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An Intrinsic Behavioural Approach to the Gap Metric

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***Abstract*— An intrinsic trajectory level approach without any recourse to an algebraic structure of a representation is utilized to develop a behavioural approach to robust stability. In particular it is shown how the controllable behaviour can be constructed at the trajectory level via Zorn's Lemma, and this is utilized to study the controllable-autonomous decomposition. The gap distance is generalised to the behavioural setting via a trajectory level definition; and a basic robust stability theorem is established for linear shift invariant behaviours.**

1. INTRODUCTION

We begin by observing that the graph topology with its various metrizations plays a fundamental role in the theory of robust stability for classical LTI systems([1], [2], [15]. The contribution of this note is to develop the basic theory of robust stability involving the gap-distance directly from a behavioural perspective, observing that recent approaches to generalisations of the gap metric [2] have been purely trajectory based and hence are easily amenable to such a approach. There has been previous interest in developing behavioural notions of the gap metric, see e.g. [6] for an example.

From a behavioural point of view ([4], [9], [10], [11]), the approach is especially fundamental. Much has been made of the intrinsic nature of behavioural definitions and the need for 'representation free' approaches. In this note, we do not recourse to representations at all, indeed all proofs are at the intrinsic trajectory level. Our basic robust stability theorem should provide a consistent basis for the robustness interpretation of the behavioural H^∞ results in [12], [7].

In relation to the classical approaches, we remark that the standard H^2 gap is a metric on transfer functions, and does not directly apply to systems which either have non-zero initial conditions or which are not minimal (i.e. have non-controllable modes). The ν-gap ([8]) metric also induces the graph topology on the transfer functions, and can handle non-zero initial conditions at zero by its intrinsic definition on the doubly infinite time-axis. However, the ν-gap is also only directly applicable to controllable systems. By defining systems to be limits of Cauchy sequences in the graph topology [8], the standard gap approaches can also be extended to non-minimal cases; a contribution of this paper from a classical perspective is to provide an alternate and slightly more general approach to these cases.

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We observe also that within the classical framework there has been a move towards representation free approaches to the gap, e.g. especially for approaches to nonlinear systems [2]. The behavioural approach considered here is one natural extension of this viewpoint.

2. BEHAVIOURAL DEFINITIONS

Let $\mathcal{T}$ denote the time set, taken throughout to be either $\mathbb{Z}$ or $\mathbb{R}$, and let $\mathcal{T}_+ = \mathbb{N}$ if $\mathcal{T} = \mathbb{Z}$ and $\mathcal{T}_+ = \mathbb{R}_+$ if $\mathcal{T} = \mathbb{R}$. For $n \geq 1$, an n-valued behaviour $\mathcal{B}$ is a subset of the set of all maps $\mathcal{T} \mapsto \mathbb{R}^n$, i.e. $\mathcal{B} \subset \{w : \mathcal{T} \to R^n\}$. The shift operator σ_t, $t \in \mathcal{T}$ is defined: $\sigma_t w(\cdot) = w(\cdot + t)$.

Definition 2.1: Let $\mathcal{B}$ be a behaviour. Then:

1) $\mathcal{B}$ is said to be linear if $\mathcal{B}$ is a vector space.
2) $\mathcal{B}$ is said to be shift invariant (time invariant) if $w \in \mathcal{B}$ implies $\sigma_t w \in B$ for all $t \in \mathcal{T}$.

Smooth differential behaviours are linear, shift invariant, continuous-time behaviours which can be expressed as the kernel of a differential operator, ie. those for which there exists a polynomial valued matrix R s.t. that

$$\mathcal{B} = \left\{ w \in C^\infty \mid R\left(\frac{d}{dt}\right) w = 0 \right\}. \tag{2.1}$$

Observe that in this note we will be interested in non-differential/difference behaviours, for example, systems incorporating a time delay.

Definition 2.2: A behaviour $\mathcal{B}$ is said to have memory $l \geq 0$ if for any $w_1, w_2 \in \mathcal{B}$, $w_1|_{[0,l]} = w_2|_{[0,l]}$, the trajectory

$$w_3(t) = \begin{cases} w_1(t) & \text{if } t \leq 0, \\ w_2(t) & \text{if } t \geq 0, \end{cases} \tag{2.2}$$

also lies in $\mathcal{B}$.

It is easy to show that

Lemma 2.3: If $\mathcal{B}$ is shift invariant, then $\mathcal{B}$ has memory $l \geq 0$ if and only if for any $w_1, w_2 \in \mathcal{B}$ with $w_1|_{[a,a+l]} = w_2|_{[a,a+l]}$ and $a \in \mathcal{T}$, the trajectory

$$w_3(t) = \begin{cases} w_1(t) & \text{if } t \leq a, \\ w_2(t) & \text{if } t \geq a, \end{cases}$$

also lies in $\mathcal{B}$.

If a behaviour has memory $0 \leq l < \infty$ it is said to have finite memory, if $l = 0$ then it is memoryless. Note that a non-memoryless continuous time differential behaviour has finite memory, and $l > 0$ can be taken to be arbitrarily small; a discrete time behaviour also has finite memory, and here $l \geq 0$ depends on the system order. The minimal memory $l_0 \geq 0$ of a behaviour $\mathcal{B}$ is the largest number s.t. $\mathcal{B}$ has memory l for all $l > l_0$. Note that the minimum is not necessarily attained.

The standard definition of autonomy is that behaviour $\mathcal{B}$ is said to be autonomous if for any $w_1, w_2 \in \mathcal{B}$, $w_1|_{(-\infty,0]} = w_2|_{(-\infty,0]}$ implies $w_1 = w_2$. We relax this definition as follows:

Definition 2.4: A behaviour $\mathcal{B}$ with minimal memory $l_0 \geq 0$ is said to be autonomous if for any $w_1, w_2 \in \mathcal{B}$, and any interval V of length greater than l_0, $w_1|_V = w_2|_V$ implies $w_1|_\mathcal{T} = w_2|_\mathcal{T}$.

Non-autonomy of a behaviour with finite memory is thus just the existence of a trajectory in the behaviour whose support has complement containing an interval of length greater than l_0, eg. a compactly supported trajectory.

The behavioural notion of controllability is defined in [4] as follows:

Definition 2.5: A behaviour $\mathcal{B}$ is said to be controllable if and only if given $w_1, w_2 \in \mathcal{B}$, there exist $w_3 \in \mathcal{B}$ and $\tau \in \mathcal{T}_+$ such that

$$w_3(t) = \begin{cases} w_1(t) & \text{if } t \leq 0, \\ w_2(t-\tau) & \text{if } t \geq \tau. \end{cases} \tag{2.3}$$

This definition requires that the patch function w_3 remains in $\mathcal{B}$ with a time delay which is hard to deal technically in some cases including generalisation into multi-dimensional systems. So we introduce the following definition.

