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Pseudospectral Techniques for Stability Computation of Linear Time Delay Systems

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Abstract

In the last few years the authors developed numerical schemes to detect the stability of different classes of systems involving delayed terms. The base of all methods is the use of pseudospectral differentiation techniques. This interactive paper aims to be a proper media either to sum up the results achieved in this field and to present a collection of case studies in order to show the main features of all the algorithms.

1. Introduction

Many real phenomena in physics, engineering, chemistry, biology, economics, etc. are better modeled and/or simulated considering time delays. Delay systems are particularly important in control theory, where the stability effects of delays are a crucial problem [23], [22]. Important applications can be found also in machining tool where the role of parameters such as spindle speed and feed are stability determining and the models are second order systems with time periodic coefficients [19], [20]. Delays are concerned also in other fields involving different models important in the context of stability, e.g. age-structured population dynamics, neutral and mixed type (advanced-retarded) functional differential equations and partial differential equations with delays.

All delay systems are characterized by the common feature of being influenced, in their present evolution, by information on their past history. Much of their interest is concerned with the stability analysis of the *linear* case. The lack of good estimates of the parameter values (e.g. delays) involved in system models leads to develop opportune criteria to determine not only whether a nominal system is stable or not, but an entire stability region of parameters due to this uncertainty. In particular, when we deal with two varying parameters, we talk about *stability charts*.

In order to introduce the basics of our approaches for determining stability, we consider as prototype problem the simplest case of linear functional differential equations [17], i.e. we refer to delay differential equations (DDEs). In particular we focus on the one delay linear system of DDEs with constant coefficients

$$y'(t) = L_0 y(t) + L_1 y(t-\tau),\ t \geq 0, \tag{1}$$

where $L_0, L_1 \in \mathbb{C}^{m\times m}$ and $\tau > 0$. It is well known [17] that the zero solution of (1) is asymptotically stable if and only if all the characteristic roots $\lambda \in \mathbb{C}$, i.e. the (infinitely many) roots of the characteristic equation $\det(\Delta(\lambda)) = 0$, where $\Delta(\lambda) = \lambda I - L_0 - L_1 e^{-\lambda\tau} = 0$, have strictly negative real part. Since in every vertical strip there is only a finite number of characteristic roots, the asymptotic stability depends on the sign of the real part of the rightmost one.

Hence we deal with an infinite dimensional problem and the fundamental idea is to reduce it to a finite one. We use this basic fact to determine stability and in the following sections we illustrate how this can be achieved for a wide variety of systems involving delays. In Section 2 we recall the analytical and numerical tools which are the cornerstones of all our strategies. In Section 3 we consider the stability computation for the general class of linear constant coefficients DDEs (3.1), we focus on delay models with time periodic coefficients (3.2), we treat extensions to other interesting classes of functional equations, in particular a population model as to show the large applicability of the method (3.3) and finally we illustrate how stability charts can be efficiently computed for all the cases considered (3.4).

2. Analytical and numerical basics

In the last few years, we proposed several numerical approaches for stability analysis of delay systems, which are based on the discretization of either the solution operator associated to (1) or the infinitesimal generator of the solution operator semigroup. We briefly recall that the solution operator $T(t)$, $t \geq 0$, associated to (1) is defined by $T(t)\varphi = y_t$, $\varphi \in X$, where $X = C([-\tau,0],\mathbb{C}^m)$ endowed with the maximum norm, y_t is the function

