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Effects of Degree Correlation on the synchronizability of networks of nonlinear oscillators

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Abstract— **This paper is concerned with the investigation of the effects that the presence of degree correlation has on the synchronization of a network of coupled nonlinear oscillator. The main result is that disassortative networks are found to synchronize better. Specifically, synchronizability is enhanced in networks with various degree distribution, when nodes characterised by low degree are more likely to be connected to nodes with higher degree.**

I. INTRODUCTION

Recently, much research effort has been spent to characterise the synchronizability of networks of nonlinear oscillators [1], [2], [3]. Typically, a network is considered consisting of N identical oscillators coupled through the edges of the network itself. Each oscillator $\{x_i, i = 1...N\}$ is characterized by its own dynamics described by a nonlinear vector field, say $f = f(x)$, and is influenced by the output function of its neighbors represented by another nonlinear term, say $h = h(x)$. Thus the equations of motion are the following:

$$\frac{dx_i}{dt} = f(x_i) - \sigma \sum_{j=1}^{N} \mathcal{L}_{ij} h(x_j), \qquad i = 1, 2, ...N, \tag{1}$$

where σ is the overall coupling strength, and $\mathcal{L} = \{\mathcal{L}_{ij}\}$ is the Laplacian associated to the network topology [4]. This model has been shown to be representative of interactions that frequently occur in many different application areas, as biology, sociology and more importantly technological and communication systems.

Recent years have witnessed a great interest in the observation and in the study of the structure of real-world networks, as the Internet, electrical power-grids, biological and social networks and food webs. The availability of new databases containing huge amounts of data on these networks has led researchers from many fields to study which are the more important statistical features characterizing the structure of real world networks. The most important discovery in this context has been that the degree distribution of almost all the analyzed networks, follows a power law, i.e. the probability of having a node of degree k, scales as $k^{-\gamma}$ where γ is a network dependent constant, [5], [6], [7]. What is particularly surprising is that this property was observed to be practically a universal feature, being common to various networks in very different contexts, ranging from social environments to technology and biology.

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Thus, an immediate question arises on whether the network structure can have any influence on its synchronizability properties. Many recent studies have shown that some topological features can indeed have such an effect, focusing in particular on the heterogeneity in the degree distribution [2], [3], particular forms of ordering at the nodes [8] and opportune weighting of the connections among the oscillators [9].

For instance, as recently discussed in [2] (see sec. 2 below for further details), scale free networks were found to show better synchronizability for increasing value of the power law exponent γ [2], [3].

Another important topological property of physical networks is that often their nodes show preferential attachment to other nodes in the network according to their degree [10], [11]. According to this property, networks are said to exhibit *assortative* mixing (or positive correlation) if nodes of a given degree tend to be attached with higher likelihood to nodes with similar degree. (Similarly disassortative networks are those with nodes of higher degree more likely to be connected with nodes of lower degree.)

The presence of correlation has been detected experimentally in many real-world networks.

The main aim of this paper is to investigate the relationship between degree correlation and network synchronizability. Our main finding is that disassortative networks (i.e. networks where nodes with low degree are more likely to be connected to nodes of high degree) synchronize better.

II. SCALE FREE NETWORKS AND SYNCHRONIZABILITY

As mentioned above, the main feature of scale free networks (typical of real world networks), is an high heterogeneity in the degree distribution (higher than in purely random networks). In [2] the relationship between network structure and synchronizability was analyzed and an interesting phenomenon was observed, which was termed "the paradox of heterogeneity"; that is, although heterogeneity in the degree distribution, leads to a reduction in the average distance between nodes (the so called *small world* effect [12]), it may suppress synchronization.

Interestingly in [2], the transition of the underlying network from scale free (power law distributed) to random (poisson distributed) was shown to have a big impact on the eigenratio R of the Laplacian eigenvalues. Namely, a decrease of the heterogeneous nature of the network was discovered to yield, as a result, a reduction of R, thus increasing the synchronizability properties of the network itself.