Definition 2.6: Given a behaviour $\mathcal{B}$, a sub-behaviour $\mathcal{B}^* \subset \mathcal{B}$ is said to be $\mathcal{B}$-controllable or controllable as abbreviation if and only if given $w_1, w_2 \in \mathcal{B}^*$ and $s \in \mathcal{T}$ there exist $w_3 \in \mathcal{B}$ and $\tau \in \mathcal{T}_+$ such that

$$w_3(t) = \begin{cases} w_1(t) & \text{if } t \leq s, \\ w_2(t) & \text{if } t \geq s+\tau. \end{cases} \tag{2.4}$$

We remark that if $\mathcal{B}* = \mathcal{B}$, then the $\mathcal{B}$-controllability of $\mathcal{B}^*$ is the same as defined by Definition 2.5 except for the shift. So controllability implies $\mathcal{B}$-controllability. But the following example shows that the converse is not true.

Example 2.7: Let $\mathcal{B} = C^\infty(\mathbb{R}, \mathbb{R})$ and $\mathcal{B}^* = \{c : c \in \mathbb{R}\}$. Then both $\mathcal{B}$ and $\mathcal{B}^*$ are linear shift invariant behaviours with finite memory and $\mathcal{B}^* \subset \mathcal{B}$. It is straightforward to check that $\mathcal{B}^*$ is $\mathcal{B}$-controllable, but not $\mathcal{B}^*$-controllable nor controllable in the sense of Definition 2.5 even without the time delay.

For shift invariant behaviours, we have

Lemma 2.8: Suppose $\mathcal{B}, \mathcal{B}^*$ are both shift invariant behaviours and $\mathcal{B}^* \subset \mathcal{B}$. $\mathcal{B}^*$ is $\mathcal{B}$-controllable if and only if given any $w_1, w_1 \in \mathcal{B}^*$, there exists $w_3 \in \mathcal{B}, \tau \in \mathcal{T}_+$ s.t.

$$w_3(t) = \begin{cases} w_1(t) & \text{if } t \leq 0, \\ w_2(t) & \text{if } t \geq \tau. \end{cases} \tag{2.5}$$

We now consider the properties of controllable behaviour. All the conclusions in this section hold for both $\mathcal{B}$-controllability and the controllability of Polderman and Willems' defined by Definition 2.5 although we present them for $\mathcal{B}$-controllability only. So in occasions, we omit the prefix " $\mathcal{B}$-".

Corresponding to the notion of controllability distinguished controllable sub-behaviours can be defined.

Lemma 2.9: Suppose $\mathcal{B}$ is a behaviour. Then there exists at least one maximal $\mathcal{B}$-controllable sub-behaviour.

Proof: Set inclusion defines an order in the set of all $\mathcal{B}$-controllable sub-behaviours. For any chain of $\mathcal{B}$-controllable sub-behaviours: $\mathcal{B}_\alpha \subset \mathcal{B}_\beta \subset \cdots \subset \mathcal{B}_\gamma \subset \ldots$ with $\alpha, \beta, \gamma \dots \in \Gamma$ and Γ the index set, it has an upper bound: $\mathcal{B}^* \subset \cup_{\beta \in \Gamma} \mathcal{B}_\beta$, where $\mathcal{B}^* \subset \mathcal{B}$ is $\mathcal{B}$-controllable since given any $w_1, w_2 \in \mathcal{B}^*$, we have $w_1 \in B_\alpha$, $w_2 \in \mathcal{B}_\beta$ for some $\alpha, \beta \in \Gamma$, hence $w_1, w_2 \in \mathcal{B}_\gamma$, $\gamma = \max\{\alpha, \beta\}$, and by the controllability of $\mathcal{B}_\gamma$ it follows that there exists $w_3 \in \mathcal{B}$ satisfying equation (2.4), thus the controllability of $\mathcal{B}^*$ follows. Zorn's lemma then gives the existence of a maximal sub-behaviour as required. ■

Note that this set-theoretic construction is extremely general: we do not require any linearity, memory or differential/difference structure on $\mathcal{B}$. In general, maximal controllable sub-behaviours are not unique. However, if the behaviour $\mathcal{B}$ is linear, then there exists a unique maximal controllable linear sub-behaviour, which we denote by $\mathcal{B}_{\text{cont}}$.

Lemma 2.10: Suppose $\mathcal{B}$ is a linear behaviour. Then there exists a unique maximal linear $\mathcal{B}$-controllable sub-behaviour $\mathcal{B}_{\text{cont}}$.

Proof: We consider the set of all linear $\mathcal{B}$-controllable sub-behaviours. With the relation induced by subset inclusion, this set is partially ordered and a maximal sub-behaviour $\mathcal{B}_{\text{cont}}$ exists which is also linear.

To show the uniqueness, let $\mathcal{B}_1$ be another non-zero linear maximal $\mathcal{B}$-controllable sub-behaviour and let $\mathcal{B}_2 = \text{span}(\mathcal{B}_{\text{cont}}, \mathcal{B}_1)$, the linear span of $\mathcal{B}_{\text{cont}}$ and $\mathcal{B}_1$. For any $w_1, w_2 \in \mathcal{B}_2$, without loss of generality, we may suppose that $w_i = \alpha_i x_i + \beta_i y_i$ with $\alpha_i, \beta_i \in \mathbb{R}, x_i \in \mathcal{B}_{\text{cont}}, y_i \in \mathcal{B}_1$ and $i = 1, 2$. Since $0 \in \mathcal{B}_{\text{cont}} \cap \mathcal{B}_1$, by the definition of controllability, for all $s \in \mathcal{T}$, there exist $\tau_1, \tau_2 > 0$ and $z_1, v_1 \in \mathcal{B}$ such that $z_1|_{(-\infty,s]} = x_1|_{(-\infty,s]}, z_1|_{[s+\tau_1,\infty)} = 0|_{[s+\tau_1,\infty)}, v_1|_{(-\infty,s]} = y_1|_{(-\infty,s]}, v_1|_{[s+\tau_2,\infty)} = 0|_{[s+\tau_2,\infty)}$. Let $\tau_3 = \max\{\tau_1, \tau_2\}$ and $w_3 = \alpha_1 z_1 + \beta_1 v_1 \in \mathcal{B}$. Then we have $w_3|_{(-\infty,s]} = \alpha_1 x_1|_{(-\infty,s]} + \beta_1 y_1|_{(-\infty,s]} = w_1|_{(-\infty,s]}$ and $w_3|_{[s+\tau_3,\infty)} = 0|_{[s+\tau_3,\infty)}$. This shows that w_1 is switched to 0 in $\mathcal{B}$. Similarly we can prove that there exists $\tau_4 > 0, w_4 \in \mathcal{B}$ such that $w_4|_{(-\infty,s]} = 0|_{(-\infty,s]}$ and $w_4|_{[s+\tau_4,\infty)} = w_2|_{[s+\tau_4,\infty)}$. Write $\tau_5 = \max\{\tau_3, \tau_4\}, w_5 = w_3 + w_4$. Then we see that $w_5 \in \mathcal{B}, w_5|_{(-\infty,s]} = w_1|_{(-\infty,s]}$ and $w_5|_{[s+\tau_5,\infty)} = w_2|_{[s+\tau_5,\infty)}$. This shows that $\mathcal{B}_2$ is $\mathcal{B}$-controllable. Since $\mathcal{B}_{\text{cont}} \subset \mathcal{B}_2$ and $\mathcal{B}_{\text{cont}} \neq \mathcal{B}_2$, it contradicts with the maximality of $\mathcal{B}_{\text{cont}}$. Hence $\mathcal{B}_{\text{cont}}$ is unique. ■