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$y_t(\theta) = y(t+\theta)$, $\theta \in [-\tau, 0]$, and y is the solution of (1) with initial data $\varphi \in X$. The family $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup with infinitesimal generator $\mathscr{A} : D(\mathscr{A}) \subseteq X \to X$ given by $\mathscr{A}\psi = \psi', \psi \in D(\mathscr{A})$, with domain $D(\mathscr{A}) = \{\psi \in X \mid \psi' \in X,\ \psi'(0) = L_0\psi(0) + L_1\psi(-\tau)\}$. So (1) can be restated as the abstract Cauchy problem [12] $\frac{dy_t}{dt} = \mathscr{A}y_t$, $t > 0$, with $y_0 = \varphi$. The important result [12], [17] $\det(\Delta(\lambda)) = 0 \Leftrightarrow \lambda \in \sigma(\mathscr{A})$, where $\sigma(\cdot)$ denotes the spectrum, suggests the idea to turn the characteristic roots approximation problem into a corresponding eigenvalue problem for suitable matrix discretization of $\mathscr{A}$ (i.e. *infinitesimal generator* approach, in the sequel **IGA**). Moreover, the further result [12], [17] $\det(\Delta(\lambda)) = 0 \Leftrightarrow \lambda = \frac{1}{t}\ln\mu$, $\mu \in \sigma(T(t)) \setminus \{0\}$, where $\mu \in \mathbb{C}$ is known as characteristic multiplier, suggests the idea to translate the stability limit in terms of $|\mu| < 1$ and therefore to turn the characteristic multipliers approximation problem into a corresponding eigenvalue problem for suitable matrix discretization of $T(t)$ (i.e. *solution operator* approach, in the sequel **SOA**).

In the following sections we recall the guidelines of the use of pseudospectral differentiation methods as numerical tool to provide both discretization matrices. Loosely speaking, these techniques are based on the exact differentiation of interpolants at selected sets of points.

2.1. IGA

For fixed N, N positive integer, let us consider the mesh Ω_N of $N+1$ Chebyshev points in $[-\tau, 0]$

$$\Omega_N = \left\{\theta_i = \frac{\tau}{2}\left(\cos\left(i\frac{\pi}{N}\right) - 1\right),\ i = 0, 1, \ldots, N\right\}$$

and replace the continuous space X by the discrete space $X_N = (\mathbb{C}^m)^{\Omega_N} \cong \mathbb{C}^{m(1+N)}$. As an approximation to the infinitesimal generator $\mathscr{A}$ consider the matrix $\mathscr{A}_N : X_N \to X_N$ accomplishing the transformation $z = \mathscr{A}_N x$ with

$$\begin{cases} z_0 = L_0 p_N(0) + L_1 p_N(-\tau) & \\ z_i = p_N'(\theta_i), & i = 1, \ldots, N \end{cases}$$

where p_N is the unique $\mathbb{C}^m$-valued polynomial of degree $\leq N$ that interpolates the entries of x at the nodes of Ω_N. By considering a function $\varphi \in D(\mathscr{A})$ such that $\varphi(\theta_i) = x_i$, $i = 0, 1, \ldots, N$, the value z_0 turns out to be the exact derivative of φ at $\theta_0 = 0$: $z_0 = \varphi'(\theta_0) = [\mathscr{A}\varphi](\theta_0)$ and corresponds to the splicing condition in $D(\mathscr{A})$. On the other hand, the values z_i are approximations to the derivatives of φ at the remaining grid points: $z_i \simeq \varphi'(\theta_i) = [\mathscr{A}\varphi](\theta_i)$ and correspond to the action of $\mathscr{A}$. Finally, the Lagrange representation of p_N leads to the $m(1+N) \times m(1+N)$ matrix

$$\mathscr{A}_N = \begin{pmatrix} L_0 & 0 & \cdots & 0 & L_1 \\ d_{10} & d_{11} & \cdots & d_{1N-1} & d_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{N0} & d_{N1} & \cdots & d_{NN-1} & d_{NN} \end{pmatrix}$$

with $d_{ij} = l_{N,j}'(\theta_i) \otimes I$, $i = 1, \ldots, N$, $j = 0, 1, \ldots, N$, and $l_{N,j}$'s the Lagrange coefficients of p_N. For further details see [5], [9].

