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# On the scalable stability of nonsymmetric heterogeneous networks

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*Abstract*— **Underlying interconnection symmetry is often a major assumption for the derivation of scalable decentralized stability certificates in networks i.e. decentralized stability guarantees for an arbitrary interconnection of heterogeneous dynamical systems. Even though such symmetry simplifies significantly the mathematical analysis it is often the case that networks can behave robustly even when deviations from protocol symmetry do occur. We show in this paper how spectral inclusion techniques can be used to certify scalable stability in classes of nonsymmetric networks with potential applications in Internet congestion control and consensus protocols. The certificates derived are decentralized and scale with the degree of nonsymmetry.**

## I. INTRODUCTION

Issues of scalability are very important in the stability analysis of interconnected dynamical systems. This is because there exist many examples of heterogeneous networks such as data networks, flocking phenomena, financial markets where the system we want to analyze is not precisely known. Therefore, it is important to have means of certifying stability which are based only on local rules and these rules are also preserved when the network is modified through the addition/removal of agents. Since we are looking for stability results that hold for an arbitrary network that obeys a certain interconnection protocol, any symmetries present in this protocol will enhance the mathematical derivation of such decentralized stability conditions.

A major example in which scalable robust stability has been extensively studied is that of Internet congestion control protocols. In [1] an optimization based framework is introduced for the analysis of such protocols in arbitrary networks. Global stability is guaranteed by seeing the network as a potential system where decentralized control laws maximize a relaxed aggregate utility. Decantralized local stability conditions in the presence of delays are given in [2], [3], [4]. Along the same lines, scalable control laws are also suggested in [5]. As discussed in [6], this optimization framework imposes a particular symmetric interconnection structure. In fact it turns out that the return ratio of the linearized system can be brought to a form $G(s)A$ where $G(s)$ is a diagonal transfer matrix, $\mathrm{diag}(g_i(s))$, and $A$ is positive definite (or in the presence of delays $A(j\omega)$ is hermitian). The same kind of return ratios appear also in consensus protocols, another important class of networks with applications in UAV formations, flocking phenomena, sensor networks (see e.g. [7], [8], [9]). The structure of the adjacency matrix $A$ allows one derive scalable stability conditions for the case of heterogeneous agent dynamics in analogy with the Internet case. These conditions involve a convexification of the the frequency responses of individual dynamics and can be given a dissipativity interpretation closely related to that in [10] (see [11]).

In practice, however, deviations, from this kind of symmetry in the interconnection do arise. For example, in a data network with variable packet sizes the stability results in [1], [2], [3], [4], [5] do not hold, since aggregate flow at the resources becomes a weighted sum of flows if the congested resource is bandwidth (thus losing the special structure in the return ratio). Similarly, when one considers consensus protocols on a directed graph the adjacency matrix $A$ is no longer symmetric. Consequently, extensions of the stability certificates in non-symmetric cases can be quite significant, or at least one needs to ensure that the designed system is not fragile to an interconnection symmetry assumption.

The major contribution of this paper is that we derive scalable decentralized stability certificates for classes of networks where the interconnection matrix is not necessarily symmetric. The certificates are generalizations of ideas in [3], [6]; they can be seen as perturbations to analogous conditions in the symmetric case and they scale with the degree of nonsymmetry.

The main result presented, which holds for single input single output linear time invariant dynamical systems on bipartite graphs, takes roughly the form that the convex hull of the frequency responses of participating dynamics, each scaled by their in degree and perturbed by a set that depends on the degree of non symmetry, must not encircle the point $-1$. It should be pointed out that converting stability certification to a spectral inclusion problem of a complex matrix by means of frequency response methods, can be very useful in the analysis of networks. Using the numerical range as a tool for spectral inclusion, the internal structure of the system is revealed even in the absence of symmetry. Moreover a number of topological properties of the complex plane with a scalable character (proved in the appendix) can lead to results which are not readily deduced from time domain arguments.

The paper is structured as follows. We give first a general form of the main stability condition and describe in graph theoretic terms the classes of networks for which it can be relevant. This is then applied to models for Internet

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congestion control[1].

## II. Preliminaries

### Notation

$\sigma(M)$ denotes the spectrum of a square matrix $M$, $\rho(M)$ its spectral radius and $|M|$ the elementwise absolute value of the matrix i.e. $|[M_{ij}]| := [|M_{ij}|]$. Given $A,B \in \mathbb{C}^{m\times n}$, $M = \max(|A|,|B|)$ is the elementwise maximum i.e. $M_{ij} = \max(|A_{ij}|,|B_{ij}|)$. $Co(S)$ denotes the convex hull of a set $S$ and $\mathrm{diag}(x_i)$ the matrix with elements $x_1, x_2, \ldots$ on the leading diagonal and zeros elsewhere. $\mathbb{C}_-$ is the open left hand plane and $\mathbb{R}_+$ the open set of positive reals. The Numerical Range or Field of Values of a matrix $M \in \mathbb{C}^{n\times n}$ is the set $N(M) := \{v^*Mv : v \in \mathbb{C}^n, v^*v = 1\}$. The property $\sigma(M) \subset N(M)$ is used in this paper (see e.g. [12] [13] for a more detailed discussion of the properties of the Numerical Range). $\mathscr{H}_\infty$ is the set of proper transfer functions analytic and bounded in $\overline{\mathbb{C}}_+$ . $\mathscr{C}_0$ is the class of functions continuous in $j\mathbb{R} \cup \{\infty\}$ and $\mathscr{A}_0 := \mathscr{H}_\infty \cap \mathscr{C}_0$ .

