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# An Electrical Interpretation of Mechanical Systems via the Pseudo-inductor in the Brayton-Moser Equations

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*Dedicato a Ester.*

***Abstract*— In this paper an analogy between mechanical and electrical systems is presented, where, in contrast to the traditional analogy, position dependence of the mass inertia matrix is allowed. In order to interpret the mechanical system in an electrical manner, a pseudo-inductor element is introduced to cope with inductor elements with voltage-dependent electromagnetic coupling. The starting point of this paper is given by systems described in terms of the Euler-Lagrange equations. Then, via the introduction of the pseudo-inductor, the Brayton-Moser equations are determined for the mechanical system.**

## I. INTRODUCTION

Analogies between mechanical and electrical systems have obtained quite some interest in the recent years. Briefly, there are two types of analogies. The first one considers the mass/condensator and spring/inductor analogy, see e.g, [1]. The second one considers the mass/inductor and spring/condensator analogy, see e.g., [2]. In this paper we are interested in the latter analogy. In particular, we are interested in rewriting motion equations for mechanical systems into motion equations for electrical circuits that can be written in terms of the Brayton-Moser framework, see e.g., [3], [4], [5]. This is a power-based framework that gives rise to a new power-based passivity based control strategy, [6], that has been applied successfully to electrical circuits.

We like to extend the above treatment for electrical circuits to mechanical systems in order to use the advantages of the power-based control method, including also the considerations about series and parallel damping as presented in [7]. The usual mass/inductor and spring/condensator analogy is not always satisfactory. For example, up to our knowledge, so far the gravity force has to be considered as an external force, while in the mechanical modeling, it is included in the potential energy and as such, can be considered as an "internal force". Furthermore, position dependency of the mass inertia matrix giving rise to coriolis and centrifugal forces is a problem in the analogy as well.

In order to interpret the gravity and position dependent mass inertia matrix in terms of an electrical circuit, we have to extend the analogy. Considerations that are similar but not physically interpretable can be found in [8]. The main problem of [8] is that it keeps the position dependency of the elements. In the Brayton-Moser framework, the equations of motion are given entirely in terms of currents and voltages. Then, position dependency can not be included in an intuitively natural way.

In [9] a similar motivation for looking for a power-based description for mechanical systems is used. However, the major difference with [9] is that here we look for the electrical interpretation in terms of the currents and voltages, while in [9] the purpose is to rewrite the standard Hamiltonian equations of motion in terms of a power-based description while not necessarily changing the coordinates and while not looking for a precise electrical interpretation. Other considerations are then in order.

We will start by briefly describing the Euler-Lagrange and Brayton-Moser frameworks, and motivate the paper by means of the example of the single pendulum. Then we continue with our main results, i.e., the Brayton-Moser description stemming from the Euler-Lagrange equations. Finally, we consider two examples, the double pendulum and the inverted pendulum on a cart.

## II. PRELIMINARIES

### *A. Euler-Lagrangian systems (EL)*

The standard Euler-Lagrange equations (e.g., [10]) for an $r$ degrees of freedom mechanical system with generalized coordinates $q \in \mathbb{R}^r$ and external forces $\tau \in \mathbb{R}^r$ are given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q} = \tau \tag{1}$$

where

$$\mathcal{L}(q,\dot{q}) \triangleq \mathcal{T}(q,\dot{q}) - \hat{\mathcal{V}}(q) \tag{2}$$

is the so-called Lagrangian function, $\mathcal{T}(q,\dot{q})$ is the kinetic energy which is of the form

$$\mathcal{T}(q,\dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q}, \tag{3}$$

where $D(q) \in \mathbb{R}^{r\times r}$ is a symmetric positive definite matrix, and $\mathcal{V}(q)$ is the potential function which is assumed to be bounded from below.

### *B. RLC-circuits: The Brayton-Moser equations (BM)*

The electrical circuits considered in this paper are complete RLC-circuits in which all the elements can be nonlinear. The standard definitions of respectively inductance and capacity matrices are given by

$$L(i_\rho) = \frac{\partial \phi_\rho(i_\rho)}{\partial i_\rho}\ ,\ C(v_\sigma) = \frac{\partial q_\sigma(v_\sigma)}{\partial v_\sigma}$$

where $i_\rho \in \mathbb{R}^r$ represents the currents flowing in the inductors and $\phi_\rho(i_\rho) \in \mathbb{R}^r$ is the related magnetic flux