In order to explain the effect of network degree distribution on synchronizability (the so called "paradox of heterogeneity" [3]), the following relationship was found in [2] that is valid for any connected network. Specifically, it was shown that

$$(1-\frac{1}{N})\frac{k_{max}}{k_{min}} \le R \le (N-1)k_{max}\ell^{e}_{max}D_{max}\overline{D}, \quad (2)$$

where k_{min} and k_{max} are the minimum and maximum node degrees; $\overline{D}$ and D_{max} are the graph characteristic path length and diameter; ℓ^{e}_{max} is the maximum of the normalized *load* on the links [13].

Equation (2) was derived assuming the absence of any degree correlation on the network. In what follows, we shall seek to analyze both the effects of heterogeneity and correlation in the degree distribution on the network synchronizability.

A. Correlation measures

Following [2], we will take as the starting point to our study the analysis of networks with an assigned degree distribution. In fact, as widely shown in the literature on complex network [14], the degree distribution can be considered one of the most relevant features characterizing the structure of graphs in dynamical terms. On the other hand, many important properties have been discovered to characterize in more detail the networks structure, other than the degree distribution. These are mostly due to particular forms of correlation or mixing among the network vertices.

Note that, once assigned the degree distribution, one usually has many remaining degrees of freedom on how coupling the vertices among them to form a network. Sometimes, the way in which vertices are coupled among themselves is influenced by some properties at the vertices, leading to various forms of mixing, according to the way nodes choose their neighbors.

Degree correlation is probably the simplest case of mixing among the network vertices one can imagine, since it can be defined by the knowledge of only the degree at the vertices (which is probably the most trivial information about the nodes one can have).

In [10], this property has been conveniently measured by means of a single normalized index, the Pearson statistic r defined as follows:

$$r = \frac{1}{\sigma_q^2}\sum_{ij} ij(e_{ij} - q_i q_j), \quad (3)$$

where q_k is the probability that a randomly chosen edge is connected to a node having remaining degree k (the remaining degree is given by the actual degree minus 1, since we are excluding the starting edge); σ_q is the standard deviation of the distribution q_k and e_{ij} represents the probability that two vertices at the endpoints of a generic edge have degree i and j respectively. Thus the correlation measure, r, as defined by (3), is given by the average distance between the effective distribution, e_{ij}, of links between nodes of degrees i and j and the theoretical distribution that would be observed in the case the network were uncorrelated, i.e. $q_i q_j$. Positive values of r indicate assortative mixing, while negative values characterize disassortatively mixed networks.

Moreover, a strategy was devised in [11] to generate networks with a given degree distribution and a desired correlation coefficient r. This strategy is based on a Monte Carlo method to change the combination of connections among vertices according to their degree in order to guarantee the network to exhibit the desired correlation properties at the end of the process (see [11] for further details).

For the sake of simplicity, in what follows, we will consider a simpler form of degree correlation where vertices are not characterized by their specific degree, but are shared in two archetypical classes, that is one, say L, containing low degree nodes and another, say H, containing high degree ones. In so doing, we will introduce an alternative measure of degree correlation, say $\hat{r}$, to estimate the assortativity or disassortativity of a given network with respect to the considered subdivision in two sole classes. Then using a technique analogous to that presented in [11], we will be able to generate networks with a desired value of the degree correlation index $\hat{r}$.

Note that, however, all the results derived in the following, could be generalized to an increasing number of considered classes, including the limiting case where each class contains exactly all and only the vertices with a given degree. Nonetheless showing what happens for a generic number of considered classes is beyond the scope of this paper and will be the object of a forthcoming publication.

Let us now partition the network vertices as follows: we include in class L, all the vertices with degree k less than or equal to some threshold value $\hat{k}$ and in the other one, H, all the vertices with degree higher than $\hat{k}$ (note that essentially, what does really matter in the way we partition the vertices, is that the average degree of nodes in L, say k_L is lower than k_H the average degree of nodes in class H. Also, let us term as n_L and n_H the number of vertices in L and H.)

According to this construction the probability of having a vertex in L and in H is equal to $p(k_L) = \frac{n_L}{N}, p(k_H) = \frac{n_H}{N}$.