### A. Graph Theoretic Setting

We consider a directed graph representation of an interconnected system and use the following notation.
$G = (V,E,A)$ is a weighted directed graph where $V = \{v_1,\ldots,v_n\}$ is the set of nodes, $E \subseteq V \times V$ the set of directed edges and $A = [a_{ij}]$ a weighted adjacency matrix. Directed edges are denoted as $e_{ij} = (v_i, v_j)$. Node $v_j$ is defined the head of the edge $e_{ij}$, node $v_i$ the tail. The adjacency matrix $A \in \mathbb{R}^{n\times n}$ satisfies $a_{ji} \neq 0 \Leftrightarrow e_{ij} \in E$. The in-neighbours of a node $v_i$ are defined as $N_i^{in} = \{v_j \in V : (v_j, v_j) \in E\}$ and its in-degree as $|N_i^{in}|$. Similarly the out-neighbours are defined as $N_i^{out} = \{v_j \in V : (v_i, v_j) \in E\}$ and the out-degree as $|N_i^{out}|$. In the digraph representation of the network each dynamical element corresponds to a node of the graph. Furthermore in a network of $n$ dynamic agents, each with scalar input $u_i(t)$, scalar output $y_i(t)$ and transfer function $g_i(s)$, the input and output vectors, $u(t) = [u_1(t),\ldots,u_n(t)]^T$ and $y(t) = [y_1(t),\ldots,y_n(t)]^T$ respectively, satisfy the relation $u(t) = Ay(t)$, where $A$ is the adjacency matrix of the graph. A graph is bipartite if its nodes can be divided into two sets such that nodes from one set are only connected to nodes of the other set.
In a digraph with $a_{ii} = 0\ \forall i$ the graph Laplacian is defined as $L = I - D^{-1}A$, where $D = \mathrm{diag}(d_i)$, $\quad d_i = \sum_{k=1,k\neq i}^{n} a_{ik}$ . Note that if $A$ is a $0-1$ matrix then $d_i$ is the in-degree.

For an interconnected system on a graph as defined above, the return ratio is of the form $G(s)A$, where $G(s) = \mathrm{diag}(g_1(s),\ldots,g_n(s))$, if the interconnection is broken at the output of each of the dynamical systems. In a bipartite graph the adjacency matrix can be chosen to be block antidiagonal by ordering the nodes in an appropriate sequence. If we denote $g_1(s),\ldots,g_n(s)$, $h_1(s),\ldots,h_m(s)$ the transfer functions of the dynamics corresponding to nodes from the two disjoint sets in the bipartition respectively, the return ratio becomes of the form $\mathrm{diag}(g_i(s))Rb^T\mathrm{diag}(h_j(s))Rf$, $\quad Rf, Rb \in \mathbb{R}^{n\times m}$ by breaking the loop at the output of each of the $g_i(s)$.

[1] Consensus protocols are also a potential application of the results presented (see Remark 1); these are not extensively discussed in the paper due to length limitations.

## III. Main Result

The main proposition in the paper is given below.

*Proposition 1:* Given an $m\times n$ transfer matrix $R(s)$ and $Rf(s)$, $Rb(s)$ where

$$Rf_{lr}(s) = \mu_{lr}^f R_{lr}(s), \quad Rb_{lr}(s) = \mu_{lr}^b R_{lr}(s)$$
$$\mu_{lr}^f,\ \mu_{lr}^b \in [0,1] \text{ for all } l,r$$

and also $M(j\omega) = \max(|Rf(j\omega)|,|Rb(j\omega)|)$ satisfies $\rho(M(j\omega)^T M(j\omega)) \leq 1$ for all $\omega \in \overline{\mathbb{R}}_+$, then for $F(s) = \mathrm{diag}(f_1(s),\ldots,f_n(s))$, $f_i(s) \in \mathscr{A}_0 \quad \forall i$, the system with return ratio $L(s) \in \mathscr{A}_0^{n\times n}$ that can be factorized as $L(s) = F(s)Rb(-s)^T Rf(s)$ is stable if

$$-1 \notin Co(\{f_r(j\omega)S_r : \omega \in \overline{\mathbb{R}}_+,\ r = 1,\ldots,n\} \tag{1}$$
$$\text{where} \quad S_r = S(\min_l \mu_{lr}^f, \min_l \mu_{lr}^b) \quad \text{and}$$

$$S(\mu_1,\mu_2) := \left\{ \frac{(v_1+\mu_1 v_2+v_3)(v_1+v_2+\mu_2 v_3)^*}{(|v_1|+|v_2|+|v_3|)^2} : v_1,v_2,v_3 \in \mathbb{C} \right\} \tag{2}$$

*Proof:* This follows directly from Theorem 1 in the appendix. The convex hull in (1) gives a bound for the eigenloci and hence the system is stable according to the multivariable Nyquist criterion. ■