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vector. On the other hand $v_\sigma \in \mathbb{R}^s$ defines the voltages across the capacitors and the vector $q_\sigma(v_\sigma) \in \mathbb{R}^s$ represents the charges stored in the capacitors. From [3] we know that the differential equations of such electrical circuits have the special form

$$\begin{bmatrix} L(i_\rho) & 0 \\ \\ 0 & -C(v_\sigma) \end{bmatrix} \begin{bmatrix} \frac{di_\rho}{dt} \\ \\ \frac{dv_\sigma}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial P}{\partial i_\rho}(i_\rho, v_\sigma) \\ \\ \frac{\partial P}{\partial v_\sigma}(i_\rho, v_\sigma) \end{bmatrix}, \quad (4)$$

with the mixed potential function $P(i_\rho, v_\sigma)$ which contains the interconnection and resistive structure of the circuit and that is defined as

$$P(i_\rho, v_\sigma) = F(i_\rho) - G(v_\sigma) + {i_\rho}^T \Lambda v_\sigma \quad (5)$$

where $F : \mathbb{R}^r \to \mathbb{R}$ and $G : \mathbb{R}^s \to \mathbb{R}$ are the current potential (content) related with the current-controlled resistors (R) and voltage sources, and the voltage potential (co-content) related with the voltage-controlled resistors (i.e., conductors, G) and current sources, respectively. More specifically, the content and co-content are defined by the integrals

$$\int_0^{i_\rho} \hat{v}_R(i'_\rho) di'_\rho, \quad \int_0^{v_\sigma} \hat{i}_G(v'_\sigma) dv'_\sigma,$$

where $\hat{v}_R(i_\rho)$ and $\hat{i}_G(v_\sigma)$ are the characteristic functions of the (current-controlled) resistors and conductors (voltage-controlled resistors), respectively. The $r \times s$ matrix $\Lambda$ is given by the interconnection of the inductors and capacitors, and the elements of $\lambda$ are in $\{-1, 0, 1\}$.

### C. The single pendulum: a motivating example

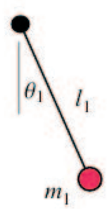


Fig. 1. Single pendulum

Consider the pendulum of Fig. 1 with mass $m_1$ connected by a rigid massless wire of length $l_1$ to a fixed reference. The angle is denoted by $\theta_1$, and the gravity constant by $g$. The potential energy $\mathcal{V}$ and the kinetic energy $\mathcal{T}$ are given by $\mathcal{V}(\theta) = m_1 g l_1 (1 - \cos\theta)$ and $\mathcal{T}(\dot{\theta}) = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$. The Lagrangian is then $\mathcal{L}(\theta, \dot{\theta}) = \mathcal{T}(\dot{\theta}) - \mathcal{V}(\theta)$. Now, we like to study a full nonlinear analogue of the pendulum to an LC electrical circuit, where we consider masses as inductive elements. Consider the electrical circuit that is shown in the Fig. 2. We consider the rotational force resulting from the gravity as a capacitive element. In the analogy, this means that this will result in a capacitor and a corresponding voltage. Assuming $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, it follows that the characteristics of the capacitor are given by

$$\theta_1 = \arcsin\left(\frac{v_\sigma}{m_1 g l_1}\right) =: -f_v(v_\sigma),$$
$$\frac{\partial f_v(v_\sigma)}{\partial v_\sigma} = \frac{1}{\sqrt{(m_1 g l_1)^2 - {v_\sigma}^2}} = C(v_\sigma).$$

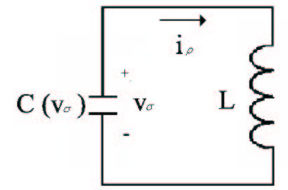


Fig. 2. Single pendulum equivalent LC circuit

Using the standard analogy where $m_1 l_1^2 = L$ and $\dot{\theta} = i_\rho$, we finally obtain directly from the EL equations that $L\frac{di_\rho}{dt} + v_\sigma = 0$ and additionally, we obtain the capacitor equation given by $C(v_\sigma)\frac{dv_\sigma}{dt} = -i_\rho$. These are the motion equations of the single pendulum expressed in an electrical fashion.

### D. Definitions

In order to introduce the electrical counter part of the position dependent mass we introduce the so-called *pseudo-inductor*. This is an inductor, but now relating the magnetic flux linkages to current and the voltage, which differs from the "usual" electrical case, i.e.,

$$\phi = -f_\phi(v_\sigma, i_\rho). \quad (6)$$

where $\phi \in \mathbb{R}^\rho$ is the flux related to the inductors. This definition lead to the following implicit relation between voltage and current

$$v_\rho = \frac{d\phi}{dt} = -\frac{\partial f_\phi}{\partial i_\rho}\frac{di_\rho}{dt} - \frac{\partial f_\phi}{\partial v_\sigma}\frac{dv_\sigma}{dt}. \quad (7)$$

Now, define the *pseudo-inductance matrix* and the *co-pseudo-inductance matrix* as

$$\widetilde{L}(i_\rho, v_\sigma) = \frac{\partial f_\phi}{\partial i_\rho} \quad , \quad \widetilde{M}(i_\rho, v_\sigma) = \frac{\partial f_\phi}{\partial v_\sigma} \quad (8)$$

respectively, then (7) can be written as

$$v_\rho = -\widetilde{L}(i_\rho, v_\sigma)\frac{di_\rho}{dt} - \widetilde{M}(i_\rho, v_\sigma)\frac{dv_\sigma}{dt}. \quad (9)$$