Thus, the probability, say $q_i, \quad i \in \{L, H\}$, of finding a vertex of remaining degree equal to k_L or k_H respectively, at the endpoint of a generic edge picked up randomly within the network, is given by $q_L = \frac{n_L k_L}{n_L k_L + n_H k_H}, q_H = \frac{n_H k_H}{n_L k_L + n_H k_H}$.

Now, we propose that the presence of degree correlation in the network can be estimated by using a new coefficient, $\hat{r}$ defined as:

$$\hat{r} = \frac{\sum_{i\in\{L,H\}, j\in\{L,H\}} k_i k_j (e_{ij} - q_i q_j)}{{\sigma_q}^2}, \quad (4)$$

where $e_{i,j}$ $(i, j = \{L, H\})$ is the probability of finding an edge connecting nodes belonging to L or H to nodes in L or H respectively (e.g. e_{LH} is the probability that an edge connects a node belonging to the class of low-degree nodes to one belonging to the class of high-degree ones).

Note that σ_q^2 can be recast in terms of k_L and k_H, after simple algebraic manipulations, as $\sigma_q^2 = (q_L k_L^2 + q_H k_H^2) - (q_L k_L + q_H k_H)^2 = \frac{n_L k_L n_H k_H (k_L - k_H)^2}{(n_L k_L + n_H k_H)^2} = q_L q_H (k_L - k_H)^2$.

Also, $\hat{r}$ can be expressed alternatively as:

$$\hat{r} = \frac{1}{\sigma_q^2}\mathbf{k}^T\left(\mathbf{E}-\mathbf{q}\mathbf{q}^T\right)\mathbf{k} \tag{5}$$

where

$$\mathbf{k} = \begin{pmatrix} k_L \\ k_H \end{pmatrix}, \qquad \mathbf{E} = \begin{pmatrix} e_{LL} & e_{LH} \\ e_{HL} & e_{HH} \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} q_L \\ q_H \end{pmatrix}$$

It is worth mentioning here that this measure provides an estimate of the correlation degree of the network but differs from the correlation index (3) defined in [10], [11]. In particular, (3) is based on the exact degree of the vertices at each of the link endpoints in the network. The quantity defined in (4), instead, accounts only for whether connections in the networks are between low degree nodes (belonging to the set L) and high degree ones (belonging to the set H).

Nonetheless, we found the new coefficient to be easier to compute in applications and to be conceptually equivalent to the one in [10], [11] in the sense that it provides a reliable indication of the assortative - disassortative nature of the network under investigation.

Now using the new coefficient $\hat{r}$, it is possible to derive an expression for the coefficients of the matrix E associated to a network with a given value of $\hat{r}$. From (5), we have that $\mathbf{k}^T\left(\mathbf{E}-\mathbf{q}\mathbf{q}^T\right)\mathbf{k} = \sigma_q^2\hat{r},$.

So, if a symmetric matrix $\mathbf{M}$ exists such that $\mathbf{k}^T\mathbf{M}\mathbf{k} = 1$, we have $\mathbf{k}^T\mathbf{E}\mathbf{k} = \mathbf{k}^T\mathbf{q}\mathbf{q}^T\mathbf{k} + \mathbf{k}^T\hat{r}\sigma_q^2\mathbf{M}\mathbf{k}$, and thus, $\mathbf{E} = \mathbf{q}\mathbf{q}^T + \hat{r}\sigma_q^2\mathbf{M}$.

Note that, as pointed out in [11], $\mathbf{M}$ can be computed using the fact that it must be symmetric, it must satisfy $\mathbf{k}^T\mathbf{M}\mathbf{k} = 1$ and moreover must not vary the total amount of links connecting the nodes belonging to each class.