*Remark 1:* The structure of the return ratio has an appealing graph theoretic interpretation in the case $R(s)$ is a constant real matrix. As discussed in the preliminaries, this could correspond to the return ratio of a bipartite weighted directed graph where dynamics are associated with only one of the two disjoint sets of nodes in the bipartition. Such return ratios appear in Internet protocols as it will be illustrated in section IV.
In addition such a return ratio could correspond to any graph where the adjacency matrix can by factorized as $RfRb$. This factorization naturally exists for the Laplacian of a digraph [14] and the Laplacian is an adjacency matrix in the case of consensus protocols.
$R(s)$ is presented in the proposition as a transfer matrix so as to include the case where delays are associated with the edges. The fact that $Rb(-s)$ conveniently appears as a scaled version of $R(-s)$, can occur in cases of constant roundtrip times as in Internet protocols, by factoring out the round trip time in the agent dynamics.

*Remark 2:* In a symmetric network ($\mu_{lr}^f = \mu_{lr}^b = 1 \quad \forall l,r$) $S(1,1) = [0,1]$ and condition (1) above becomes

$$-1 \notin Co(\{f_r(j\omega) : \omega \in \mathbb{R}_+,\ r = 1,\ldots,n\} \cup 0)$$

Hence, set $S_r$ can be seen as a perturbation set that scales with the degree of nonsymmetry. It is important to note that $S_r$ depends for each agent, only on the smallest scaling factors in the corresponding rows of $Rf(s)$ and $Rb(s)$. It is shown in Lemma 3 in the Appendix that $S(\mu_1,l_1) \subset S(\mu_2,l_2)$ for $0 \leq \mu_1 < \mu_2 \leq 1,\ 0 \leq l_1 < l_2 \leq 1$. Therefore as $\mu, l$ tend

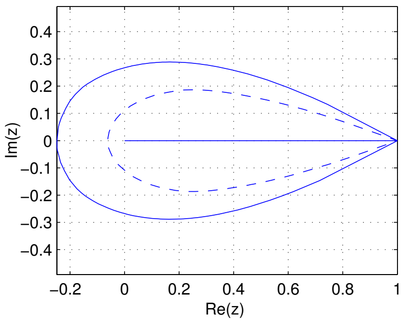


Fig. 1. The boundary of $S(\mu,l)$ for $(\mu,l)=(0,0),(0.5,0.5),(1,1)$. Observe that $S(\mu,l)\to[0,1]$ as $\mu,l\to 1$.

from 0 to 1 $S(0,0)$ 'shrinks' to $[0,1]$. The set $S(\mu,l)$ has a well defined shape as shown in figure 1.

*Remark 3:* The convex hull condition (1) can be given a decentralized interpretation by constraining the perturbed frequency response of each of the agents not to intersect a predefined hyperplane through the point $-1$. Note that they will all be on the same side of the hyperplane since $0\in S_r$. Scalability follows from the fact that a new transfer function $f_i(s)$ will introduce only an additional such condition.
The distance of each perturbed frequency response from the hyperplane gives also a measure of robustness i.e. it is guaranteed that the system will remain stable for an additive perturbation on the agent dynamics with infinity norm smaller than this corresponding distance.

*Remark 4:* The fact that the bound of the eigenloci in (1) always includes zero, implies that the same stability condition holds when the return ratio is being permuted since for matrices $A\in\mathbb{C}^{m\times n}$ and $B\in\mathbb{C}^{n\times m}$, $AB$ and $BA$ have the same non-zero eigenvalues.

*Remark 5:* The spectral bound on $M$ can be obtained by appropriate scaling at the input of the dynamics. One can exploit the fact that for $M\in\mathbb{C}^{n\times n}$ $\rho(M)\leq ||M||_\infty=\max_i\sum_j|M_{ij}|$.

## IV. Internet congestion control

We consider TCP like protocols where users/sources implement the control law based on the aggregate price from the resources/links along the route and the resource prices are generated as functions of the aggregate flow through the resource. Note that the underlying graph is bipartite as users communicate directly only with resources and resources only with users.

The symmetry lies in the fact that the resources do not discriminate between users when producing congestion signals (prices) as a function of the aggregate flow through the resource and equivalently users do not discriminate between resources when determining their data flow as a function of the aggregate prices they receive. Nevertheless in practice such discrimination does occur. Consider, for example, a differentiated services scheme where packet drop probabilities at some routers depend on the priority given to a particular flow. Then the 'price' received from the router will not be the same for all users. In addition, the presence of variable packet sizes can compromise symmetry. In RED, for example, packets are dropped based on the queue size in packets (e.g. WRED the RED implementation on CISCO routers) even though the queue size could grow because the congested resource is bitrate; symmetry is thus lost if RED is activated by some routers. The results presented in this section are an extension of the approach in [6] where a special kind of nonsymetry in the forward path was considered and with a more general nonsymmetric setting as in this paper being conjectured.