Similarly, we will consider a *capacitor* as a function relating the charge and the voltage, i.e.,

$$q_{\sigma j} = -f_v^j(v_{\sigma j}), \quad j = 1, \ldots, s. \quad (10)$$

By defining the non-negative capacity matrix $C(v_\sigma) = diag\left(\frac{\partial f_v^j(v_{\sigma j})}{\partial v_{\sigma j}}\right), j = 1, \ldots, s$, we have from differentiation of (10) that

$$i_\sigma = -C(v_\sigma)\frac{dv_\sigma}{dt}. \quad (11)$$

*Remark 1:* The voltage dependency of the pseudo-inductor results in an incremental pseudo-inductance matrix $\widetilde{L}$ given in (8) that can often be taken as follows

$$\widetilde{L}(v_\sigma) = \begin{bmatrix} L_1 & \cdots & M_{1\rho}(v_\sigma) \\ \vdots & \ddots & \vdots \\ M_{\rho 1}(v_\sigma) & \cdots & L_\rho \end{bmatrix} \quad (12)$$

where $M_{ij}(v_\sigma)$ represents a magnetic effect between the inductors $L_i$ and $L_j$ that can be written as

$$M_{ij}(v_\sigma) = k_{ij}(v_\sigma)\sqrt{L_i L_j}. \tag{13}$$

We thus consider a coupling coefficient $k_{ij}(v_\sigma)$, that is voltage controlled.

*Remark 2:* We consider topologically complete RLC-circuits. Then, the current flowing through the inductors $i_\rho$ and the voltages across the capacitors $v_\sigma$ are independent and fully descriptive. It means that, even in the case of pseudo-inductors we keep the following relations

$$v_\rho = f_\rho(i_\rho, v_\sigma) = -\frac{\partial P(i_\rho, v_\sigma)}{\partial i_\rho}$$
$$i_\sigma = f_\sigma(i_\rho, v_\sigma) = \frac{\partial P(i_\rho, v_\sigma)}{\partial v_\sigma}$$

In other words, the voltage across any inductor $v_\rho$ is given by a combination of the voltage across each capacitor $v_\sigma$ and the current flowing in each inductor $i_\rho$. The same holds for the current $i_\sigma$ through the branches of the capacitors.

## III. Main result

EL equations for mechanical systems represent the force balance evaluated on each mass. Indeed, the generalized coordinates $q_i$ and $\dot{q}_i$ characterize the position and velocity of the mass $m_i$. For a few number of simple motion systems, replacing the position by the charge stored in a capacitor and the velocity by the current flowing through the inductors, leads to a set of equations that also describes the behavior of an electrical circuit. For more complex mechanical systems that have a coriolis and centrifugal term this is not possible anymore. Our main result gives conditions such that the analogy between mechanical and electrical systems holds even for more complex mechanical behaviors. Our results are currently only valid for mechanical systems that move in the plane. Furthermore, we present some examples of mechanical systems where an electrical interpretation is provided.

### A. From EL to BM

Compare the functions in (2) and (5). It is then easily observed that a main difference between the two functions is the relation between the state variables, in the first case $(q, \dot{q})$ and in the second one $(i_\rho, v_\sigma)$. Notice that $(q, \dot{q}) \in \mathbb{R}^{2r}$ while $(i_\rho, v_\sigma) \in \mathbb{R}^{r+s}$, which corresponds to the fact that mechanical systems are "nodical" (see e.g., [2]), where as electrical systems are not. In order to cope with this, the Kirchhoff current law may be needed (see e.g., [11]). However, it can also be translated into an integral version and will be represented in one of the conditions of our theorem. In the following Lemma 1, we show that, if the $q_\rho \in \mathbb{R}^r$ dependency of the general mass matrix $D(q_\rho)$ can be expressed in terms of a new variable $q_\sigma \in \mathbb{R}^s$, assumption A1, and if there exists a a direct force-position link between $q_\sigma$ and $v_\sigma \in \mathbb{R}^s$, assumption A2, then, via the use of the pseudo-inductor defined in the previous section, the behavior of mechanical systems that exhibit coriolis and centrifugal terms can be still electrically interpretable. As a consequence, a re-styling of the BM framework, where a cross-term will be added into the $Q(i_\rho, v_\sigma)$ matrix, is also provided.

*Lemma 1:* Consider the general Lagrangian function (2). Assume that:

**A1** (*interconnection*) $i_\rho = i_\sigma$, [1]

**A2** (*force-position link*) $q_{\sigma j} = -f_v^j(v_{\sigma j}) \in \mathcal{C}^1$ with $j = 1, ..., r$ is a set of invertible functions such that:

- -) $\frac{\partial f_v^j(v_{\sigma j})}{\partial v_{\sigma j}} = C_j(v_{\sigma j})$,
- -) $f_q^j(q_{\sigma j}) = v_{\sigma j}$.