Specifically in our case, it is straightforward to show that there is a unique matrix $\mathbf{M}$ satisfying these constraints:

$$\mathbf{M} = \{m_{ij}\} = \begin{pmatrix} \frac{1}{(k_L-k_H)^2} & \frac{-1}{(k_L-k_H)^2} \\ \frac{-1}{(k_L-k_H)^2} & \frac{1}{(k_L-k_H)^2} \end{pmatrix} \tag{6}$$

Thus $\mathbf{E} = \{e_{LH}\}$ can be rewritten as follows:

$$\mathbf{E} = \begin{pmatrix} q_L^2 + \hat{r}q_Lq_H & q_Lq_H - \hat{r}q_Lq_H \\ q_Lq_H - \hat{r}q_Lq_H & q_H^2 + \hat{r}q_Lq_H \end{pmatrix} \tag{7}$$

We can now use the new coefficient $\hat{r}$ and the matrix E associated to a desired value of $\hat{r}$ to generate networks with a given degree distribution and a desired value of the correlation coefficient $\hat{r}$.

To start with, we will use the network building model presented in [15], analogous to the one analyzed in [2]. Firstly, assigned a degree distribution of the form $p(k) \sim k^{-\gamma}$, we generate a sequence of numbers $k_i \geq k_{min}$ drawn from the given distribution (the random configuration can be recovered in the limit of $\gamma = \infty$). Each number represents the degree of a node in the network. Then, we choose pairs of nodes at random and connect them, avoiding repeated links and self-connections.

The next stage is to perform a series of moves on the network edges in order for the resulting network to exhibit the desired value of the correlation index $\hat{r}$. To this aim, given $\hat{r}$, we partition the network nodes in the two sets L and H according to their degree and compute the average node degrees, k_L and k_H, of nodes belonging to the two classes respectively. Using these quantities we then compute the probability matrix $\mathbf{E}$ using (7).

Then, in order to achieve the desired correlation, we apply a procedure similar to the one presented in [10], [11]. Specifically, we choose two links connecting pairs of nodes and evaluate to which groups the nodes at their endpoints belong; for each edge we enumerate four different possibilities: L-L, L-H, H-L, H-H. Then we consider a possible rewiring of this two edges. For example suppose to have selected two edges of types $L-L$, $H-H$ and to examine a rewiring yielding to replace these connections with two new ones: $L-H$, $H-L$. Then we make the move with probability (see [11],[16] for more details)$P = \min\left(\frac{e_{LH}e_{HL}}{e_{LL}e_{HH}}, 1\right)$. Note that each move affects e_{ij} $(i,j = \{L,H\})$ and thus the correlation parameter $\hat{r}$, while it does not modify the degree distribution, which was fixed at the beginning of the rewiring procedure. Therefore, this procedure is an excellent method to investigate the effects of varying correlation on the network, without interfering with the degree distribution.

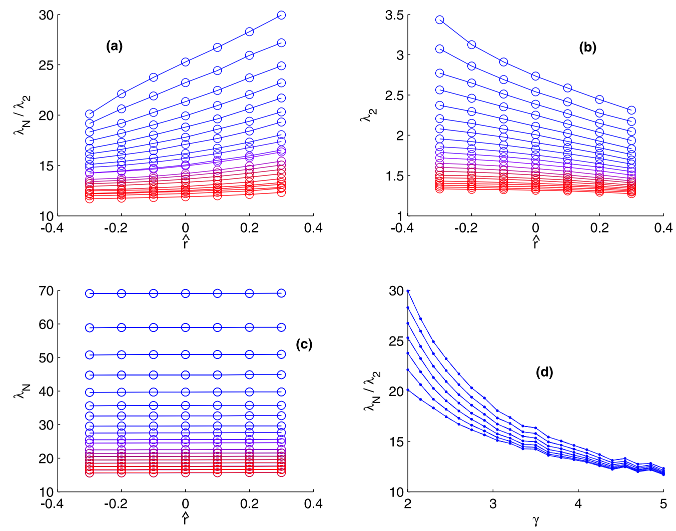

Fig. 1. Synchronizability of degree correlated networks of size 10^3 nodes, $k_{min} = 5$. Behavior of the eigenratio λ_N/λ_2 (a), of the second lowest eigenvalue λ_2 (b) and of the highest one λ_N (c), as functions of the correlation coefficient $\hat{r}$, for γ varying from 2 (blue line) to 5 (red line) in steps of 0.2. (d) The eigenratio as function of γ as varying the correlation coefficient $\hat{r}$ from -0.3 (bottom line) to 0.3 (top line) in steps of 0.1. All the plots are averaged over 10^2 realizations.