We use the notation in [4], [1] and define the following: $x_r$ is the flow rate associated with route $r$. $U_r(x_r)$ is the utility of the user/source on route $r$, which is a continuously differentiable, strictly concave, increasing function of the flow $x_r$ . $T_r=\tau_{lr}+\tau_{rl}$ is the round trip delay of the $r$th route, with $\tau_{rl}$ being the propagation delay from source $r$ to link/resource $l$ and $\tau_{lr}$ the return delay from link $l$ to source $r$.

$$y_l(t)=\sum_{r:r\text{ uses }l}x_r(t-\tau_{rl}) \tag{3}$$

is the aggregate flow through link $l$. $p_l=f_l(y_l)$ is the link price per unit flow, which is a non-negative, strictly increasing function of the aggregate flow through the resource $l$. We assume link prices to be static functions of the flow (this is valid for low length queues, and large capacities) as in [1].

$$q_r(t)=\sum_{l:l\text{ used by }r}p_l(t-\tau_{lr}) \tag{4}$$

is the aggregate price along route $r$. The control law is performed by the users according to

$$\dot{x}_r(t)=k_rx_r(t-T_r)\left(1-\frac{q_r(t)}{U_r'(x_r(t))}\right) \tag{5}$$

Taking Laplace transforms we can write (3) in the symmetric protocol as a vector equation

$$\bar{y}(s)=R(s)\bar{x}(s) \tag{6}$$

$$\text{where}\quad R_{lr}=\begin{cases}e^{-s\tau_{rl}} & \text{if route } r \text{ uses link } l\\ 0 & \text{otherwise}\end{cases}$$

$$\text{and (4) as}\qquad \bar{q}(s)=\text{diag}(e^{-sT_r})R^T(-s)\bar{p}(s) \tag{7}$$

We now consider a non-symmetric protocol where $R(s)$ is replaced by scaled matrices $Rf(s)$ in (6) and $Rb(s)$ in (7), where

$$Rf_{lr}=\mu^f_{lr}R_{lr},\quad Rb_{lr}=\mu^b_{lr}R_{lr}$$
$$\mu^f_{lr},\ \mu^b_{lr}\in[0,1]\text{ for all }l,r \tag{8}$$

For small perturbations about equilibrium flow and prices

$$y(t)=\hat{y}+\delta y(t),\quad q(t)=\hat{q}+\delta q(t)\quad\text{etc.} \tag{9}$$

$$\overline{\delta y}=Rf(s)\overline{\delta x},\quad \overline{\delta q}(s)=\text{diag}(e^{-sT_r})Rb^T(-s)\overline{\delta p}(s) \tag{10}$$

and the equilibrium relations

$$\hat{y}=Rf(0)\hat{x},\quad \hat{q}=Rb^T(0)\hat{p} \tag{11}$$

Linearization of the source law (5) and the static link price gives

$$\overline{\delta x}_r(s)=-k_r\frac{\hat{x}_r}{\hat{q}_r}\frac{1}{s+k_r\alpha_r}\overline{\delta q}_r(s),\quad \alpha_r=-\frac{\hat{x}_r}{\hat{q}_r}U_r''(\hat{x}_r) \tag{12}$$

$$\overline{\delta p}_l(s)=f_l'(\hat{y})\overline{\delta y}_l(s) \tag{13}$$

Breaking the loop at the source leads to the following return ratio

$$G(s) = \text{diag}\left(k_r \frac{\hat{x}_r}{\hat{q}_r} \frac{e^{-sT_r}}{s + k_r\alpha_r}\right) \times Rb^T(-s)\text{diag}(f_l')Rf(s) \tag{14}$$

The following condition for stability folows from Proposi-

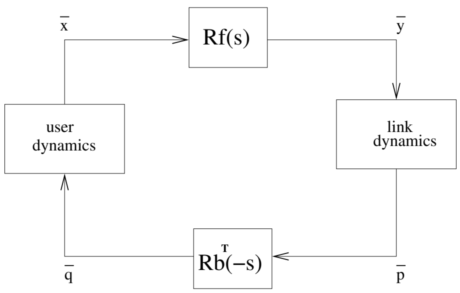


Fig. 2. Block diagram of interconnected network.

tion 1. In the subsequent Corollary a decentralized interpretation is given with local network parameters.

*Proposition 2:* The interconnection described by (3-8),(11) with $n$ users and $k_r \geq 0$ $\forall r$ is locally asymptotically stable around its equilibrium if there exists $B$ such that the inequalities below are satisfied

$$f_l'(\hat{y}_l) \leq \frac{\hat{p}_l}{\bar{\hat{y}}_l} B \quad \forall l \tag{15}$$

$$-1 \notin Co\left\{B\bar{k}_r \frac{e^{-j\omega T_r}}{j\omega + \bar{k}_r\bar{\alpha}_r} S_r : \omega \in \mathbb{R}_+,\ r = 1\ldots,n\right\} \tag{16}$$

$$\text{where} \qquad \bar{k}_r = \frac{\bar{\hat{q}}}{\hat{q}} k_r,\ \bar{\alpha}_r = -\frac{\hat{x}_r}{\bar{\hat{q}}_r} U_r''(\hat{x}_r) > 0,$$

$$\bar{\hat{q}}_r = \sum_l \hat{p}_l \max(\mu_{lr}^f, \mu_{lr}^b),\ \bar{\hat{y}}_l = \sum_r \hat{x}_r \max(\mu_{lr}^f, \mu_{lr}^b),$$

$$S_r = S(\min_l \mu_{lr}^f, \min_l \mu_{lr}^b)$$

where set $S(\mu_1, \mu_2)$ is as defined in (2).