2.2. SOA

For the same Ω_N and X_N given in Section 2.1, we can obtain an approximation $x^{(n)} \in X_N$ to the state solution y_{t_n} of (1) at time $t_n = n\tau$, $n \in \mathbb{N}$, on the interval $[t_n - \tau, t_n] = [t_{n-1}, t_n]$ considering the polynomial of collocation $p_N^{(n)}$ for (1) relevant to the nodes $t_n + \Omega_N = \{t_n + \theta_i,\ i = 0, 1, \ldots, N\}$. Setting $x^{(n)} = \left(\left(p_N^{(n)}(\theta_0)\right)^T, \left(p_N^{(n)}(\theta_1)\right)^T, \ldots, \left(p_N^{(n)}(\theta_N)\right)^T\right)^T$, such an approximation $x_i^{(n)} \simeq y_{t_n}(\theta_i)$ follows from the collocation at θ_i, $i = 0, \ldots, N-1$, and from the continuity at the extremes:

$$\begin{cases} \left(p_N^{(n)}\right)'(\theta_i) = L_0 p_N^{(n)}(\theta_i) + L_1 p_N^{(n-1)}(\theta_i), \\ p_N^{(n)}(\theta_N) = p_N^{(n-1)}(\theta_0). \end{cases}$$

In order to derive a relation between the numerical solutions at two successive steps, say t_{n-1} and t_n, we use the Lagrange representation of $p_N^{(n)}$ whose coefficients $l_{N,j}$'s are independent of n. This leads to $x^{(n)} = S_N x^{(n-1)}$, $n \in \mathbb{N}$, where $S_N = A_N^{-1} B_N$ and A_N and B_N are the $m(1+N) \times m(1+N)$ matrices given by

$$A_N = \begin{pmatrix} l_{N,0}'(\theta_0) \otimes I - L_0 & \cdots & l_{N,N}'(\theta_0) \otimes I \\ \vdots & \ddots & \vdots \\ l_{N,0}'(\theta_{N-1}) \otimes I & \cdots & l_{N,N}'(\theta_{N-1}) \otimes I \\ 0 & \cdots & I \end{pmatrix}$$

and

$$B_N = \begin{pmatrix} L_1 & 0 & \cdots & 0 & 0 \\ 0 & L_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & L_1 & 0 \\ I & 0 & \cdots & 0 & 0 \end{pmatrix},$$

respectively. Since $x^{(0)}$ corresponds to the discretization of the initial data $\varphi = y_0$ for (1) on Ω_N, by recursion it clearly follows that $x^{(n)} = S_N^n x^{(0)}$, $n \in \mathbb{N}$, hence $T_N = S_N^n$ is the discretization of $T(t_n)$. For further details see [5].

3. Applications

3.1. Constant coefficients DDEs

Engelborghs and Roose propose in [15] the SOA via linear multistep (LMS) time integration for generalization of (1) to the multiple discrete delay case. Their method is implemented in the MATLAB package DDE-BIFTOOL [14]. The distributed delay case is considered in [21] by using LMS methods and in [4] by using Runge-Kutta (RK) methods. The complete development of the IGA first appears in [11], [6], [9] where a matrix approximation to $\mathscr{A}$ is obtained discretizing the derivative by RK, LMS and pseudospectral differencing methods, respectively. With the last technique we can take advantage of the well-known "spectral accuracy" to obtain very accurate approximation with small matrix dimension [5], [9]. This behavior, i.e. $O(N^{-N})$ error, represents in fact, for sufficiently small tolerance, the outstanding advantage of this method compared to RK and LMS schemes where the error is $O(N^{-p})$, p the convergence order. That is why we choose this last one as the core algorithm for stability detection. For further details and examples see [9].

With this method we are able to face the stability computation for the general class of DDEs represented by

$$y'(t) = L_0 y(t) + \sum_{l=1}^{k} \left(L_l y(t-\tau_l) + \int_{-\tau_l}^{-\tau_{l-1}} M_l(\theta) y(t+\theta) d\theta \right)$$

where $L_0, L_1, \dots, L_k \in \mathbb{C}^{m\times m}$, $0 = \tau_0 < \tau_1 < \cdots < \tau_k = \tau$ and $M_l : [-\tau, 0] \to \mathbb{C}^{m\times m}$, $l = 1, \dots, k$, are smooth functions. As examples consider the DDEs

$$\begin{cases} y_1'(t) &= -0.5y_1(t) - \tanh(y_1(t-1.57)) \\ &+ \tanh(y_2(t-0.2)) \\ y_2'(t) &= -0.5y_2(t) + 2.34\tanh(y_1(t-0.2)) \\ &- \tanh(y_2(t-1.57)) \end{cases}, \qquad (2)$$