B. Effects of correlation on synchronizability

Equipped with a method to generate networks with the same degree distribution but different correlation properties, we can now explore the effects that the presence of correlation has on the network synchronizability. Here, we investigate these effects numerically. Analytical bounds to

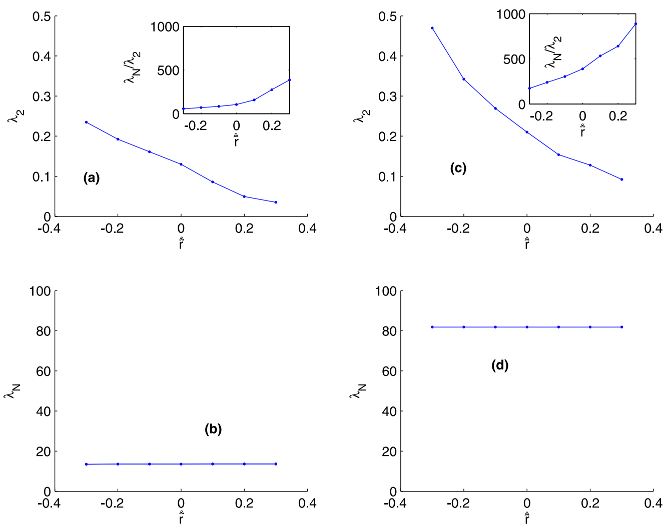

Fig. 2. Left side panels: behavior of the second lowest eigenvalue λ_2 (a) and of the highest one λ_N (b) as varying the degree correlation index $\hat{r}$ in Erdos Renyi random graphs with 10^3 nodes, $2 \cdot 10^3$ edges. In the left side inset is also reported the behavior of the related eigenratio λ_n/λ_2. Right side panels: behavior of the second lowest eigenvalue λ_2 (c), of the highest one λ_N (d) and of the eigenratio (in the inset) as varying the degree correlation index $\hat{r}$ in Barabasi-Albert scale free graphs with 10^3 nodes, $2 \cdot 10^3$ edges.

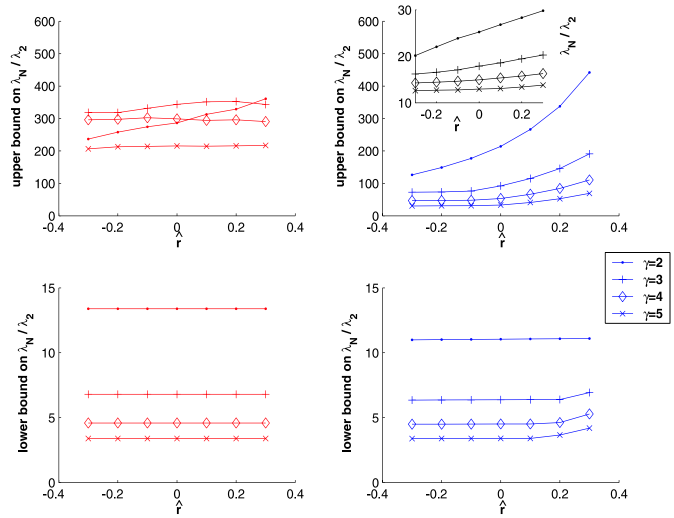

Fig. 3. Plots of the upper and lower bounds on the eigenratio λ_N/λ_2 as functions of $\hat{r}$ for different values of $\gamma = [2, 3, 4, 5]$. The figure shows the comparison between the bounds given by (2) (on the left panels) and those (18) (on the right panels). The inset shows the observed behavior of the eigenratio. $N = 10^3$, $k_{min} = 5$.

explain the numerical observations will be given in the next section.