*Remark 6:* In a symmetric network ($\mu_{lr}^f = \mu_{lr}^b = 1 \quad \forall l, r$) $\bar{\hat{y}} = \hat{y}$, $\bar{\hat{q}} = \hat{q}$ and $S(1,1) = [0,1]$. Hence the conditions above converge to those in [4] for symmetric networks i.e.

$$f_l'(\hat{y}_l) \leq \frac{\hat{p}_l}{\hat{y}_l} B \quad \forall l \ , \qquad k_r T_r < \frac{\pi}{2} \frac{1}{B} \quad \forall r \tag{17}$$

*Proof:* [of Proposition 2] Proposition 1 is applied by reducing the return ratio $G(s)$ to the similar form

$$\hat{G}(j\omega) = \text{diag}\left(Bk_r \frac{\bar{\hat{q}}_r}{\hat{q}_r} \frac{e^{-j\omega T_i}}{j\omega + k_r\alpha_r}\right) \hat{Rb}^T(-j\omega)\hat{Rf}(j\omega) \tag{18}$$

$$\hat{Rb}^T(-j\omega) = \text{diag}\left(\sqrt{\frac{\hat{x}_r}{\bar{\hat{q}}_r}}\right) Rb^T(-j\omega)\text{diag}\left(\sqrt{\frac{f_l'}{B}}\right)$$

$$\hat{Rf}(j\omega) = \text{diag}\left(\sqrt{\frac{f_l'}{B}}\right) Rf(j\omega)\text{diag}\left(\sqrt{\frac{\hat{x}_r}{\bar{\hat{q}}_r}}\right)$$

The spectral radius bound is achieved by means of Remark 5 and noting the equilibrium relations (11) (see [4]). ■

In the following Corollary we satisfy condition (16) with a delay dependent bound on the gain $\bar{k}_r$ and a delay independent bound on $B$. Before stating the Corollary we define the following parameter given a set $S_r$ as in Proposition 2

$$\lambda(S_r) := \min_{\gamma > 0}\left[\gamma \text{ s.t. } \Re(z) > -1 \forall\ z \in \left\{\frac{e^{-jx}}{jx + \gamma} S_r : x \in \overline{\mathbb{R}}_+\right\}\right]$$

*Corollary 1:* The interconnection described by (3-8),(11) with $n$ users and $k_r \geq 0$ $\forall r$ is locally asymptotically stable around its equilibrium if there exists $B$ such that

$$f_l'(\hat{y}_l) \leq \frac{\hat{p}_l}{\bar{\hat{y}}_l} B \quad \forall l \tag{19}$$

$$\text{AND} \quad \forall r \quad \text{EITHER} \qquad B < \bar{\alpha}_r \tag{20}$$

$$\text{OR} \quad \text{the inequalities below are satisfied}$$

$$\bar{k}_r T_r < \frac{1}{B}, \quad B < \frac{\bar{\alpha}_r}{\lambda(S_r)} \tag{21}$$

where $\bar{k}_r$, $\bar{\alpha}_r$, $\bar{\hat{q}}$, $\bar{\hat{y}}, S_r$ are as defined in Proposition 2.

*Remark 7:* Since $S(0,0)$ is the worst case perturbation set that includes all other sets $S_r$ (see Lemma 3), (21) is true for all $S_r$ if $B < \frac{\bar{\alpha}_r}{\lambda(S(0,0))}$ .

*Remark 8:* Condition (20) is delay independent, nevertheless, it can be rather conservative.

*Remark 9:* It should be emphasized that the importance of the bounds in Corollary 1 lie in the fact that they are decentralized and hold for arbitrary interconnections like the results for symmetric protocols. Once we deviate from symmetry, the symmetric bounds still hold, i.e. feedback gain depends on delay and the nature of the price functions. An extra delay independent bound is, however, also introduced, that depends on the degree of non symmetry (through $S_r$) as well as the nature of the utility functions of the users (through $\alpha_r$). Notice that this bound only affects users behaving in a non-symmetric way since for symmetric users ($S_r = S(1,1) = [1,0]$) the delay independent condition in (21) becomes redundant.