taken from [15] and

$$y'(t) = L_0 y(t) + L_1 y(t-1) + \int_{-1}^{0} M(\theta) y(t+\theta) d\theta \qquad (3)$$

taken from [16] with coefficients matrices

$$L_0 = \begin{pmatrix} -3 & 1 \\ -24.646 & -35.430 \end{pmatrix}, L_1 = \begin{pmatrix} 1 & 0 \\ 2.35553 & 2.00365 \end{pmatrix}$$

$$M(\theta) = \begin{pmatrix} 1 & -\theta \\ 1 & -\theta \end{pmatrix},$$

whose approximated characteristic roots are shown in Fig. 1 and Fig. 2 using the IGA.

3.2. Time periodic coefficients DDEs

Consider the system of DDEs with time dependent coefficients

$$\begin{aligned} y'(t) &= L_0(t) y(t) + \sum_{l=1}^{k} \Bigg(L_l(t) y(t-\tau_l) \\ &+ \int_{-\tau_l}^{-\tau_{l-1}} M_l(t,\theta) y(t+\theta) d\theta \Bigg) \end{aligned} \qquad (4)$$

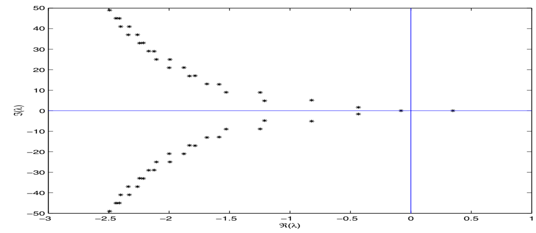

Figure 1. First fifty rightmost roots of (2).

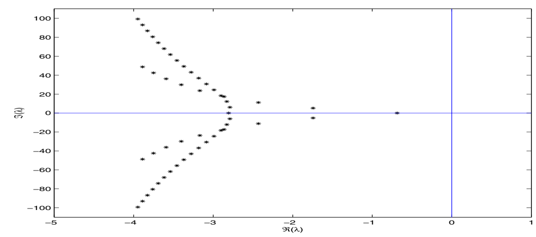

Figure 2. First fifty rightmost roots of (3).

with coefficients varying periodically in time with period P. Stability of the zero solution is determined, according to the Floquet theory, by the characteristic multipliers: if they all lie inside the unit disc of the complex plane then the system is asymptotically stable [17], [19]. In particular the stability determining role is played by the eigenvalues of the so-called fundamental matrix $U = T(P)$ where $T(t)$, $t \geq 0$, is the solution operator semigroup associated to (4). Due to this, the SOA must be used after providing extension to the time dependent coefficients case as in [3] for the case $P = \tau$ or by the authors for the general case. As an example consider the delayed damped Mathieu equation [20]

$$y''(t) + b_0 y'(t) + c_0(t) y(t) = c_1 \int_{-1}^{0} w(\theta) y(t+\theta) d\theta \qquad (5)$$

with $c_0(t) = c_{0\delta} + c_{0\varepsilon}\cos(2\pi t/P)$ and $w(\theta) = -(\pi/2)\sin(\pi\theta)$ whose approximated characteristic multipliers are shown in Fig. 3 computed with the SOA.

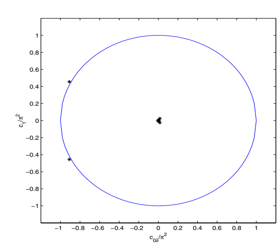

Figure 3. First fifty characteristic multipliers of (5) with $b_0 = 0$, $P = 1/2$ and $c_{0\varepsilon} = 20$.

3.3. Applications to other delay systems

The IGA with pseudospectral differentiation can be applied to more general classes of linear functional differential systems in order to numerically compute the (stability determining) eigenvalues of related derivative operators with non-local boundary conditions such as the infinitesimal generator for the DDEs case. Examples are given in the following sections where we set $X = C([\alpha,\beta],\mathbb{C}^m)$ as the state space and $y_t \in X$ as the state at time $t \geq 0$. $[\alpha,\beta]$ varies accordingly to the class of functional differential equations considered and y_t is given by $y_t(\theta) = y(t+\theta)$, $\theta \in [\alpha,\beta]$. For a deeper analysis see [10].