Fig. 1(a) shows the effects of varying the degree correlation on the Laplacian eigenratio R for different values of the degree distribution exponent γ. For all values of γ, we observe a reduction of R for decreasing values of the correlation. This means that disassortative mixing enhances the network synchronizability. Interestingly, as depicted in Fig. 1(b) and Fig. 1(c), we observe that, under variations of the correlation parameter, the changes in R seem to be due only to variations of λ_2 while λ_N, the largest eigenvalue of the Laplacian, is found to be practically independent from $\hat{r}$. Fig. 1(d) shows the evolution of R under variations of γ for different values of the correlation coefficient r. As discussed in [2], in the case of uncorrelated networks ($\hat{r} = 0$), synchronizability improves for increasing values of γ. We find that, as shown in Fig. 1(d), this is still the case when degree correlation is introduced. A consistent decrease of the values of R for the same value of γ is observed when the network is disassortative, i.e. $\hat{r} < 0$.

So far, we have applied our numerical simulations to scale free networks, reproduced by the algorithm introduced in [15], the so-called *configuration model* (see also [17]). In order to give evidence of the broad scope of the results presented in this paper, in Fig. 2, we have shown the behavior of the highest and the lowest eigenvalue of the Laplacian as varying the correlation index $\hat{r}$ (according to the method proposed above) in $Erd\ddot{o}s - Renyi$ random graphs [18], and in Barabasi-Albert scale free networks (with $\gamma = 3$) [5]. Note that in all the different cases considered, λ_2 is found to decrease as increasing the degree of assortativity in the network, while λ_N is found to be practically independent from it.

In what follows, we propose a way of estimating analytical bounds on the Laplacian eigenvalues which provide a theoretical explanation for the numerical results presented in this section.

III. Analytical Bounds on the Laplacian Eigenvalues

Analytical estimates of the Laplacian eigenvalues have been already used in the literature to explain the effects of the network topological features on its synchronizability properties. In [1] it was shown that the eigenratio $R = \frac{\lambda_N}{\lambda_2}$ between the highest eigenvalue λ_N and the lowest eigenvalue λ_2 of the Laplacian associated to the network structure is an essential measure of the network synchronizability, i.e. the smaller the eigenratio, the larger the interval of the values of the coupling gain σ, for which the stability of the synchronous state is achieved. It is therefore important to characterise how the network topological features affect the Laplacian eigenratio.

For example, the analytical bounds given by (2), were used in [2] to explain the changes observed in the eigenratio R as the parameter γ was varied in a scale-free network topology. We found that, although these bounds should hold for any generic network topology, they seem to be inappropriate to account for the changes in R observed in the network when degree correlation is introduced. In particular, as shown in Fig. 3, the values of the upper bounds on R computed according to (2) give estimates which (i) are further away from the observed values of the eigenratio and more importantly (ii) do not reproduce the behavior of the eigenratio under variation of the network degree correlation.

Therefore, in order to explain what happens physically when changing the correlation and why it affects the measures of networks synchronizability, we shall seek to define new analytical bounds based on the mathematical theory of graph spectra. In particular we will focus on estimating the effects of correlation on the eigenvalue λ_2, the parameter known as *algebraic connectivity of graphs* [4]. In so doing, we will make use of some fundamental results in the field of *algebraic graph theory*. Specifically, we will use the so-called Cheeger inequalities that find application in the solution of isoperimetric problems [19], [20]. Before presenting the derivation of the new bounds on λ_2, we recall briefly in what follows the main aspects of such inequalities.

A. Cheeger Inequalities: an overview

Given a graph, consider a subset of edges which disconnects the graph in two parts, also termed as a *cut*. Isoperimetric problems examine optimal relations between the size of the cut and the size of the separated parts. For a given subset of vertices, say S, we define $h_G(S)$ as the quantity given by:

$$h_G(S) = \frac{\mathcal{D}(S)N}{|S|(N-|S|)}, \tag{8}$$

where $\mathcal{D}(S)$ is the number of edges in the boundary of S and $|S| < \frac{N}{2}$ is the number of vertices in S. The Cheeger constant of a graph is defined as follows:

$$h_G = \min_S h_G(S), \tag{9}$$

Remarkably, it can be shown that the following *Cheeger inequality* holds [19]. Namely, we have:

$$\lambda_2 \leq h_G. \tag{10}$$

Note that finding the subset S such as to achieve the minimization of $h_G(S)$ is known to be an NP-hard problem [21]. We will show below that this problem can be overcome to compute the bounds of interest.