*Proof:* [of Corollary 1] This follows directly from Proposition 2. Note that

$$\left\{B\bar{k}_r \frac{e^{-j\omega T_r}}{j\omega + \bar{k}_r\bar{\alpha}_r} S_r : \omega \in \overline{\mathbb{R}}_+,\right\} = \left\{\frac{e^{-jxB\bar{k}_rT_r}}{jx + \frac{\bar{\alpha}_r}{B}} S_r : x \in \overline{\mathbb{R}}_+,\right\}$$

If all sets above lie to the right of the point $-1$, so does the convex hull of their union over all $r$. Using Lemma 4 and the fact that $S_r$ is star shaped with respect to 0 (see Lemma 1) it is sufficient to consider the maximum and minimum values of $B\bar{k}_r T_r$ and $\frac{\bar{\alpha}_r}{B}$ respectively. Hence inequalities (21) are sufficient for (16) to be satisfied.
Condition (20) is also sufficient for (16) since the set we want to convexify in this case lies in a unit ball centred at the origin. ■

## V. Conclusions

A way has been suggested for the relaxation of interconnection symmetry assumptions in the derivation of scalable decentralized robust stability certificates for heterogeneous networks. The result presented holds for a class of networks including Internet congestion control models and consensus protocols. The stability conditions can be seen as a perturbation of analogous ones in symmetric networks and scale with the degree of nonsymmetry. The analysis is primarily based on the use of the Numerical range as a spectral inclusion tool that helps to certify stability. It is thus illustrated how such spectral inclusion techniques can reveal the internal structure of the system even without symmetry.

## APPENDIX

Lemmas 1 and 2 are used to prove Theorem 1, the main Theorem in the paper.

*Lemma 1 ($S(\mu,l)$ is star):* Given $\mu, l \in [0,1]$ the set

$$S(\mu,l) := \left\{ \frac{(a+\mu b+c)(a+b+lc)^*}{(|a|+|b|+|c|)^2} : a,b,c \in \mathbb{C} \right\} \quad (22)$$

is star shaped w.r.t. 0 i.e. if $v \in S(\mu,l)$ then $kv \in S(\mu,l)$ for any $k \in [0,1]$

*Proof:* Let

$$f(a,b,c) = \frac{(a+\mu b+c)(a+b+lc)^*}{(|a|+|b|+|c|)^2}$$

choose $b_1, c_1$ s.t.

$$b_1 + lc_1 = \mu b_1 + c_1 \Longleftrightarrow (1-\mu)b_1 = (1-l)c_1$$

$$\text{and let} \qquad \bar{a} = a - b_1 - lc_1 = a - \mu b_1 - c_1$$
$$\bar{b} = b + b_1, \quad \bar{c} = c + c_1$$

$$\text{Note that} \qquad a+\mu b+c = \bar{a}+\mu\bar{b}+\bar{c}$$
$$a+b+lc = \bar{a}+\bar{b}+l\bar{c}$$

$$\text{Hence} \quad kf(a,b,c) = f(\bar{a},\bar{b},\bar{c}), \text{ where } k = \left(\frac{|a|+|b|+|c|}{|\bar{a}|+|\bar{b}|+|\bar{c}|}\right)^2$$

By choosing $|b_1|, |c_1|$ sufficiently large $k$ can be chosen by continuity to be any number in $(0,1]$. The fact $0 \in S$ is obvious e.g. $a=-c, b=0$. ■

*Lemma 2 (Characterizing $S(\mu,l)$ ):* Given $\mu_i, l_i \in [0,1]$ and $\mu_i, l_i$ are not all equal to zero

$$\left\{ \frac{(\sum_i \mu_i v_i)(\sum_i l_i v_i)^*}{(\sum_i \max(\mu_i,l_i)|v_i|)^2} : v_i \in \mathbb{C} \right\} \subseteq S(\mu_{min}, l_{min})$$

where $\mu_{min} = \min_i \mu_i$, $l_{min} = \min_i l_i$ and $S(\mu_{min}, l_{min})$ is as defined in (22).

*Proof:* Let

$$A = \left\{ \frac{(\sum_i \mu_i v_i)(\sum_i l_i v_i)^*}{(\sum_i \max(\mu_i,l_i)|v_i|)^2} : v_i \in \mathbb{C} \right\}$$

$$\text{and} \quad B = \left\{ \frac{(a+\mu_{min} b+c)(a+b+l_{min}c)^*}{(|a|+|b|+|c|)^2} : a,b,c \in \mathbb{C} \right\}$$

$$\text{if} \quad \mu_i < l_i \quad \text{let} \quad \mu_i v_i = k_i \mu_{min} v_i - k_i \mu_{min} v_i + \mu_i v_i$$

$$\text{s.t.} \quad k_i + \mu_i - k_i \mu_{min} = l_i \quad \text{i.e.} \quad k_i = \frac{l_i - \mu_i}{1-\mu_{min}} \quad (23)$$

$$\text{Similarly, if} \quad l_i < \mu_i \quad \text{let} \quad l_i v_i = l_{min} k_i v_i - l_{min} k_i v_i + l_i v_i$$

$$\text{s.t.} \quad k_i - l_{min} k_i + l_i = \mu_i \quad \text{i.e.} \quad k_i = \frac{\mu_i - l_i}{1-l_{min}}$$

$$\text{So} \quad \sum_i \mu_i v_i = \underbrace{\sum_{i:\mu_i=l_i} \mu_i v_i}_{f} + \underbrace{\sum_{i:\mu_i>l_i} \mu_i v_i}_{d+e} + \underbrace{\sum_{i:\mu_i<l_i} \mu_i v_i - k_i \mu_{min} v_i}_{b} + \underbrace{\sum_{i:\mu_i<l_i} \mu_{min} k_i v_i}_{\mu_{min} c}$$

