\left(Q_i(1), \ldots, Q_i(N-1), \overline{Q}_i\right)$ in which $\overline{Q}_i$ is the solution to the Lyapunov equation $A_i^T \overline{Q}_i A_i - \overline{Q}_i = -Q_i$

then the origin is an exponentially stable equilibrium for the closed-loop system $x(k+1) = Ax(k) + Bu(x(k))$, in which $u(x(k)) = \left[u_1^{p(k)}(x(k), k)^T, \ldots, u_M^{p(k)}(x(k), k)^T\right]^T$, for all $x(k) \in \mathcal{X}$ and all $p(k) = 1, 2, \ldots$.

Remark 1: If $Q_i \geq 0$ (not > 0) for some $i \in \{1, M\}$ and $(Q_i^{1/2}, A_i)$ detectable $\forall \ i \in \{1, M\}$ then the closed-loop system $x(k+1) = Ax(k) + Bu(x(k))$ is asymptotically stable under the state feedback cooperation-based distributed MPC control law.

We re-examine the 7 input-7 output, two subsystem plant considered in Section III-A. It was observed (Fig. 2) that the communication-based iterates did not converge and the controller defined by terminating the communication-based MPC algorithm after 100 iterates caused unstable closed-loop behavior. We note from Fig. 2 that the distributed controller obtained by terminating the FC-MPC algorithm after just 1 iterate is sufficient to stabilize the closed-loop system and achieves performance comparable to centralized MPC. The behavior of the cumulative communication-based MPC and FC-MPC cost function at time 1.75 min is shown in Fig. 3. While the communication-based MPC cost function shows sustained oscillatory behavior with iteration number, the FC-MPC cost function approaches the centralized MPC cost function monotonically with iteration number.

Remark 2: When the constraints (10) is specified in terms of a set of algebraic inequalities and equalities such that the active constraints are linearly independent for each $i \in \{1, M\}$, then the distributed control law defined by Theorem 2 is a Lipschitz continuous function of the system state $x(k)$.

Define $\mathcal{X}_N$ to be the N-step constrained stabilizable set. For systems with unstable decentralized modes, we have the following theorem on closed-loop stability under the distributed MPC control law.

Theorem 3 (Unstable decentralized modes): Given Algorithm 1 and the state feedback distributed MPC formulation (9), (10) with $N \geq r, \ r = \max_{1 \leq i \leq M} r_i$ and end constraint $z_{ii}(k+N|k) = U_{u_i}^T x_i(k+N|k) = 0$ enforced on the unstable decentralized modes. If all interaction models are stable and if $\forall \ i \in \{1, M\}$

- (A_{ii}, B_{ii}) stabilizable with $r_i \geq 0$ unstable modes.
- $Q_i(0) = Q_i(1) = \ldots = Q_i(N-1) = Q_i > 0$.
- $R_i(0) = R_i(1) = \ldots = R_i(N-1) = R_i > 0$.
- $\mathbb{Q}_i = \text{diag}\left(Q_i(1), \ldots, Q_i(N-1), \overline{Q}_i\right)$ in which $\overline{Q}_i = U_{s_i} \Sigma_i U_{s_i}^T$. Σ_i is the solution to the Lyapunov equation $A_{s_i}^T \Sigma_i A_{s_i} - \Sigma_i = -U_{s_i}^T Q_i U_{s_i}$ with $A_i = [U_{s_i}, U_{u_i}] \begin{bmatrix} A_{s_i} & \mathbb{A}_i \\ & A_{u_i} \end{bmatrix} [U_{s_i}, U_{u_i}]^T$. A_{u_i} and A_{s_i} represent the unstable and stable eigenvalue blocks obtained through a real Schur decomposition of A_i.

then the origin is an asymptotically stable equilibrium point for the closed-loop system $x(k+1) = Ax(k) + Bu(x(k))$, in which $u(x(k)) = \left[u_1^{p(k)}(x(k), k)^T, \ldots, u_M^{p(k)}(x(k), k)^T\right]^T$, for all $x(k) \in \mathcal{X}_N$ and all $p(k) = 1, 2, \ldots$.

IV. Distributed MPC: Output feedback

In practice, the states of each subsystem are not all measurable and are typically estimated from local measurements. The most commonly used state estimator for linear systems is the Kalman filter. It is assumed in the output feedback context that a stable Kalman filter, based on the composite model, is available for each subsystem. We denote the steady-state filter gain for each local observer as L_i; the subsystem observer poles are given by the eigenvalues of $(A_i - L_i C_i), i \in \{1, M\}$.

At time k, let the output feedback distributed MPC be terminated after $p(k) = l$ cooperative iterates. For each subsystem $i \in \{1, M\}$, the composite model states $x_i(k)$ are estimated using a steady-state Kalman filter designed based on the composite model. The notation e_i denotes the state estimate error for subsystem i. Given the open-loop input trajectory $\boldsymbol{u}_i^l(\widehat{x}(k))$, for each subsystem $i, i \in \{1, M\}$, obtained after l cooperative iterates (Algorithm 1) in which $\widehat{x}^T = \left[\widehat{x}_1^T, \widehat{x}_2^T, \ldots, \widehat{x}_M^T\right]$, we define the distributed MPC control law under output feedback as

$$u_i(k) = u_i^l(k|k) \equiv u_i^l(\widehat{x}(k), k), \quad i \in \{1, M\} \tag{16}$$

For stable systems, exponential stability of the closed-loop system under the output feedback distributed MPC control law is assured by the following theorem which requires that the local observers are exponentially stable but makes no assumptions on the optimality of the obtained state estimates.