The above proof also shows that the span of any two linear controllable sub-behaviours is controllable and, therefore, so is the span of all linear controllable sub-behaviours. Hence

$$\mathcal{B}_{\text{cont}} = \text{span}\{B \subset \mathcal{B} : B \text{ is linear and controllable}\}.$$

To the best knowledge of the authors, this direct set-theoretic construction of $\mathcal{B}_{\text{cont}}$ does not appear in the literature. Within the behavioural literature, the controllable sub-behaviour is typically constructed algebraically given the equations governing the behaviour, and it is shown via the duality between the behaviour and the algebraic structure that the controllable sub-behaviour is the 'largest' such subset. It is noteworthy to observe that in some settings (e.g. one dimensional differential systems), the existence of a suitable maximal algebraic object appears constructively.

Next we show that shift invariance is preserved for unique maximal controllable sub-behaviours elements $\mathcal{B}^* \subset \mathcal{B}$. In particular, in the shift invariant linear setting, $\mathcal{B}_{\text{cont}}$ is linear and shift invariant.

Lemma 2.11: Suppose $\mathcal{B}$ is shift invariant and has a unique maximal $\mathcal{B}$-controllable sub-behaviour $\mathcal{B}^*$. Then $\mathcal{B}^*$ is shift invariant.

Proof: Let $r, s \in \mathcal{T}, \sigma_r w_1, \sigma_r w_2 \in \sigma_r \mathcal{B}^*$ with $w_1, w_2 \in \mathcal{B}^*$. Then there exist $w_3 \in \mathcal{B}^*$ and $\tau > 0$ such that $w_3(t) = w_1(t)$ for $t \le s + r$ and $w_3(t) = w_2(t)$ for $t \ge s + r + \tau$. Hence

$$\sigma_r w_3(t) = \begin{cases} \sigma_r w_1(t), & \text{if } t \le s \\ \sigma_r w_2(t), & \text{if } t \ge s + \tau. \end{cases} \quad \text{for any } r \in \mathcal{T}.$$

Since $\sigma_r w_3 \in \sigma_r \mathcal{B}^*$, we see $\sigma_r \mathcal{B}^*$ is $\mathcal{B}$-controllable and hence $\sigma_r \mathcal{B}^* \subset \mathcal{B}^*$ as $\mathcal{B}^*$ is the unique maximal $\mathcal{B}$-controllable sub-behaviour.

Similarly, $\mathcal{B}^* = \sigma_{-r}\sigma_r \mathcal{B}^* \subset \sigma_r \mathcal{B}^*$. This completes the proof. ■

Corollary 2.12: Suppose $\mathcal{B}$ is linear and shift invariant. Then, $\mathcal{B}_{\text{cont}}$ is linear and shift invariant.

We conclude this section by showing that $\mathcal{B}$-controllable linear sub-behaviours inherit memory properties from the original behaviour $\mathcal{B}$:

Lemma 2.13: Let $\mathcal{B}$ be a linear shift invariant behaviour with finite memory $l \ge 0$. Then $\mathcal{B}_{\text{cont}}$ has memory $l \ge 0$.

Proof: First of all, we need a new notion: a sub-behaviour $\mathcal{B}^*$ is 0-$\mathcal{B}$-controllable if given any $w_1, w_2 \in \mathcal{B}^*$, there exist $w_3 \in \mathcal{B}$ and $\tau \in \mathcal{T}_+$ satisfying 2.5. Using the same procedure as used in Lemma 2.10, we can see that a maximal linear 0-$\mathcal{B}$-controllable sub-behaviour of $\mathcal{B}$ exists which is denoted by $\mathcal{B}^0_{\text{cont}}$. Using the same procedure as used in Lemma 2.11, we can see that $\mathcal{B}^0_{\text{cont}}$ is shift invariant. By Lemma 2.8, $\mathcal{B}_{\text{cont}} = \mathcal{B}^0_{\text{cont}}$.

Now, let $w_1, w_2 \in \mathcal{B}^*$ with $w_1|_{[0,l]} = w_2|_{[0,l]}$. Then

$$w_3(t) = \begin{cases} w_1(t), & \text{if } t < 0 \\ w_2(t), & \text{if } t \ge 0 \end{cases} \in \mathcal{B}.$$

Since $\mathcal{B}_{\text{cont}}$ is $\mathcal{B}$-controllable, for any $w \in \mathcal{B}_{\text{cont}}$, there exist $\tau_1 > 0$ and $v_1 \in \mathcal{B}$ such that $v_1|_{(-\infty,0]} = w_1|_{(-\infty,0]} = w_3|_{(-\infty,0]}$ and $v_1|_{[\tau_1,\infty)} = w|_{[\tau_1,\infty)}$, that is, w_3 can be switched to w. Similarly, there exist $\tau_2 > 0$ and $v_2 \in \mathcal{B}$ such that $v_2|_{(-\infty,0]} = w|_{(-\infty,0]}$ and $v_2|_{[\tau_2,\infty)} = w_2|_{[\tau_2,\infty)} = w_3|_{[\tau_2,\infty)}$, that is, w can be switched to w_3. Hence $w_3 \in \mathcal{B}^0_{\text{cont}}$ by its maximality and therefore $w_3 \in \mathcal{B}_{\text{cont}}$ as shown above. This completes the proof. ■

3. The Autonomous-Controllable Decomposition

For 1D (one dimensional) differential/difference behaviours $\mathcal{B}$ it is well known that $\mathcal{B}$ can be split into a direct sum of the controllable and an autonomous part:

$$\mathcal{B} = \mathcal{B}_{\text{cont}} \oplus \mathcal{B}_{\text{aut}}. \tag{3.6}$$

where $\mathcal{B}_{\text{cont}} \subset \mathcal{B}$ is the maximal controllable sub-behaviour. It is known that in the nD differential/difference setting, this sum is not direct for $n > 1$ (see [13]).

This section aims to provide characterisations for both direct and non-direct sum decompositions at the trajectory level. We consider the linear, shift invariant, autonomous behaviours with finite memory:

Lemma 3.1: Let $\mathcal{B}, \mathcal{B}^*$ be linear, shift invariant behaviours with finite memory and $\mathcal{B}^* \subset \mathcal{B}$. Let $(\mathcal{B}^*)_{cont}$ be the maximal linear $\mathcal{B}^*$-controllable sub-behaviour of $\mathcal{B}^*$ and $\mathcal{B}^*_{\text{cont}}$ be the maximal linear $\mathcal{B}$-controllable sub-behaviour of $\mathcal{B}^*$.

(i) If $\mathcal{B}^*$ is autonomous, then $(\mathcal{B}^*)_{cont} = \{0\}$.