Another interesting inequality in spectral geometry is due to Mohar [20]:

$$\lambda_2 \geq k_{max} - \sqrt{{k_{max}}^2 - {h'_G}^2}, \tag{11}$$

where $h'_G = \min_S \frac{\mathcal{D}(S)}{|S|}$.

We will show next how inequalities (10) and (11) can be used to estimate more accurate upper and lower bounds on λ_2.

B. Estimating bounds on λ_2

We propose to obtain an upper bound on λ_2 by using (10). To overcome the limitations due to the computation of the subset S that minimizes $h_G(S)$, we will follow a stochastic approach in order to estimate $h_G(S)$, starting from the available information we have on the network. We will assume, using similar assumptions to those taken to generate a network with a given degree correlation, that the network presents as noticeable features only the degree distribution and the correlation specified; all other aspects being completely random.

Then, having fixed the degree sequence, we would be able to give a full characterization of a randomly chosen subset S in terms of the number of nodes $\mathcal{N} = |S|$ in it, the number of nodes, say $\mathcal{N}_H$, among them belonging to the group of nodes H with high degree, and the network correlation measure $\hat{r}$. Notice that, under these conditions, the number of nodes in the low degree set, L, is given by $\mathcal{N}_L = \mathcal{N} - \mathcal{N}_H$, that is, we have completely defined the *degree distribution* of S. Moreover it is worth noting that the subset S is not supposed to satisfy any particular condition, not even of being connected.

Let us introduce $x_L = \frac{\mathcal{N}_\mathcal{L}}{n_L}$ and $x_H = \frac{\mathcal{N}_\mathcal{H}}{n_H}$, the fraction of nodes in S drawn from each of the two class. Now, we observe that the number of edges in $\mathcal{D}(S)$ is given by the total number of edges starting from the vertices in S, less the ones, say $\mathcal{I}$, that are contained in S, i.e. connecting endpoints both belonging to S.

Thus we can estimate $\mathcal{I}(\mathcal{S})$ and $\mathcal{D}(S)$ as follows:

$$\begin{aligned}\mathcal{I}(S) &= \mathcal{I}(x_L, x_H, \hat{r}) \\ &= \frac{(k_L+k_H)N}{2}(e_{HH}x_H^2 + 2e_{HL}x_Lx_H + e_{LL}x_L^2),\end{aligned} \tag{12}$$

$$\mathcal{D}(S) = \mathcal{D}(x_L, x_H, \hat{r}) = x_Ln_Lk_L + x_Hn_Hk_H - 2\mathcal{I}(S). \tag{13}$$

Then, $h_G(S)$ can be computed using (8). Note that, using (12) and (13), $h_G(S)$ can be thought of as a function $h_G(x_L, x_H, \hat{r})$. Hence, assigned a certain correlation $\hat{r}$, a numerical optimization technique can be used to solve the problem of finding h_G such that

$$h_G = \min_{x_L, x_H} h_G(x_L, x_H, \hat{r}), \tag{14}$$

under the constraints that $0 \leq x_L, x_H \leq 1$ and $x_Ln_L + x_Hn_H < \frac{N}{2}$.

Therefore, the minimum value h_G, can be found that according to (10) provides an upper bound for λ_2.

Moreover, after some algebraic manipulations, we also find that

$$\frac{\partial h_G(S)}{\partial \hat{r}} \propto \frac{\partial \mathcal{D}(S)}{\partial \hat{r}} = -2q_Lq_H(x_L - x_H)^2 \leq 0 \tag{15}$$

Since, for all x_L and x_H, $\frac{\partial h_G(x_L, x_H, \hat{r})}{\partial \hat{r}} \leq 0$, then we have that $\frac{\partial(\min_{x_L, x_H} h_G(x_L, x_H, \hat{r}))}{\partial \hat{r}} \leq 0$, and thus:

$$\frac{\partial h_G}{\partial \hat{r}} \leq 0. \tag{16}$$

Therefore, we can predict analytically that h_G, which represents an higher bound on λ_2, will decrease as the degree correlation is increased.