$$\sum_i l_i v_i = \underbrace{\sum_{i:l_i=\mu_i} l_i v_i}_{f} + \underbrace{\sum_{i:l_i>\mu_i} l_i v_i}_{b+c} + \underbrace{\sum_{i:l_i<\mu_i} l_i v_i - k_i l_{min} v_i}_{d} + \underbrace{\sum_{i:l_i<\mu_i} l_{min} k_i v_i}_{l_{min} e}$$

$$\sum_i \mu_i v_i = \underbrace{f+d+b}_{k} + e + \mu_{min} c = k + e + \mu_{min} c \quad (24)$$

$$\sum_i l_i v_i = \underbrace{f+d+b}_{k} + c + l_{min} e = k + c + l_{min} e \quad (25)$$

$$\begin{aligned} \text{Also} \quad & |k|+|e|+|c| \le |f|+|d|+|b|+|e|+|c| \\ &= \sum_{i:\mu_i \ge l_i} \mu_i |v_i| + \sum_{i:\mu_i<l_i} (|\mu_i - k_i \mu_{min}||v_i| + |k_i||v_i|) \\ &\le \sum_{i:\mu_i \ge l_i} \mu_i |v_i| + \sum_{i:\mu_i<l_i} l_i |v_i| \le \sum_i \max(\mu_i, l_i)|v_i| \quad (26) \end{aligned}$$

The second inequality is true because

$$|\mu_i - k_i \mu_{min}| + |k_i| \le l_i - k_i + k_i = l_i \quad \text{using (23)}$$

Hence from (24-26) we deduce that for each $x \in A$ there exists $y \in B$ s.t. $x = \eta y$ where

$$\eta = \frac{|k|+|e|+|c|}{\sum \max(\mu_i, l_i)|v_i|} \le 1$$

Therefore $A \subseteq B$ using the fact that the set $B$ is star shaped w.r.t. 0 from Lemma 1. ■

*Theorem 1:* Given $R \in \mathbb{C}^{m \times n}$ and $Rf$, $Rb$ where

$$Rf_{lr} = \mu^f_{lr} R_{lr}, \quad Rb_{lr} = \mu^b_{lr} R_{lr}, \quad \mu^f_{lr}, \ \mu^b_{lr} \in [0,1] \text{ for all } l,r$$

and also $M = \max(|Rf|,|Rb|)$ satisfies $\rho(M^T M) \le 1$, then for $F = \mathrm{diag}(f_1,\ldots,f_n)$, $f_i \in \mathbb{C}$, we can bound the spectrum of $FRb^*Rf$ as follows:

$$\sigma(FRb^*Rf) \subset Co(\{f_r S_r : r = 1,\ldots,n\})$$

where $S_r = S(\min_l \mu^f_{lr}, \min_l \mu^b_{lr})$ and $S(\mu_1,\mu_2)$ is as in (22).

*Proof:* $\sigma(FRb^*Rf) = \sigma(RfFRb^*)$ if we ignore zero eigenvalues. This is not a problem since the bounding region in Theorem 1 always includes zero (0 is always in $S_r$).

$$\rho(MM^T) \le 1 \Rightarrow \quad v^* M M^T v \le 1 \ \forall v \in \mathbb{C}^m \text{ s.t. } v^* v = 1$$

$$\text{since} \quad \rho(MM^T) = \|M\|_2^2 = \sup_{v \in \mathbb{C}^m, v \ne 0} \frac{\|M^T v\|_2^2}{\|v\|_2^2}$$

Expanding $v^* M M^T v$ we get

$$\sum_j (|v_1 M_{1j} + v_2 M_{2j} + \ldots|)^2 \le 1 \ \forall v \in \mathbb{C}^m \text{ s.t. } v^* v = 1$$

And since this is true for all such $v$

$$\sum_j (|v_1 M_{1j}| + |v_2 M_{2j}| + \ldots)^2 \le 1 \ \forall v \in \mathbb{C}^m \text{ s.t. } v^* v = 1 \quad (27)$$

We then bound the spectrum with the field of values of the corresponding matrix.

$$\sigma(RfFRb^*) \subset N(RfFRb^*)$$
$$:= \{v^*RfFRb^*v : v \in \mathbb{C}^m \;\; v^*v = 1\} \tag{28}$$

Now
$$v^*RfFRb^*v = \sum_{k=1}^{n} f_k \left(\sum_{i=1}^{m} v_i^* R f_{ik}\right)\left(\sum_{i=1}^{m} v_i R b_{ik}^*\right)$$
$$= \sum_{k=1}^{n} f_k \left(\sum_i \max(\mu_{ik}^f, \mu_{ik}^b)|v_i R_{ik}|\right)^2 \cdot \frac{\left(\sum_{i=1}^m v_i^* \mu_{ik}^f R_{ik}\right)\left(\sum_{i=1}^m v_i \mu_{ik}^b R_{ik}^*\right)}{\left(\sum_i \max(\mu_{ik}^f, \mu_{ik}^b)|v_i R_{ik}|\right)^2}$$
$$\in \sum_{k=1}^{n} \left(\sum_i \max(\mu_{ik}^f, \mu_{ik}^b)|v_i R_{ik}|\right)^2 f_k S_k \tag{29}$$
$$= \sum_{k=1}^{n} \left(\sum_i |v_i M_{ik}|\right)^2 f_k S_k$$
$$\subset \left(\sum_{k=1}^{n} \left(\sum_i |v_i M_{ik}|\right)^2\right) Co(\{f_k S_k : k = 1, \ldots, n\}$$
$$\subset Co(\{f_k S_k : k = 1, \ldots, n\} \tag{30}$$