Then:

$$\frac{\partial \mathcal{L}(q_\rho, \dot{q}_\rho)}{\partial q_\rho} = -\hat{D}(i_\rho, v_\sigma)C(v_\sigma)\frac{dv_\sigma}{dt} + v_\sigma \tag{14}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q_\rho, \dot{q}_\rho)}{\partial \dot{q}_\rho}\right) = \widetilde{D}(v_\sigma)\frac{di_\rho}{dt} - \dot{\bar{D}}(v_\sigma)C(v_\sigma)\frac{dv_\sigma}{dt} \tag{15}$$

where

$$\hat{D}(i_\rho, v_\sigma) = \begin{bmatrix} \frac{1}{2}{i_\rho}^T \frac{\partial D(q_\rho)}{\partial q_{\rho_1}}|_{q_\rho = -f_v(v_\sigma)} \\ \cdots \\ \frac{1}{2}{i_\rho}^T \frac{\partial D(q_\rho)}{\partial q_{\rho_r}}|_{q_\rho = -f_v(v_\sigma)} \end{bmatrix} \tag{16}$$

$$C(v_\sigma) = diag\left(\frac{\partial f_v^j(v_{\sigma j})}{\partial v_{\sigma j}}, j = 1, ..., r\right) \tag{17}$$

$$\widetilde{D}(v_\sigma) = D(q_\rho)|_{q_\rho = -f_v(v_\sigma)} \tag{18}$$

$$\bar{D}(i_\rho, v_\sigma) = \begin{bmatrix} a_{11}(i_\rho, v_\sigma) & \cdots & a_{1r}(i_\rho, v_\sigma) \\ \vdots & \ddots & \vdots \\ a_{r1}(i_\rho, v_\sigma) & \cdots & a_{rr}(i_\rho, v_\sigma) \end{bmatrix} \tag{19}$$

with $a_{ij}(i_\rho, v_\sigma) = {i_\rho}^T C^{-1}(v_\sigma) \nabla_{v_\sigma} \widetilde{D}_{ij}(v_\sigma)$ for $i, j \in \{1, r\}$.

*Proof:* Consider $\mathcal{T}(q_\rho, i_\rho)$ in (3), then for $j = 1, \ldots, r$

$$\frac{\partial \mathcal{T}(q_\rho, i_\rho)}{\partial q_{\rho_j}} = \frac{1}{2}{i_\rho}^T D_q^j(q_\rho) i_\rho. \tag{20}$$

where

$$D_q^j(q_\rho) := \frac{\partial D(q_\rho)}{\partial q_{\rho_j}}. \tag{21}$$

By using assumptions A1 and A2, where we adopt the integrated version of A1 with equal integral constants, then

$$q_\rho = q_\sigma = -f_v(v_\sigma). \tag{22}$$

Replacing (22) into (21), we define the following new matrix

$$D_v^j(v_\sigma) := D_q^j(q_\rho)|_{q_\rho = -f_v(v_\sigma)}. \tag{23}$$

Now, substituting (11) and (23) into (20), we obtain

$$\frac{\partial \mathcal{T}(q_\rho, i_\rho)}{\partial q_{\rho_j}} = -\frac{1}{2}{i_\rho}^T D_v^j(v_\sigma) C \frac{dv_\sigma}{dt} =$$
$$= -\hat{D}^j(i_\rho, v_\sigma) C \frac{dv_\sigma}{dt} \tag{24}$$

[1] Implying that $s = r$ and $\Lambda = I$. See Remark 4 for the physical implications.

where $\hat{D}^j(i_\rho, v_\sigma) := \frac{1}{2} {i_\rho}^T D_v^j(v_\sigma)$ is the $j^{th}$ row of (16). Furthermore, we have that

$$\frac{\partial \hat{\mathcal{V}}(q_\rho)}{\partial q_\rho} = \frac{\partial \mathcal{V}(q_\sigma)}{\partial q_\rho} = \left(\frac{\partial q_\sigma}{\partial q_\rho}\right)^T \frac{\partial \mathcal{V}(q_\sigma)}{\partial q_\sigma} = v_\sigma$$

This proofs (14). On the other hand, considering (22) and (18) it follows that

$$\frac{\partial \mathcal{L}(q_\rho, i_\rho)}{\partial i_\rho} = D(q_\rho) i_\rho = \widetilde{D}(v_\sigma) i_\rho \tag{25}$$

which, once differentiated w.r.t. time, yields

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q_\rho, i_\rho)}{\partial i_\rho}\right) = \widetilde{D}(v_\sigma)\frac{di_\rho}{dt} + \bar{D}(i_\rho, v_\sigma) i_\rho. \tag{26}$$

with $\dot{\widetilde{D}}(v_\sigma)$ given by (19). Then (15) is obtained by using A1 and (11). ■