(ii) If $\mathcal{B}^*_{\text{cont}} = \{0\}$, then $\mathcal{B}^*$ is autonomous.

Proof: (i) First of all, $0 \in (\mathcal{B}^*)_{cont}$ is obvious. Suppose there exists $w \ne 0, w \in (\mathcal{B}^*)_{cont}$. By $\mathcal{B}^*$-controllability, there exist trajectories $w_1, w_2 \in \mathcal{B}^*$ and $\tau_1, \tau_2 > 0$ s.t. $w_1|_{(-\infty,0]} = 0$, $w_1|_{(\tau_1,\infty)} = w|_{(\tau_1,\infty)}$ and $w_2|_{(-\infty,0]} = w|_{(-\infty,0]}$, $w_2|_{(\tau_2,\infty)} = 0$. By shift invariance,$\sigma_{-\tau_2-l_0} w_1 \in \mathcal{B}^*$ where l_0 is the minimum finite memory. Since $\sigma_{-\tau_2-l_0} w_1|_{[\tau_2,\tau_2+l_0]} = 0 = w_2|_{[\tau_2,\tau_2+l_0]}$, it follows from the autonomous assumption that $\sigma_{-\tau_2-l_0} w_1(t) = w_2(t)$ for all $t \in \mathcal{T}$. This tells $w_2(t) = 0$ for all $t \in \mathcal{T}$ and therefore $w_1 = \sigma_{\tau_2+l_0} w_2 = 0$. So $w \equiv 0$ which is a contradiction.

(ii) Suppose $\mathcal{B}^*$ is not autonomous. Then there exists $w_1, w_2 \in \mathcal{B}^*$ such that $w_1|_V = w_2|_V$ for some interval $V = [r, r + l_0]$ with $r \in \mathcal{T}$ and for which $w_1 \ne w_2$. Let $0 \ne v = \sigma_r(w_1 - w_2)$. Then, $v|_{[0,r]} = 0, v \in \mathcal{B}$ and by the finite memory assumption, it follows that $w_3, w_4 \in \mathcal{B}$ where the signals w_3, w_4 are defined by $w_3|_{(-\infty,0]} = 0$, $w_3|_{[l_0,\infty)} = v|_{[l_0,\infty)}$ and $w_4|_{(-\infty,0]} = v|_{(-\infty,0]}$, $w_4|_{[l_0,\infty)} = 0$. By Lemma 2.8, $\text{span}\{0, v\}$ is a 0-$\mathcal{B}$-controllable sub-behaviour of $\mathcal{B}$. Since $v \ne 0$, a non-empty maximal linear 0-$\mathcal{B}$-controllable sub-behaviour, denoted by $(\mathcal{B}^*_{\text{cont}})^0$, exists. By Lemma 2.8, $\mathcal{B}^*_{\text{cont}} = (\mathcal{B}^*_{\text{cont}})^0$. This contradicts the assumption $\mathcal{B}^*_{\text{cont}} = \{0\}$. ■

Lemma 3.2: Let $\mathcal{B}$ be a linear, shift invariant behaviour with finite memory and suppose $\mathcal{B} = \mathcal{B}_{\text{cont}} \oplus \mathcal{B}^*$ where $\mathcal{B}^*$ has finite memory and is shift invariant. Then $\mathcal{B}^*$ is autonomous.

Proof: Let $\mathcal{B}^*_{\text{cont}}$ be the maximal linear $\mathcal{B}$-controllable sub-behaviour of $\mathcal{B}^*$. Then $\mathcal{B}_{\text{cont}} \subset \mathcal{B}_{\text{cont}} + \mathcal{B}^*_{\text{cont}} \subset \mathcal{B}_{\text{cont}} + \mathcal{B}^*$. Since $\mathcal{B}_{\text{cont}} + \mathcal{B}^*_{\text{cont}}$ is $\mathcal{B}$-controllable and $\mathcal{B}_{\text{cont}}$ is maximal, it follows that $\mathcal{B}^*_{\text{cont}} = \{0\}$. Since $\mathcal{B}^*$ has finite memory, $\mathcal{B}^*$ is autonomous by Lemma 3.1. ■

We observe that if $\mathcal{B}_{\text{cont}}$ has finite co-dimension (as in the differential ([5]) and commensurate delay ([3])settings), then it is known that $\mathcal{B}_{\text{cont}}$ splits $\mathcal{B}$.

Let P_V, $V \subset \mathbb{R}$ denote the natural projection (restriction) of signals defined on $\mathbb{R}$ to signals defined on V. As a shorthand we write P_+ for $P_{\mathbb{R}_+}$ and P_- for $P_{\mathbb{R}_-}$.

Lemma 3.3: 1) For any two behaviours $\mathcal{B}_1, \mathcal{B}_2$, we have $P_V(\mathcal{B}_1 + \mathcal{B}_2) = P_V\mathcal{B}_1 + P_V\mathcal{B}_2$.

2) If $\mathcal{B}$ is a linear, shift invariant behaviour with finite memory $l > 0$ and $\mathcal{B} = \mathcal{B}_{\text{cont}} \oplus \mathcal{B}^*$ then $P_V\mathcal{B} = P_V\mathcal{B}_{\text{cont}} \oplus P_V\mathcal{B}^*$ for all intervals V of length greater than l.

Proof: Claim 1) is rather obvious. For claim 2), we need only to prove the sum $P_V\mathcal{B}_{\text{cont}} \oplus P_V\mathcal{B}^*$ is direct.

Let $w_1 \in \mathcal{B}^*$, $w_2 \in \mathcal{B}_{\text{cont}}$ such that $w_1|_V = w_2|_V$, i.e. $w_1|_V = w_2|_V \in P_V\mathcal{B}_{\text{cont}} \cap P_V\mathcal{B}^*$ Consider any $w_3 \in \mathcal{B}_{\text{cont}}$. Then by the memory property w_1 can be patched to w_2 and conversely, and by $\mathcal{B}$-controllability w_2 can be patched to w_3 and conversely. This tells that w_1 can be patched to w_3 in $\mathcal{B}$ and conversely. Hence $w_1 \in \mathcal{B}_{\text{cont}}$. By the direct sum property it follows that $w_1 = 0$, hence $w_1|_V = w_2|_V = 0$. ■

4. Stability

Stability is closely related to the signal spaces involved. Since this section, the behaviours considered will be restricted to be within the extended signal spaces $L_e^p := L_e^p(\mathcal{T}, \mathbb{R}^n), 1 \le p \le \infty$, i.e., L^p behaviours or subsets of L_e^p. Here and after, given a general normed signal space (say) Y of signals from $\mathcal{T}$ or $\mathcal{T}_+$ to $\mathbb{R}^n$, its extended space Y_e is defined as:

$$Y_e = \{y \mid \mathcal{I} \to \mathbb{R}^n : T_\tau y \in Y \text{ for all } \tau > 0\},$$

where $I = \mathcal{T}$ or $\mathcal{T}_+$ subject to on which set Y is defined, and T_τ is the truncation operator, that is

$$(T_\tau y)(t) = \begin{cases} y(t) & \text{for } t \le \tau \\ 0 & \text{for } t > \tau. \end{cases}$$

As a shorthand we denote by

$$X = L^p(\mathcal{T}_+) =: L^p(\mathcal{T}_+, \mathbb{R}^n), 1 \le p \le \infty.$$

So $X_e = L_e^p(\mathcal{T}_+)$. We remark that some of our discussions remain valid for C^∞ behaviours.