Using (11), we can also derive the following relationship for the lower bound on λ_2:

$$\frac{\partial(k_{max} - \sqrt{k_{max}^2 - h'^2_G})}{\partial \hat{r}} = \left(\frac{h'_G}{\sqrt{k_{max}^2 - {h'_G}^2}}\right)\frac{\partial h'_G}{\partial \hat{r}}, \tag{17}$$

where the first term in the product is a quantity intrinsically positive (note also that when making the correlation change, the degree distribution is fixed and thus, k_{max} cannot vary with $\hat{r}$). Then following an approach similar to the one used above to compute the upper bound, it is easy to show that the lower bound in (11) has to decrease with $\hat{r}$. Then since both the upper and the lower bounds have to decrease with $\hat{r}$, also λ_2 is expected to have the same trend.

On the other hand, from graph theoretical results (see for example [21]), it can be shown that the following relationship on λ_N holds: $\frac{N}{N-1}k_{max} \leq \lambda_N \leq 2k_{max}$.

This confirms our numerical finding that λ_N is only dependent on the network maximum degree k_{max}, and thus independent from $\hat{r}$ (see Fig. 1). Therefore we have that, as a consequence, the eigenratio $\frac{\lambda_N}{\lambda_2}$ will increase for higher values of the correlation coefficient. As shown in Fig. 3, this is indeed what is observed with the new bound on λ_2 giving a much better estimate of the behavior of the eigenratio with respect to both changes in the degree distribution and the degree correlation. (In Fig. 3 the bound is computed taking the minimum of $h_G(\hat{r})$ on both x_L and x_H).

Note that these estimates do not take into account higher order effects, such as the clustering and the formation of closed loops within the subset S, that are known to be more frequent in assortative mixed networks [22] [23]. Indeed, considering also these in the computation of the upper and lower bounds (10) (11), would yield as a result a steeper slope of the trend of h_G vs. $\hat{r}$ (and hence to a higher slope of the bounds on $\frac{\lambda_N}{\lambda_2}$ in Fig.3).

Interestingly, fixing x_L and x_H, the slope $\frac{\partial h_G(x_L,x_H,\hat{r})}{\partial \hat{r}}$ is proportional to[1] $q_L q_H$ that analogously to $\frac{\partial \lambda_2}{\partial r}$ is a quantity decreasing with γ.

C. Estimating bounds on $\frac{\lambda_N}{\lambda_2}$

In order to compare directly the bounds (2) with the ones introduced in this paper, we will show how they differently predict the behavior of the eigenratio R.

From the bounds on λ_N given above and the ones on λ_2 in (10) and (11) we easily get the following analytical bounds on the Laplacian eigenratio:

$$\frac{N}{N-1}\frac{k_{max}}{h_G} \leq \frac{\lambda_N}{\lambda_2} \leq \frac{2}{1-\sqrt{1-\frac{h_G'^2}{k_{max}^2}}}. \qquad (18)$$

The comparison between these bounds computed as explained above and those proposed in (2) are shown in Fig. 3. We observe that the upper bounds in (18) provide better estimates of changes in the eigenratio under variations of γ and more importantly $\hat{r}$ while the opposite happens in most cases for the lower bounds. We wish to emphasize that, when compared to the bounds given by (2), the novel bounds computed according to (18) seem to better replicate the observed behavior of the eigenratio under variations of both $\hat{r}$ and γ.

[1]Actually what we observe is that the minimum is always achieved either for $x_L = 0$ and $x_H = 1$ or for $x_L = 1$ and thus $x_H = 0$

IV. Conclusion

In this paper we have studied the effect of degree correlation on the eigenratio parameter defined in [1] as a measure of the synchronizability of a network of coupled nonlinear oscillators.

We have found that disassortative mixing, which is typical of biological and technological networks, plays a positive role in enhancing network synchronizability.

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