*Theorem 1:* If the conditions of Lemma 1 are fulfilled, then the Euler-Lagrange (1) equations can be rewritten in terms of the Brayton-Moser framework as follows

$$-Q(i_\rho, v_\sigma)\begin{bmatrix} \frac{di_\rho}{dt} \\ \\ \frac{dv_\sigma}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial P(i_\rho, v_\sigma)}{\partial i_\rho} \\ \\ \frac{\partial P(i_\rho, v_\sigma)}{\partial v_\sigma} \end{bmatrix} \tag{27}$$

with

$$Q(i_\rho, v_\sigma) = \begin{bmatrix} -\widetilde{D}(v_\sigma) & (\bar{D}(i_\rho, v_\sigma) - \hat{D}(i_\rho, v_\sigma))C(v_\sigma) \\ \\ 0 & C(v_\sigma) \end{bmatrix},$$

$$P(i_\rho, v_\sigma) = F(i_\rho) - G(v_\sigma) + {i_\rho}^T v_\sigma$$

, and the functions $F(i_\rho)$ and $G(i_\sigma)$ being the content and the co-content of the mixed potential function $P(i_\rho, v_\sigma)$, respectively.

*Proof:* Substitute (15) and (14) the EL equations. Then

$$\begin{aligned} &\widetilde{D}(v_\sigma)\frac{di_\rho}{dt} - \bar{D}(i_\rho, v_\sigma) C(v_\sigma)\frac{dv_\sigma}{dt} + \\ &+ \hat{D}(i_\rho, v_\sigma) C \frac{dv_\sigma}{dt} = \tau + v_\sigma = \frac{\partial P}{\partial i_\rho} \end{aligned} \tag{28}$$

where $\frac{\partial F(i_\rho)}{\partial i_\rho} = \tau$ is a velocity controlled force that includes external forces and velocity controlled dissipation. As a consequence of assumption A1 we have that $G(v_\sigma) = 0$— see Remark 4—. Thus, $\frac{\partial P}{\partial v_\sigma} = i_\rho = i_\sigma$. Combined with (11) we obtain

$$\frac{\partial P}{\partial v_\sigma} = -C(v_\sigma)\frac{dv_\sigma}{dt} \tag{29}$$

which concludes the proof. ■

*Remark 3:* The flux related to the inductor element (the mass analogue) is given by

$$\phi = [L(v_\sigma) + M(v_\sigma)] i_\rho.$$

Time differentiation yields the dynamical behavior of a coupled pseudo-inductor, as given in (9), where

$$\begin{aligned} \widetilde{L}(v_\sigma) &= \widetilde{D}(v_\sigma) \\ \widetilde{M}(i_\rho, v_\sigma) &= -(\bar{D}(i_\rho, v_\sigma) - \hat{D}_v(i_\rho, v_\sigma))C(v_\sigma) \end{aligned}$$

In physical terms, it means that the kinetic energy $\mathcal{T}(q_\rho, i_\rho)$ in electrical terms corresponds to the energy stored into coupled inductors, where the coupling coefficient $k_{ij} = k_{ij}(v_\sigma)$ is assumed to be voltage-controlled.

*Remark 4:* As seen in the proof of Theorem 1, a consequence of assumption A1 is that $G(v_\sigma) = 0$, i.e., there are no purely resistive branches or current sources in the circuit. Moreover, the equivalent circuit presents an equal number of inductors and capacitors that are series-connected. To overcome this drawback we refer to our second example.

### *B. Example: the double pendulum*

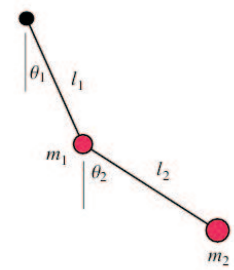


Fig. 3. Double pendulum

Consider a double pendulum with masses $m_1$,$m_2$ and rigid massless wires of lengths $l_1$ and $l_2$. The angles with the vertical are denoted by $\theta_1$ and $\theta_2$, as illustrated in Fig. 3, i.e., $\theta = (\theta_1, \theta_2)^T$. The gravity constant is given by g. Then the potential energy of the system is given as

$$\mathcal{V}(\theta) = -K_1 \cos\theta_1 - K_2 \cos\theta_2 + C_g$$

where $K_i = (\sum_{k=i}^2 m_k) l_i g$ and $C_g = (m_1 l_1 + m_2(l_1 + l_2))g$. The kinetic energy is given by

$$\mathcal{T}(\theta, \dot{\theta}) = \frac{1}{2}(D_{11}\dot{\theta}_1^2 + D_{22}\dot{\theta}_2^2) + D_{12}(\theta)\dot{\theta}_1\dot{\theta}_2$$