The standard behavioural definition of stability for autonomous systems is as follows:

Definition 4.1: An autonomous system $\mathcal{B}_{\text{aut}}$ is said to be X-stable if and only if for any $w \in \mathcal{B}_{\text{aut}}$, $w|_{[0,\infty)} \in X$.

The stability can be equivalently expressed as the statement that $\mathcal{B}_{\text{aut}}$ is stable if and only if $P_+\mathcal{B}_{\text{aut}} \cap X = P_+\mathcal{B}_{\text{aut}}$.

The results hold in both cases. But in the next two sections, the underlying signal spaces are required to have a norm and therefore the consideration in the next two sections will be restricted to L^p behaviours.

Definition 4.2: A behaviour $\mathcal{B}$ with i/o partition $u|y$ is stable if and only if for all $u \in X$ and for all $w \in \mathcal{B}$ for which $w|_{\mathbb{R}_+} = (u, y)$ then $y \in X$.

Associated to any behaviour are the stable sub-behaviours which correspond to the behaviour taking zero values up to time $t = 0$:

Definition 4.3: The graph $\mathcal{G}_\mathcal{B}$ of a behaviour $\mathcal{B}$ is defined to be:

$$\mathcal{G}_\mathcal{B} := \left\{ w \in \mathcal{B}|_{\mathbb{R}_+} \;\middle|\; \begin{array}{l} v \in \mathcal{B},\ w|_{\mathbb{R}_+} = v|_{\mathbb{R}_+}, \\ v|_{(-\infty,0]} = 0,\ v|_{\mathbb{R}_+} \in X \end{array} \right\}.$$

The extended graph $\mathcal{G}_\mathcal{B}$ of $\mathcal{B}$ is defined to be

$$\mathcal{Z}_\mathcal{B} := \{w \in \mathcal{B}|_{\mathbb{R}_+} \mid v \in \mathcal{B},\ w|_{\mathbb{R}_+} = v|_{\mathbb{R}_+},\ v|_{(-\infty,0]} = 0\}.$$

Note that when $X = L^2$, $\mathcal{G}_\mathcal{B}$ corresponds to the classical H^2 graph.

Lemma 4.4: Let $\mathcal{B} = \mathcal{B}_{\text{cont}} \oplus \mathcal{B}_{\text{aut}}$ be a linear, shift invariant behaviour with finite memory. Then $\mathcal{Z}_{\mathcal{B}_{\text{cont}}} = \mathcal{Z}_\mathcal{B}$ and $\mathcal{G}_{\mathcal{B}_{\text{cont}}} = \mathcal{G}_\mathcal{B}$.

In the case when the sum of $\mathcal{B} = \mathcal{B}_{\text{cont}} + \mathcal{B}_{\text{aut}}$ is not a direct sum, this lemma is hardly true since it depends on Lemma 3.3 (ii).

Definition 4.5: A linear behaviour with i/o partition $u|y$ is uniformly stable if and only if

1) $\mathcal{B}$ is stable.
2) There exists a bounded operator $\Psi \colon X \to X$ such that for all $(u, y) \in \mathcal{B}$ such that $u|_{\mathbb{R}_+} \in X$, $(u, y)|_{\mathbb{R}_-} = 0$ it follows that $y = \Psi(u)$.

Note that the existence of a single stable autonomous sub-behaviour $\mathcal{B}_{\text{aut}}$ s.t. $\mathcal{B} = \mathcal{B}_{\text{cont}} \oplus \mathcal{B}_{\text{aut}}$ does *not* imply stability. e.g. consider $\dot{x} = x + u$, $\dot{z} = -z$, $y = x + z$. Then the sub-behaviour generated by $\dot{z} = -z$, $u = x = 0$, $y = z$ is stable and has the direct sum property and yet the behaviour is not stable. This property however, characterizes *stabilizability*.

Definition 4.6: A behaviour $\mathcal{B}$ is said to be stabilizable if and only if for all $w_1 \in \mathcal{B}$, there exists $w_2 \in \mathcal{B}$ s.t. $w_1|_{(-\infty,0]} = w_2|_{(-\infty,0]}$ and $w_2|_{[0,\infty)} \in X$.

Lemma 4.7: Let $\mathcal{B}$ be a linear shift invariant behaviour with finite memory. If $\mathcal{B} = \mathcal{B}_{\text{cont}} + \mathcal{B}_{\text{aut}}$ and $\mathcal{B}_{\text{aut}}$ is stable, then $\mathcal{B}$ is stabilizable.

Proof: Suppose here exists a stable autonomous sub-behaviour $\mathcal{B}_{\text{aut}}$ s.t. $\mathcal{B} = \mathcal{B}_{\text{cont}} + \mathcal{B}_{\text{aut}}$. Then given any $w \in \mathcal{B}$, there exist $w_1 \in \mathcal{B}_{\text{cont}}, w_2 \in \mathcal{B}_{\text{aut}}$ s.t. $w = w_1 + w_2$. By controllability and shift invariance of $\mathcal{B}_{\text{cont}}$, w_1 can be patched with $0 \in \mathcal{B}_{\text{cont}}$ so there exists $w_1' \in X$ s.t. $w_1|_{(-\infty,0]} = w_1'|_{(-\infty,0]}$. By the stability of $\mathcal{B}_{\text{aut}}$, $w_2 \in X$ Hence $w' = w_1' + w_2 \in X$ and $w|_{(-\infty,0]} = w'|_{(-\infty,0]}$, hence $\mathcal{B}$ is stabilizable. ■

Note that stabilizability is a property independent of any choice of i/o partition: indeed Lemma 4.7 says stabilizability only requires the stability of an autonomous behaviour, and this does not require a i/o partition.

5. Interconnections

Definition 5.1: Given a plant behaviour $\mathcal{B}^P$, a controller behaviour $\mathcal{B}^C$ and interconnection equations $\mathcal{B}^I$:

$$\mathcal{B}^I = \{(w_0, w_1, w_2)^T \in X_e \mid w_0 = w_1 + w_2\} \qquad (5.7)$$

we define the closed loop behaviour $\mathcal{B}^{P \wedge_I C}$ as follows:

$$\mathcal{B}^{P \wedge_I C} = \{(w_0, w_1, w_2)^T \in \mathcal{B}^I \mid w_1 \in \mathcal{B}^P,\ w_2 \in \mathcal{B}^C\}.$$

To ensure uniqueness of solutions of the closed loop (modulo the autonomous part of the behaviour) we adopt the following definition:

Definition 5.2: Given a plant behaviour $\mathcal{B}^P$, a controller behaviour $\mathcal{B}^C$ and interconnection equations $\mathcal{B}^I$ (5.7), the behaviour $\mathcal{B}^{P\wedge_I C}$ is said to be well posed if and only if

$$X_e = \mathcal{Z}_{\mathcal{B}^P} \oplus \mathcal{Z}_{\mathcal{B}^C} \tag{5.8}$$

This captures the idea that for the interconnection of behaviours with zero past, 'w_0 is an input, and for any input w_0, there exists unique internal signals w_1, w_2.