The inclusion in (29) follows from Lemma 2 and inclusion (30) follows from (27). ∎

*Lemma 3 ($S(\mu,l)$ scales with nonsymmetry):* The set $S(\mu,l)$ as defined in (22) satisfies

$$S(\mu_2,l_2) \subset S(\mu_1,l_1) \quad \text{if} \quad 1 \geq \mu_2 > \mu_1 \geq 0,\; 1 \geq l_2 > l_1 \geq 0$$

*Proof:* Let $v_1, v_2, v_3 \in \mathbb{C}$. Note

$$\mu_2 v_2 = (\mu_2 - k_\mu \mu_1)v_2 + \mu_1 k_\mu v_2$$

and choose $k_\mu$ s.t.

$$\mu_2 - k_\mu \mu_1 + k_\mu = 1 \quad \text{i.e.} \quad k_\mu = \frac{1-\mu_2}{1-\mu_1}, \quad 0 \leq k_\mu < 1 \tag{31}$$

Similarly $\quad l_2 v_3 = (l_2 - k_l l_1)v_3 + l_1 k_l v_3 \quad$ where $\quad k_l = \dfrac{1-l_2}{1-l_1}$

So
$$v_1 + \mu_2 v_2 + v_3 = v_1 + (\mu_2 - k_\mu \mu_1)v_2 + (l_2 - k_l l_1)v_3 + \mu_1 k_\mu v_2 + k_l v_3 \tag{32}$$
$$v_1 + v_2 + l_2 v_3 = \underbrace{v_1 + (\mu_2 - k_\mu \mu_1)v_2 + (l_2 - k_l l_1)v_3}_{\bar{v}_1} + \underbrace{k_\mu v_2}_{\bar{v}_2} + l_1 \underbrace{k_l v_3}_{\bar{v}_3} \tag{33}$$

$$|\bar{v}_1| + |\bar{v}_2| + |\bar{v}_3| =$$
$$|v_1 + (\mu_2 - k_\mu \mu_1)v_2 + (l_2 - k_l l_1 v_3)| + |k_\mu v_2| + |k_l v_3|$$
$$\leq |v_1| + (|\mu_2 - k_\mu \mu_1| + |k_\mu|)|v_2| + (|l_2 - k_l l_1 v_3| + |k_l|)|v_3|$$

From (31) $\mu_2 - k_\mu \mu_1 = 1 - k_\mu$ and $0 < 1 - k_\mu \leq 1$ (similarly for $k_l$). So all real parameter quantities in the absolute values in the last expression are positive and hence

$$|\bar{v}_1| + |\bar{v}_2| + |\bar{v}_3| \leq |v_1| + |v_2| + |v_3| \tag{34}$$

Since $v_i$ are arbitrary for some choice of $v_i$ the inequality above is strict. Now for $v \in \mathbb{C}^3$ and $\mu, l \in [0,1]$ let

$$f(v,\mu,l) := \frac{(v_1 + \mu v_2 + v_3)(v_1 + v_2 + l v_3)^*}{(|v_1)| + |v_2| + |v_3|)^2}$$

From (32),(33),(34) for every $v \in \mathbb{C}^3$ $\exists \bar{v} \in \mathbb{C}^3$ s.t.

$$f(v,\mu_2,l_2) = \underbrace{\left(\frac{\sum|\bar{v}_i|}{\sum|v_i|}\right)^2}_{\eta} f(\bar{v},\mu_1,l_1)$$

where $\eta \leq 1$ and strictly less than 1 for some $v_i$. Hence $S(\mu_2,l_2) \subset S(\mu_1,l_1)$ since $S(\mu,l)$ is star from Lemma 1. ∎

*Lemma 4:* Let

$$P(n,\gamma) := \left\{\frac{e^{-jxn}}{jx+\gamma} l_1 : x \in \overline{\mathbb{R}}_+\right\} \quad \text{for some } n,\gamma \in \overline{\mathbb{R}}_+$$

and set $l_1$ being the line segment $l_1 = [0,1]$, then

$$P(n_1,\gamma) \subseteq P(n_2,\gamma) \quad \text{for} \quad n_1 \leq n_2 \tag{35}$$
$$P(n,\gamma_1) \subseteq P(n,\gamma_2) \quad \text{for} \quad \gamma_2 \leq \gamma_1 \tag{36}$$

*Proof:* Let $a \in P(n_2,\gamma)$, $b \in P(n_2,\gamma)$, $n_1 \leq n_2$ such that $a,b$ have the same phase. Then parameter $x$ for $a$ is greater than that for $b$ hence $|a| < |b|$. Similarly for case (36) if $a \in P(n,\gamma_1)$, $b \in P(n,\gamma_2)$, $\gamma_2 \leq \gamma_1$ such that $a,b$ have the same phase then $|a| < |b|$ . ∎

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