where $D_{ii} = (\sum_{k=i}^2 m_k) l_i^2$ and $D_{12}(\theta) = D_{21}(\theta) = m_2 l_1 l_2 \cos(\theta_2 - \theta_1)$ are the terms of the mass-matrix $D(\theta)$ defined in (3). Hence, the position dependence of the mass inertia matrix is only present in $D_{12} = D_{21}$. This term represents the coupling induced by the second mass – or the other way around –. The Euler-Lagrange equations of motion are given by

$$D_{11}\ddot{\theta}_1 + D_{12}(\theta)\ddot{\theta}_1 - \frac{\partial D_{12}(\theta)}{\partial \theta_1}\dot{\theta}_2^2 + K_1 \sin\theta_1 = \tau_1 \tag{30}$$

$$D_{22}\ddot{\theta}_2 + D_{12}(\theta)\ddot{\theta}_1 + \frac{\partial D_{12}(\theta)}{\partial \theta_2}\dot{\theta}_2^2 + K_2 \sin\theta_1 = \tau_2 \tag{31}$$

where the vector $\tau = (\tau_1, \tau_2)^T$ represent the external torques applied on each joint. Clearly, the angular position $\theta$ corresponds to $q_\rho$ in the mass matrix configuration. The angular velocity $\dot{\theta}$ is related to the current vector $i_\rho$. Thus,

$$\mathcal{L}(q_\rho, i_\rho) = \frac{1}{2} {i_\rho}^T D(q_\rho) i_\rho - \sum_{j=1}^{2} \int f_q^j(q_{\rho j}) dq_{\rho j} \tag{32}$$

where

$$\int f_q^1(q_{\rho 1})dq_{\sigma 1} = -K_1 \cos q_{\rho 1},$$
$$\int f_q^2(q_{\rho 2})dq_2 = -K_2 \cos q_{\rho 2} + C_g \tag{33}$$

In this case, there are two masses that both experience an independent gravity force that accounts for two potential energy sources. Hence, $i_\rho = i_\sigma$. Clearly, assumption 2 is fulfilled as well and

$$f_q^j(q_{\sigma j}) = K_j \sin q_{\sigma j} \quad , \quad j = 1, 2.$$

As far as they are locally invertible for $q_{\sigma j} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, naming $f_q^j(q_{\sigma j}) = -v_{\sigma j}$, we have that

$$q_{\sigma j} = \arcsin\left(-\frac{v_{\sigma j}}{K_j}\right) = -f_v^j(v_{\sigma j}) \quad , \quad j = 1, 2$$

and

$$\frac{\partial f_v^j(v_{\sigma j})}{\partial v_{\sigma j}} = \frac{1}{\sqrt{K_j^2 - v_{\sigma j}^2}} = C_j(v_{\sigma j})$$

are well defined in the open interval $(-K_j, K_j)$. Now, apply Theorem 1 to obtain the following set of equations in the BM fashion

$$-Q(i_\rho, v_\sigma)\begin{bmatrix} \frac{di_\rho}{dt} \\ \\ \frac{dv_\sigma}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial P(i_\rho, v_\sigma)}{\partial i_\rho} \\ \\ \frac{\partial P(i_\rho, v_\sigma)}{\partial v_\sigma} \end{bmatrix}, \tag{34}$$

where, with for ease of notation we write $f_v^j(v_{\sigma j}) = f_v^j$. For $i = 1, 2$, we have

$$\widetilde{D}_{ii}(v_\sigma) = D_{ii} = \left(\sum_{k=i}^{2} m_k\right) l_i^2,$$
$$\widetilde{D}_{12}(v_\sigma) = m_2 l_1 l_2 \cos(f_v^1 - f_v^2),$$
$$\bar{D}_{ii}(i_\rho, v_\sigma) = \hat{D}_{ii}(i_\rho, v_\sigma) = 0,$$
$$\bar{D}_{12}(i_\rho, v_\sigma) = m_2 l_1 l_2 \sin(f_v^1 - f_v^2) i_{\rho 1} - m_2 l_1 l_2 \sin(f_v^1 - f_v^2) i_{\rho 2},$$
$$\hat{D}_{12}(i_\rho, v_\sigma) = m_2 l_1 l_2 \sin(f_v^1 - f_v^2) i_{\rho 1},$$
$$\hat{D}_{21}(i_\rho, v_\sigma) = -m_2 l_1 l_2 \sin(f_v^1 - f_v^2) i_{\rho 2}.$$

Furthermore $v_\sigma = \left[-(m_1 + m_2) l_1 g \sin f_v^1 \ , \ -m_2 l_2 g \sin f_v^2\right]^T$ and $P(i_\rho, v_\sigma) = \tau^T i_\rho + {i_\rho}^T v_\sigma$. The matrix $C(v_\sigma)$ follows straightforward from the definition in assumption A2.