By (5.8), any $w_0 \in X_e$ has a unique decomposition $w = w_1 + w_2$ with $w_1 \in \mathcal{Z}_{\mathcal{B}^P}$ and $w_2 \in \mathcal{Z}_{\mathcal{B}^C}$. Hence two projection operators can be defined as below:

$$\Pi_{P//C} w_0 = w_1, \Pi_{C//P} w_0 = w_2 \tag{5.9}$$

which satisfy

$$\Pi_{P//C} + \Pi_{C//P} = I. \tag{5.10}$$

6. A Behavioural Generalisation of the Gap Metric and a Robust Stability Theorem

In this section we will be concerned with deriving the behavioural version of the central robust stability theorem for LTI systems. Our concern, for now, is with behaviours whose underlying signal space is equipped with a norm $\|\cdot\|$, that is X is a vector space and all behaviours $\mathcal{B}$ are such that $P_+\mathcal{B} \subset X_e$. Furthermore, we assume that $(X, \|\cdot\|)$ has the property that $\|T_\tau x\| \le a$ with $a > 0$ for all $\tau \ge 0$ implies $x \in X$. The classical spaces, e.g. $X = L^p, 1 \le p \le \infty$ satisfy this condition.

Definition 6.1: A mapping $\Psi : X_e \to X_e$ is said to be causal if and only if $T_\tau \Psi w = T_\tau \Psi T_\tau w$ for all $w \in X_e$ and $\tau > 0$. Its induced norm, denoted by $\|\Psi\|$, is defined as

$$\|\Psi\| = \sup\left\{ \frac{\|T_\tau \Psi w\|}{\|T_\tau w\|} \;\middle|\; w \in X_e, \tau > 0, T_\tau w \neq 0 \right\}.$$

Observe that $\|\Pi_{P//C}\| \ge 1$ since for any $w_0 \in \mathcal{G}_{\mathcal{B}_P}$, $\Pi_{P//C} w_0 = w_0$.

Definition 6.2: Given two behaviours $\mathcal{B}^1$, $\mathcal{B}^2$ define a gap function:

$$\begin{aligned} \vec{\delta}(\mathcal{B}^1, \mathcal{B}^2) &= \begin{cases} \inf_{\Phi\in\mathcal{O}} \|(I-\Phi)|_{\mathcal{G}_{\mathcal{B}^1}}\| & \text{if } \mathcal{B}^2 \in \Omega \\ 1 & \text{if not,} \end{cases} \\ \delta(\mathcal{B}^1, \mathcal{B}^2) &= \max\left\{ \vec{\delta}(\mathcal{B}^1, \mathcal{B}^2), \vec{\delta}(\mathcal{B}^2, \mathcal{B}^1) \right\} \end{aligned}$$

where

$$\Omega = \{\mathcal{B} : \exists \mathcal{B}_{\text{aut}} \text{ stable s.t. } \mathcal{B} = \mathcal{B}_{\text{cont}} + \mathcal{B}_{\text{aut}}\},$$

$$\mathcal{O} = \left\{ \Phi\colon D \subset \mathcal{G}_{\mathcal{B}^1} \to \mathcal{G}_{\mathcal{B}^2} \;\middle|\; \begin{array}{l} \Phi \text{ surjective, causal} \\ \text{and } \Phi(0) = 0 \end{array} \right\}.$$

Observe that this a 'real' behavioural definition: everything is defined in terms of trajectories, and all subbehaviours involved can be expressed in set-theoretic terms from the original behaviour $\mathcal{B}$. From a behavioural perspective, it should also be noted that the definition does not require a distinguished i/o partition.

The central reason for consideration of gap distances in systems theory is to obtain robust stability results. In particular we want δ to capture the idea that any reasonable stabilizing controller for $\mathcal{B}^P$ will also stabilize $\mathcal{B}^{P_1}$ provided $\delta(\mathcal{B}^P, \mathcal{B}^{P_1})$ is small. We remark that $\mathcal{B}^2 \in \Omega$ implies the stabilizability of $\mathcal{B}^2$. In the differential case, the reverse is also true and the distance between $\mathcal{B}$ and $\mathcal{B}_{\text{cont}}$ is zero if $\mathcal{B}$ is stabilizable – this is reasonable since *any* stabilizing controller for $\mathcal{B}$ will automatically stabilize $\mathcal{B}_{\text{cont}}$. Consequently δ is necessarily at most a pseudo-metric; indeed the distance between two stabilizable state-space systems with the same transfer function will be 0.

We first consider the controllable-autonomous decomposition of the interconnected behaviour.

Lemma 6.3: Suppose $\mathcal{B}^P$, B^C are linear behaviours and $\mathcal{B}^P = \mathcal{B}^P_{\text{cont}} + \mathcal{B}^P_{\text{aut}}$, $\mathcal{B}^C = \mathcal{B}^C_{\text{cont}} + \mathcal{B}^C_{\text{aut}}$. Then $\mathcal{B}^{P\wedge_I C} = \mathcal{B}^{P\wedge_I C}_{\text{cont}} + \mathcal{B}^{P\wedge_I C}_{\text{aut}}$. If $\mathcal{B}^P = \mathcal{B}^P_{\text{cont}} \oplus \mathcal{B}^P_{\text{aut}}$, $\mathcal{B}^C = \mathcal{B}^C_{\text{cont}} \oplus \mathcal{B}^C_{\text{aut}}$, Then $\mathcal{B}^{P\wedge_I C} = \mathcal{B}^{P\wedge_I C}_{\text{cont}} \oplus \mathcal{B}^{P\wedge_I C}_{\text{aut}}$. Here

$$\begin{aligned} \mathcal{B}^{P\wedge_I C}_{\text{cont}} &= \big\{(w_1+w_2, w_1, w_2) \mid w_1 \in \mathcal{B}^P_{\text{cont}},\ w_2 \in \mathcal{B}^C_{\text{cont}}\big\} \\ &\subset (\mathcal{B}^P_{\text{cont}} + \mathcal{B}^C_{\text{cont}}) \times \mathcal{B}^P_{\text{cont}} \times \mathcal{B}^C_{\text{cont}}, \\ \mathcal{B}^{P\wedge_I C}_{\text{aut}} &= \big\{(w_1+w_2, w_1, w_2) \mid w_1 \in \mathcal{B}^P_{\text{aut}},\ w_2 \in \mathcal{B}^C_{\text{aut}}\big\} \\ &\subset (\mathcal{B}^P_{\text{aut}} + \mathcal{B}^C_{\text{aut}}) \times \mathcal{B}^P_{\text{aut}} \times \mathcal{B}^C_{\text{aut}}, \end{aligned} \tag{6.11}$$

and $\mathcal{B}^{P\wedge_I C}_{\text{aut}}$ is autonomous.