3.3.1. Neutral DDEs. Let $[\alpha,\beta] = [-\tau,0]$, $\tau > 0$, and consider the linear autonomous system of neutral DDEs with both point and distributed delays $y'(t) = \mathscr{L}(y_t) + \mathscr{N}(y'_t)$, $t \geq 0$, where $\mathscr{N},\mathscr{L} : X \to \mathbb{C}^m$ are defined by

$$\mathscr{N}(\psi) = \sum_{l=1}^{k}\left(N_l\psi(-\tau_l) + \int_{-\tau_l}^{-\tau_{l-1}} D_l(\theta)\psi(\theta)d\theta\right)$$

$$\mathscr{L}(\psi) = L_0\psi(0) + \sum_{l=1}^{k}\left(L_l\psi(-\tau_l) + \int_{-\tau_l}^{-\tau_{l-1}} M_l(\theta)\psi(\theta)d\theta\right)$$

with $0 = \tau_0 < \tau_1 < \cdots < \tau_k = \tau$, $L_0 \in \mathbb{C}^{m\times m}$ and, for $l = 1,\ldots,k$, $N_l, L_l \in \mathbb{C}^{m\times m}$ and $D_l, M_l : [-\tau_{l-1}, -\tau_l] \to \mathbb{C}^{m\times m}$ are sufficiently smooth. With this setting the derivative operator $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ is given by $\mathscr{A}\psi = \psi'$ with domain $D(\mathscr{A}) = \{\psi \in X \mid \psi' \in X,\ \psi'(0) = \mathscr{L}(\psi) + \mathscr{N}(\psi')\}$. As an example consider [13]

$$y'(t) = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix} y(t) - \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix} y'(t-\tau). \qquad (6)$$

whose approximated characteristic roots are shown in Fig. 4 using the IGA for different choices of the parameters.

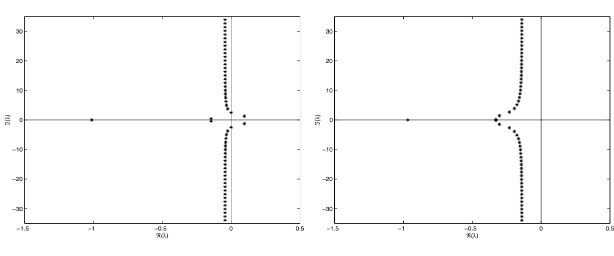

Figure 4. First characteristic roots of (6): $a = 1$, $b = 0.5$, $h = 0.8$, $\tau = 5$ (left); $a = 1$, $b = 4$, $h = 0.5$, $\tau = 5$ (right).

3.3.2. Age-structured population models. Let $[\alpha,\beta] = [-a_\dagger, 0]$, $a_\dagger > 0$, and consider the age-structured population model [18] $y(t) = K(y_t)$, $t \geq 0$, where $K : X \to \mathbb{C}^m$ is the (linear) piecewise integral operator defined by $K(\psi) = \sum_{l=1}^{d} \int_{-a_{l-1}}^{-a_l} k^{(l)}(a)\psi(a)da$, $0 = a_0 < \cdots < a_d = a_\dagger$, with sufficiently smooth kernels $k^{(l)}$, $l = 1,\ldots,d$. With this setting the derivative operator $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ is given by $\mathscr{A}\psi = \psi'$ with domain $D(\mathscr{A}) = \{\psi \in X \mid \psi' \in X,\ \psi(0) = K(\psi)\}$. As an example consider [7]

$$y(t) = \int_0^{a_\dagger} 8[1 - \ln R_0](1-a)\chi_{[\frac{1}{2},1]}(a)y_t(a)da. \qquad (7)$$

whose approximated characteristic roots are shown in Fig. 5 using the IGA with $R_0 = 1$.

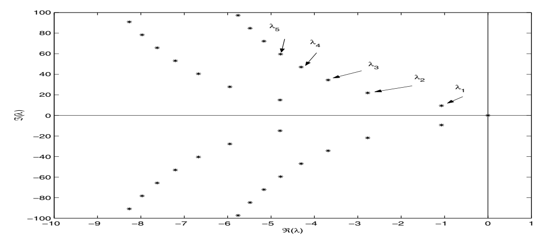

Figure 5. First characteristic roots of (7) with $R_0 = 1$.