An electrical interpretation of this set of equations is given

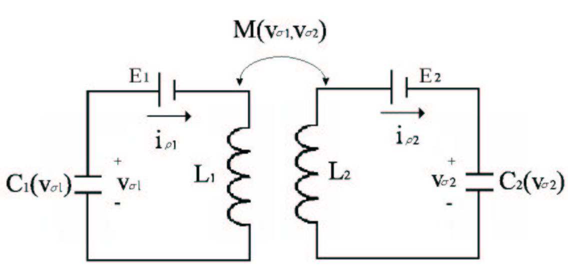


Fig. 4. Double pendulum equivalent C–L electrical circuit

in Fig. 4 where we have the mass matrix $\widetilde{D}(v_\sigma) = \widetilde{L}(v_\sigma)$, i.e., the pseudo inductor matrix as given in (12). The torque vector $\tau$ is equivalent to the two voltage sources $E = [E_1, E_2]^T$.

*C. Example: the inverted pendulum on a cart*

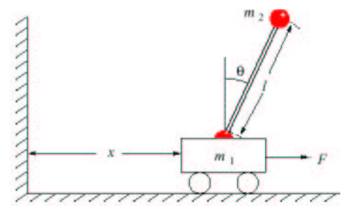


Fig. 5. Inverted pendulum on a cart.

Another interesting example of mechanical system to study is the inverted pendulum with rigid massless rod (of length $l$) placed on a cart as shown if Fig. 5. It is often present in the literature of non-linear control to test the performance of command laws in order to stabilize the pendulum mass $m_2$ in its natural unstable equilibrium point through a force $F$ acting just on the cart of mass $m_1$. The equations describing the dynamics of the to masses could be computed considering as state variables the angular position of the row with the vertical axis $\theta$ and the cart distance $x - x_0$ to a fixed reference ($x_0 = 0$). The motion equations can be determined via the Euler-Lagrange equations. Considering the energetic terms as follows

$$\begin{aligned} \mathcal{T}(\theta, \dot{\theta}, \dot{x}) &= \frac{1}{2} D_{11} \dot{x}^2 + \frac{1}{2} D_{22} \dot{\theta}^2 + D_{12}(\theta) \dot{\theta} \dot{x} \\ \mathcal{V}(\theta) &= K \cos \theta \end{aligned}$$

where $D_{11} = m_1 + m_2$, $D_{22} = m_2 l^2$, $D_{12} = m_2 l \cos \theta$ and $K = m_2 g l$, the related Lagrangian is $\mathcal{L}(\theta, \dot{\theta}, \dot{x}) = \mathcal{T}(\theta, \dot{\theta}, \dot{x}) - \mathcal{V}(\theta)$ and consequently the EL equations, considering the external force vector $\tau = [F, 0]^T$, are given by

$$D_{11}\ddot{x} + D_{12}(\theta)\ddot{\theta} + \frac{\partial D_{12}(\theta)}{\partial \theta}\dot{\theta}^2 = F \tag{35}$$
$$D_{22}\ddot{\theta} + D_{12}(\theta)\ddot{x} - K \sin \theta = 0 \tag{36}$$

As in the previous example, just replace $[\theta, \dot{\theta}, \dot{x}]$ by $[q_{\rho 1}, i_{\rho 1}, i_{\rho 2}]$

$$\mathcal{L}(q_{\rho 1}, i_\rho) = \frac{1}{2} {i_\rho}^T D(q_{\rho 1}) i_\rho - \int f_v^1(q_{\rho 1}) dq_{\rho 1} \tag{37}$$

where $i_\rho = [i_{\rho 1}, i_{\rho 2}]^T$. If we take $s$ to be the number of capacitive elements, or in other words, as the amount of potential energy sources, we find in this case that $s = 1$, and thus assumption A1 is violated. For that reason, we add a virtual potential energy source

$$\int f_v^2(v_{\sigma 2}) dv_{\sigma 2}$$

such that his energetic contribution is approximately 0. So far, we choose $f_v^2(v_{\sigma 2}) = C_2 v_{\sigma 2}$, where we assume $C_2$ very large, i.e., so that

$$\lim_{C_2 \to \infty} \frac{1}{2C_2} {q_{\sigma 2}}^2 = 0$$

For simplicity but without loosing in generality, notice that we assumed $C_2(v_{\sigma 2}) = C_2$. The other potential energy term is

$$\int f_v^1(q_{\sigma 1}) dq_{\sigma 1} = K \cos q_{\sigma 1}.$$

Then, we obtain the following relations

$$q_{\sigma 1} = \arcsin - \frac{v_{\sigma 1}}{K} = -f_v^1(v_{\sigma 1}),$$
$$\frac{\partial f_v^1(v_{\sigma 1})}{\partial v_{\sigma 1}} = \frac{1}{\sqrt{K^2 - v_{\sigma 1}^2}} = C_1(v_{\sigma 1})$$

with $f_v^1(v_{\sigma 1})$ and $C_1(v_{\sigma 1})$ invertible functions in the open interval $(-K, K)$. Since A1 and A2 of Lemma 1 are now satisfied for the system with virtual capacitor, from Theorem 1 we have