Proof: It is straightforward to verify that $\mathcal{B}^{P\wedge_I C}_{\text{cont}}$ is the maximal controllable behaviour, and that $\mathcal{B}^{P\wedge_I C}_{\text{aut}}$ is autonomous. Let $w \in \mathcal{B}^{P\wedge_I C}$. Then $w = (v_1+v_2, v_1, v_2)$, and by the direct sum decompositions of $\mathcal{B}^P$, $\mathcal{B}^C$, there exist elements $x_1 \in \mathcal{B}^P_{\text{cont}}$, $x_2 \in \mathcal{B}^P_{\text{aut}}$, and $y_1 \in \mathcal{B}^C_{\text{cont}}$, $y_2 \in \mathcal{B}^C_{\text{aut}}$ such that $v_1 = x_1 + x_2$, $v_2 = y_1 + y_2$. Consequently, there exists a decomposition of $w = z_1 + z_2$ where $z_1 \in (\mathcal{B}^P_{\text{cont}} + \mathcal{B}^C_{\text{cont}}) \times \mathcal{B}^P_{\text{cont}} \times \mathcal{B}^C_{\text{cont}}$ and $z_2 \in (\mathcal{B}^P_{\text{aut}} + \mathcal{B}^C_{\text{aut}}) \times \mathcal{B}^P_{\text{aut}} \times \mathcal{B}^C_{\text{aut}}$, namely $z_1 = (x_1+y_1, x_1, y_1)$, $z_2 = (x_2+y_2, x_2, y_2)$.

When $\mathcal{B}^P = \mathcal{B}^P_{\text{cont}} \oplus \mathcal{B}^P_{\text{aut}}$, $\mathcal{B}^C = \mathcal{B}^C_{\text{cont}} \oplus \mathcal{B}^C_{\text{aut}}$, the existence for x_1, x_2, y_1, y_2 and z_1, z_1 are all unique. Hence $\mathcal{B}^{P\wedge_I C} = \mathcal{B}^{P\wedge_I C}_{\text{cont}} \oplus \mathcal{B}^{P\wedge_I C}_{\text{aut}}$. ■

The following key proposition relates a condition of stability of a particular half-line projection to stability of the entire system behaviour.

Proposition 6.4: Let $\mathcal{B}^P$, B^C be linear, shift invariant behaviours with finite memory and $\mathcal{B}^P = \mathcal{B}^P_{\text{cont}} + \mathcal{B}^P_{\text{aut}}$, $\mathcal{B}^C = \mathcal{B}^C_{\text{cont}} + \mathcal{B}^C_{\text{aut}}$. Suppose $\mathcal{B}^P_{\text{aut}}$ and $\mathcal{B}^C_{\text{aut}}$ are stable and $\mathcal{B}^{P\wedge_I C}$ is well posed. Suppose further that $X = \mathcal{G}_{\mathcal{B}^P} \oplus \mathcal{G}_{\mathcal{B}^C}$. Then $\mathcal{B}^{P\wedge_I C}$ is stable.

Proof: Suppose $w \in \mathcal{B}^{P\wedge_I C}$, and $w|_{[0,\infty)} = (w_0, w_1, w_2)$. We have to show if $w_0 \in X$ then $w_1, w_2 \in X$. So let $w_0 \in X$. Since $\mathcal{B}^{P\wedge_I C} = \mathcal{B}^{P\wedge_I C}_{\text{cont}} + \mathcal{B}^{P\wedge_I C}_{\text{aut}}$, it follows that $w = x + y$, where $x = (x_0, x_1, x_2) \in \mathcal{B}^{P\wedge_I C}_{\text{cont}}$ and $y = (y_0, y_1, y_2) \in \mathcal{B}^{P\wedge_I C}_{\text{aut}}$. By stability of $\mathcal{B}^P_{\text{aut}}$, $\mathcal{B}^C_{\text{aut}}$ we know $x_0|_{[0,\infty)} = w_0 - y_0|_{[0,\infty)} \in X$. Since $X = \mathcal{G}_{\mathcal{B}^P} \oplus \mathcal{G}_{\mathcal{B}^C}$, we see $x_0|_{[0,\infty)} = \tilde{x}_1|_{[0,\infty)} + \tilde{x}_2|_{[0,\infty)}$ for some $\tilde{x}_1 \in \mathcal{G}_{\mathcal{B}^P}, \tilde{x}_2 \in \mathcal{G}_{\mathcal{B}^C}$. By the controllability of $\mathcal{B}^{P\wedge_I C}_{\text{cont}}$ it follows that there exists $z \in \mathcal{B}^{P\wedge_I C}$such that $z|_{(-\infty,-\tau]} = 0$, $z|_{[0,\infty)} = x|_{[0,\infty)}$ and therefore there exist

$z_1 \in \mathcal{B}^P, z_2 \in \mathcal{B}^C$ such that $z_1|_{(-\infty,-\tau]} = 0$, $z_1|_{[0,\infty)} = x_1|_{[0,\infty)}$ and $z_2|_{(-\infty,-\tau]} = 0$, $z_2|_{[0,\infty)} = x_2|_{[0,\infty)}$. By shift invariance of $\mathcal{B}^P$, $\sigma_{-\tau} z_1 \in \mathcal{B}^P$. Since $(\sigma_{-\tau} z_1)|_{(-\infty,0)} = 0$, we know $\sigma_{-\tau} z_1 \in \mathcal{Z}_{\mathcal{B}^P}$. Similarly, $\sigma_{-\tau} z_2 \in \mathcal{Z}_{\mathcal{B}^C}$. Since $\sigma_{-\tau} x_i|_{[\tau,\infty)} = \sigma_{-\tau} z_i|_{[\tau,\infty)} (i = 1, 2)$ and $x_0 = x_1 + x_2$, we have $\sigma_{-\tau} x_0|_{[\tau,\infty)} = \sigma_{-\tau} x_1|_{[\tau,\infty)} + \sigma_{-\tau} x_2|_{[\tau,\infty)} = \sigma_{-\tau} z_1|_{[\tau,\infty)} + \sigma_{-\tau} z_2|_{[\tau,\infty)}$. Since $x_0|_{[0,\infty)} = \tilde{x}_1|_{[0,\infty)} + \tilde{x}_2|_{[0,\infty)}$ and the well posedness assumption, we see $(\sigma_{-\tau} z_1)|_{[\tau,\infty)} = \sigma_{-\tau} x_1|_{[\tau,\infty)} = \sigma_{-\tau} \tilde{x}_1|_{[\tau,\infty)}$ and $(\sigma_{-\tau} z_2)|_{[\tau,\infty)} = \sigma_{-\tau} x_2|_{[\tau,\infty)} = \sigma_{-\tau} \tilde{x}_2|_{[\tau,\infty)}$, which indicate that $\sigma_{-\tau} x_1|_{[\tau,\infty)}, \sigma_{-\tau} x_2|_{[\tau,\infty)} \in X$ and hence $x_1|_{[0,\infty)}, x_2|_{[0,\infty)} \in X$. Hence $w_1 = x_1|_{[0,\infty)} + y_1|_{[0,\infty)} \in X, w_2 = x_2|_{[0,\infty)} + y_2|_{[0,\infty)} \in X$ as required. ∎

We can now give the proof of the main robust stability result. Before giving the proof we remark that the result follows straightforwardly from Proposition 6.4 once it has been shown that $X = \mathcal{G}_{\mathcal{B}^P_{\text{cont}}} \oplus \mathcal{G}_{\mathcal{B}^C_{\text{cont}}}$, and that this classical condition is obtained directly using the technique of [1]: we have included this part of the proof from [1] for completeness.