Theorem 4 (Stable decentralized modes): Given Algorithm 1 and the distributed MPC formulation (9), (10) with the subsystem states $x_i(k), i \in \{1, M\}$ estimated using a steady-state Kalman filter based on the composite model. If $N \geq 1$, all interaction models are stable and in addition if $\forall\ i\ \in \{1, M\}$

- $A_{ii}, (A_i - L_i C_i)$ stable.
- $Q_i(0) = Q_i(1) = \ldots = Q_i(N-1) = Q_i > 0$.
- $R_i(0) = R_i(1) = \cdots = R_i(N-1) = R_i > 0$.
- $\mathbb{Q}_i = \text{diag}\left(Q_i(1), \ldots, Q_i(N-1), \overline{Q}_i\right)$ in which $\overline{Q}_i$ is the solution of the Lyapunov equation $A_i^T \overline{Q}_i A_i - \overline{Q}_i = -Q_i$

then the origin is an exponentially stable equilibrium for the closed-loop system $x(k+1) = Ax(k) + Bu(\widehat{x}(k))$, in which $u(\widehat{x}(k)) = \left[u_1^{p(k)}(\widehat{x}(k), k)^T, \ldots, u_M^{p(k)}(\widehat{x}(k), k)^T\right]^T$, for all $\widehat{x}(k) \in \mathcal{X}$ and all $p(k) = 1, 2, \ldots$.

Define $\mathcal{X}_{\mathrm{e,N}}^{\mathrm{i}}$ to be the set of admissible state estimate errors e_i *i.e.*, $\mathcal{X}_{\mathrm{e,N}}^{\mathrm{i}} = \{e_i(k) |\ \exists\ \{u_i(k + j|k+1)\}_{j=1}^{N-1} \text{with } U_{u_i}^T \widehat{x}_i(k+N|k) = 0\}$. Let $e(k) = \left[e_1(k)^T, e_2(k)^T, \ldots e_M(k)^T\right]^T$ and $\mathcal{X}_{\mathrm{e,N}} = \mathcal{X}_{\mathrm{e,N}}^1 \times \mathcal{X}_{\mathrm{e,N}}^2 \times \ldots \times \mathcal{X}_{\mathrm{e,N}}^{\mathrm{M}}$. The following theorem guarantees closed-loop stability under output feedback for systems with unstable decentralized modes.

Theorem 5 (Unstable decentralized modes): Given Algorithm 1, the distributed MPC formulation (9), (10) with $N \geq r$ and the subsystem states $x_i(k), i \in \{1, M\}$ estimated using a steady-state Kalman filter based on the composite model. If all interaction models are stable, an end constraint $z_{ii}(k+N|k) = U_{u_i}^T x_i(k+N|k) = 0$ is enforced on the unstable decentralized modes and in addition if $\forall\ i\ \in \{1, M\}$

- (A_{ii}, B_{ii}) stabilizable, $(A_i - L_i C_i)$ stable.
- $Q_i(0) = Q_i(1) = \ldots = Q_i(N-1) = Q_i > 0$.
- $R_i(0) = R_i(1) = \cdots = R_i(N-1) = R_i > 0$.
- $\mathbb{Q}_i = \text{diag}\left(Q_i(1), \ldots, Q_i(N-1), \overline{Q}_i\right)$ in which $\overline{Q}_i = U_{s_i} \Sigma_i U_{s_i}^T$ with Σ_i obtained as the solution of the Lyapunov equation $A_{s_i}^T \Sigma_i A_{s_i} - \Sigma_i = -U_{s_i}^T Q_i U_{s_i}$

then the origin is an exponentially stable equilibrium for the closed-loop system $x(k+1) = Ax(k) + Bu(\widehat{x}(k))$, in which $u(\widehat{x}(k)) = \left[u_1^{p(k)}(\widehat{x}(k), k)^T, \ldots, u_M^{p(k)}(\widehat{x}(k), k)^T\right]^T$, for all $\widehat{x}(k) \in \mathcal{X}_{\mathrm{N}},\ e(k) \in \mathcal{X}_{\mathrm{e,N}}$ and all $p(k) = 1, 2, \ldots$.

V. CONCLUSIONS

In this work, a strategy was presented for coordinating the subsystems' MPCs to achieve guaranteed stability and performance properties. Existing distributed MPC formulations in the literature are suboptimal strategies with unproven nominal properties. For the proposed cooperation-based MPC algorithm, convergence, optimality and nominal closed-loop stability properties are established. All intermediate iterates generated by the FC-MPC algorithm are feasible. Further, the distributed control law derived by terminating the FC-MPC algorithm at any intermediate iterate stabilizes the (nominal) closed-loop system. These two features allow the termination of the control algorithm at the end of the sampling interval, even if convergence is not attained.

VI. ACKNOWLEDGMENT

The authors gratefully acknowledge the financial support of the industrial members of the Texas-Wisconsin Modeling and Control Consortium, and NSF through grant #CTS-0456694.

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