3.3.3. Mixed type functional differential equations. Let $[\alpha,\beta] = [-q,p]$, $q,p > 0$, and consider the linear autonomous system of mixed type functional differential equations $y'(t) = \mathscr{Q}(y_t) + \mathscr{P}(y_t)$, $t \geq 0$, where $\mathscr{Q},\mathscr{P} : X \to \mathbb{C}^m$ are defined by

$$\mathscr{Q}(\psi) = Q_0\psi(0) + \sum_{l=1}^{k}\left(Q_l\psi(-q_l) + \int_{-q_l}^{-q_{l-1}} R_l(\theta)\psi(\theta)d\theta\right)$$

$$\mathscr{P}(\psi) = \sum_{u=1}^{h}\left(P_u\psi(p_u) + \int_{p_{u-1}}^{p_u} S_u(\theta)\psi(\theta)d\theta\right)$$

with $0 = q_0 < q_1 < \cdots < q_k = q$, $0 = p_0 < p_1 < \cdots < p_h = p$, $Q_0 \in \mathbb{C}^{m\times m}$ and, for $l = 1,\ldots,k$ and $m = 1,\ldots,h$, $Q_l, P_u \in \mathbb{C}^{m\times m}$ and $R_l : [-q_{l-1}, -q_l] \to \mathbb{C}^{m\times m}$ and $S_u : [p_{u-1}, p_u] \to \mathbb{C}^{m\times m}$ are sufficiently smooth. With this setting the derivative operator $\mathscr{A} : D(\mathscr{A}) \subset X \to$ is given by $\mathscr{A}\psi = \psi'$ with domain $D(\mathscr{A}) = \{\psi \in X \mid \psi' \in X,\ \psi'(0) = \mathscr{Q}(\psi) + \mathscr{P}(\psi)\}$. As an example consider [1]

$$y'(t) = -0.714y(t+1) + 7.5y(t) - 0.714y(t-1) \qquad (8)$$

whose approximated characteristic roots are shown in Fig. 6 using the IGA.

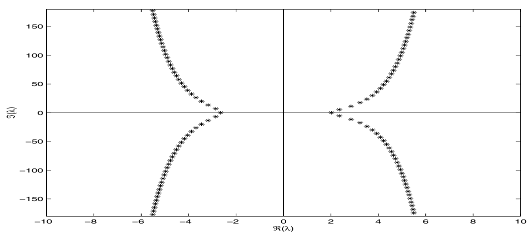

Figure 6. First characteristic roots of (8).

3.3.4. Partial differential equations with delay. Consider the following model describing the flow of a viscoelastic fluid [15]

$$\begin{aligned}\frac{dy(x,t)}{dt} &= \frac{1-\alpha}{\beta}\left(y_{xx}(x,t)-y_{xx}(x,t-\tau)\right)\\ &+ \alpha y_{xx}(x,t)+Ry(x,t)-y^3(x,t)\end{aligned} \tag{9}$$

on $x \in [0,\pi]$ with boundary conditions $y(0,t)=y(\pi,t)=0$. By using second order finite difference schemes in space we reduce to case (1) with tridiagonal $m \times m$ coefficients

$$L_0 = \frac{1}{h^2}\left(\frac{1-\alpha}{\beta}+\alpha\right)\begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix} + RI,$$

$$L_1 = \frac{1}{h^2}\frac{1-\alpha}{\beta}\begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}$$

where α, β and R are real parameters and $h=\pi/(m+1)$ with $m \in \mathbb{N}$. So stability of the zero solution of (9) can be evaluated via the IGA and results are shown in Fig. 7.

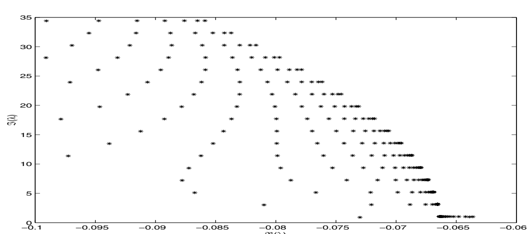

Figure 7. Some characteristic roots of (9).