Theorem 6.5: Suppose $\mathcal{B}^P$, $\mathcal{B}^{P_1}$, B^C are linear, shift invariant behaviours with finite memory. If:

1) there exist stable $\mathcal{B}^P_{\text{aut}}, \mathcal{B}^C_{\text{aut}}$ such that $\mathcal{B}^P = \mathcal{B}^P_{\text{cont}} + \mathcal{B}^P_{\text{aut}}$ and $\mathcal{B}^C = \mathcal{B}^C_{\text{cont}} + \mathcal{B}^C_{\text{aut}}$
2) $\mathcal{B}^{P \wedge C}$ is uniformly stable,
3) $\mathcal{B}^{P_1 \wedge C}$ is well-posed, and,
4) $\vec{\delta}(\mathcal{B}^P, \mathcal{B}^{P_1}) \|\Pi_{P//C}\| < 1$,

then $\mathcal{B}^{P_1 \wedge C}$ is uniformly stable.

Proof: Condition 4 implies that there exists a stable $\mathcal{B}^{P_1}_{\text{aut}}$ such that $\mathcal{B}^{P_1} = \mathcal{B}^{P_1}_{\text{cont}} + \mathcal{B}^{P_1}_{\text{aut}}$ by definition of the gap and since $\|\Pi_{P//C}\| \geq 1$. By condition 4, there exists a surjective mapping $\Phi : D \subset \mathcal{G}_{\mathcal{B}^P} \to \mathcal{G}_{\mathcal{B}^{P_1}}$ such $\|\Phi - I\| \|\Pi_{P//C}\| < 1$.

Let $w_0 \in L^2(\mathbb{R})$, $w_0|_{\mathbb{R}_-} = 0$. By the well-posedness of $\mathcal{B}^{P_1 \wedge C}$ (condition 3), we may let $w_0|_{\mathbb{R}_+} = w_1 + w_2$ with $w_1 \in \mathcal{Z}_{\mathcal{B}^{P_1}}, w_2 \in \mathcal{Z}_{\mathcal{B}^C}$. Since $\mathcal{B}^{P_1}, \mathcal{B}^C$ are shift invariant and $\mathcal{B}^{P_1}_{\text{aut}}, \mathcal{B}^P_{\text{aut}}$ are stable, for any $\tau > 0$, there exist $w_1^\tau \in \mathcal{G}_{\mathcal{B}^{P_1}}, w_2^\tau \in \mathcal{G}_{\mathcal{B}^C}$ such that $T_\tau w_1 = T_\tau w_1^\tau, T_\tau w_2 = T_\tau w_2^\tau$. Since Φ is surjective from $\mathcal{G}_{\mathcal{B}^P}$ to $\mathcal{G}_{\mathcal{B}^{P_1}}$, there exists $w_3^\tau \in \mathcal{G}_{\mathcal{B}^P}$ and $w_1^\tau = \Phi w_3^\tau$. Write $x_\tau = w_3^\tau + w_2^\tau$. Then by condition 2, for all $x_\tau \in X$, $\Pi_{P//C} x_\tau = w_3^\tau \in X, \Pi_{C//P} x_\tau = w_2^\tau \in X$ and

$$\begin{aligned} T_\tau w_0 &= T_\tau w_1 + T_\tau w_2 = T_\tau w_1^\tau + T_\tau w_2^\tau \\ &= T_\tau \Phi w_3^\tau + T_\tau w_2^\tau \\ &= T_\tau \Phi \Pi_{P//C} x_\tau + T_\tau \Pi_{C//P} x_\tau \\ &= T_\tau \Phi \Pi_{P//C} T_\tau x_\tau + T_\tau \Pi_{C//P} T_\tau x_\tau \\ &= T_\tau (\Phi - I) \Pi_{P//C} T_\tau x_\tau + T_\tau x_\tau, \end{aligned} \tag{6.12}$$

$$\begin{aligned} T_\tau \Pi_{P_1//C} w_0 &= T_\tau w_1 = T_\tau \Phi w_3^\tau = T_\tau \Phi \Pi_{P//C} x_\tau \\ &= T_\tau \Phi \Pi_{P//C} T_\tau x_\tau. \end{aligned} \tag{6.13}$$

By (6.12), we have

$$\begin{aligned} \|T_\tau x_\tau\| &\leq \|T_\tau w_0\| + \|T_\tau (\Phi - I) \Pi_{P//C} T_\tau x_\tau\| \\ &\leq \|w_0\| + \|\Phi - I\| \|\Pi_{P//C}\| \|T_\tau x_\tau\|, \end{aligned}$$

which gives

$$\|T_\tau x_\tau\| \leq \frac{\|w_0\|}{1 - \|\Phi - I\| \|\Pi_{P//C}\|}.$$

By (6.13), we have

$$\begin{aligned} &\|T_\tau \Pi_{P_1//C} w_0\| \\ &\leq \|T_\tau (\Phi - I) \Pi_{P//C} T_\tau x_\tau\| + \|T_\tau \Phi \Pi_{P//C} T_\tau x_\tau\| \\ &\leq (1 + \|\Phi - I\|) \|\Pi_{P//C}\| \frac{\|w_0\|}{1 - \|\Phi - I\| \|\Pi_{P//C}\|}. \end{aligned}$$

Hence $w_1 = \Pi_{P_1//C} w_0 \in X$ and therefore, by (5.10), $w_2 = \Pi_{C//P_1} w_0 \in X$.

So, for any $w_0 \in X$, $w_0|_{\mathbb{R}_-} = 0$, we have shown there exists $w_1 \in \mathcal{G}_{\mathcal{B}^{P_1}}$, $w_2 \in \mathcal{G}_{\mathcal{B}^C}$ such that $w_0 = w_1 + w_2$, i.e. $X = \mathcal{G}_{\mathcal{B}^{P_1}} \oplus \mathcal{G}_{\mathcal{B}^C}$. The proof is completed by an application of Proposition 6.4. ∎

Notice in the case of differential systems, that the converse of Lemma 4.7 holds, and hence condition 1) in Theorem 6.5 and Ω in Definition 6.2 can be replaced by stabilizability conditions. See [3] for a discussion of Lemma 4.7 in the commensurate delay setting.

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