3.4. Stability charts

Once we have a tool to determine the stability of a delay system as presented in the previous sections, we can perform a robust study with respect to its parameters (e.g. coefficients and/or delays). In the simplest case only one parameter varies on a given range. For instance for (7) it is of interest [18], [7] to study the evolution of the asymptotic stability as the parameter R_0 assumes values greater than the nominal $R_0 = 1$. Hence one can follow the behavior of the rightmost characteristic roots with respect to R_0 and this is shown in Fig. 8. More challenging is the situation when two varying parameters are involved, i.e. the so-called *stability charts* case. Efficient computation of stability charts is extensively treated in [8] and the algorithm proposed there is applied here to the following $8d$-system with five discrete delays depending on only two parameters τ_1 and τ_2 ([2] and courtesy of Prof. N. Olgac and Dr. R. Sipahi, University of Connecticut, Mech. Eng. Dept.):

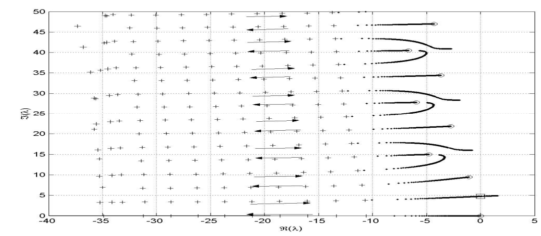

Figure 8. Characteristic roots for (7) as R_0 varies.

$$\begin{aligned} y'(t) &= L_0 y(t) + L_1\left(y(t-\tau_1)+y(t-\tau_2)\right)\\ &+ L_2\left(y(t-2\tau_1)+y(t-2\tau_2)\right)\\ &+ L_3 y(t-\tau_1-\tau_2)\end{aligned} \tag{10}$$

and to the delayed damped Mathieu equation

$$y''(t)+ky'(t)+(\delta+\varepsilon\cos 2\pi t/P)y(t) = by(t-2\pi) \tag{11}$$

with varying parameters δ and b [19]. Stability charts are shown in Fig. 9 and Fig. 10, respectively.

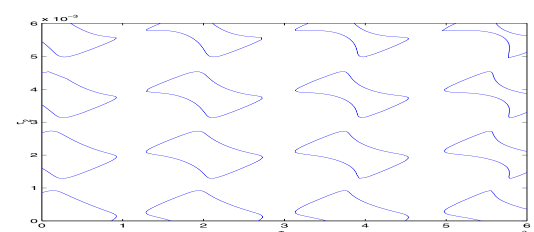

Figure 9. Stability chart for (10).

4. Conclusions

In the recent years several schemes to numerically detect the stability of linear time delay systems were proposed by the authors. All methods are based on the use of pseudospectral differentiation techniques to approximate

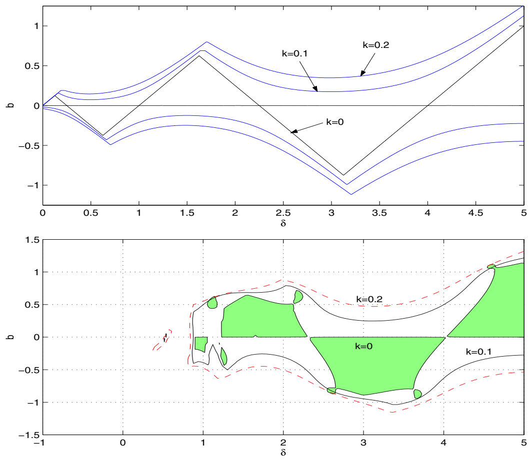

Figure 10. Stability chart for (11): $\varepsilon = 0$, $k = 0, 0.1, 0.2$ (top); $\varepsilon = 1$, $k = 0, 0.1, 0.2$, $P = 4\pi$ (bottom).

the stability determining eigenvalues of infinite dimensional operators associated to the system. Applicability is wide as shown by the numerous case studies presented and a user-friendly software is in progress.

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