$$-Q(i_\rho, v_\sigma) \begin{bmatrix} \frac{di_\rho}{dt} \\ \\ \frac{dv_\sigma}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial P}{\partial i_\rho} \\ \\ \frac{\partial P}{\partial v_\sigma} \end{bmatrix}, \tag{38}$$

where, with a slight abuse of notation $f_v^i(v_{\sigma i}) = f_v^i$, $i = 1, 2$, we have

$$\widetilde{D}_{ii}(v_\sigma) = D_{ii} \ , \ \widetilde{D}_{12}(v_\sigma) = m_2 l \cos f_v^1,$$

$$\bar{D}_{ii}(i_\rho, v_\sigma) = 0 \ , \ \bar{D}_{12}(i_\rho, v_\sigma) = -m_2 l \sin(f_v^1) i_{\rho 1},$$

$$\hat{D}_{ii}(i_\rho, v_\sigma) = \hat{D}_{21}(i_\rho, v_\sigma) = 0 \ ,$$

$$\hat{D}_{12}(i_\rho, v_\sigma) = -m_2 l \sin(f_v^1) i_{\rho 1}.$$

Moreover $P(i_\rho, v_\sigma) = \tau^T i_\rho + {i_\rho}^T v_\sigma$. Regarding the matrix

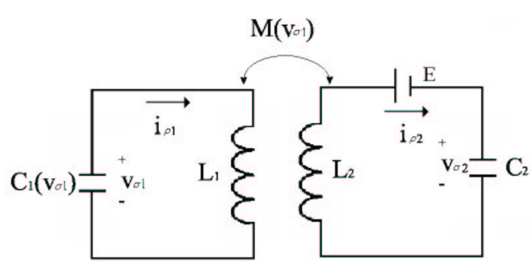


Fig. 6. Electrical interpretation of pendulum on the cart motion equations.

$C(v_\sigma)$, just consider the definition given in the previous section. Also in this case, replacing $\widetilde{D}(v_{\sigma 1})$ by the pseudo-inductance matrix $\widetilde{L}(v_{\sigma 1})$ and considering $F = E$, an electrical interpretation of (38) is provided and can be depicted in Fig. 7. Then, if we let $C_2 \to \infty$,[2] we obtain the equivalent electrical circuit for the inverted pendulum on a cart, that yields the electrical configuration shown in Fig. 7.

[2]Since $\frac{dv_{\sigma 2}}{dt} = C_2^{-1} i_{\sigma 2}$, as far as $i_{\sigma 2} = i_{\rho 2}$, the right term being limited by the presence of the inductor $L_2$, we are just assuming $v_{\sigma 2}$ as a constant which we consider to be $= 0$.

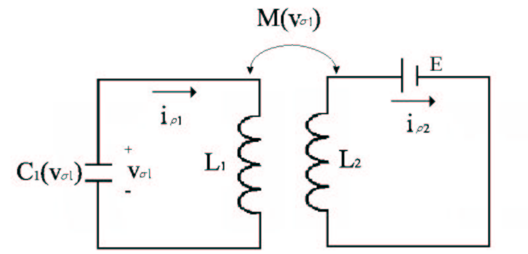


Fig. 7. Equivalent electrical circuit provided $C_2 \to \infty$.

## IV. Conclusion

In this paper, we have presented an electrical interpretation that fits within the Brayton-Moser framework of mechanical systems that move in the plane. First, we gave a new form to the Brayton-Moser framework introducing a new definition of coupled pseudo-inductors (9) where the mutual inductance is voltage controlled via the coupling coefficient $k_{ij} = k_{ij}(v_\sigma)$ defined in (13). Indeed, under the assumptions A1 and A2 of Lemma 1, applying Theorem 1 and considering the BM equations can now be written as

$$\begin{bmatrix} \widetilde{L}(v_\sigma) & \widetilde{M}(i_\rho, v_\sigma) \\ \\ 0 & -C(v_\sigma) \end{bmatrix} \begin{bmatrix} \frac{di_\rho}{dt} \\ \\ \frac{dv_\sigma}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial P}{\partial i_\rho} \\ \\ \frac{\partial P}{\partial v_\sigma} \end{bmatrix}. \tag{39}$$

where $\widetilde{L}(v_\sigma) = \widetilde{D}(v_\sigma)$ and $\widetilde{M}(i_\rho, v_\sigma) = -(\dot{\widetilde{D}}(v_\sigma) - \hat{D}(i_\rho, v_\sigma)$ justify the electrical interpretation presented for the given examples.


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## References


[1] M. Smith, "Synthesis of mechanical networks: the interter," *IEEE Trans. Aut. Contr.*, vol. 47, no. 10, pp. 1648–1662, october 2002.

[2] S. Stramigioli, *Modeling and IPC Control of Interactive Mechanical Systems: a coordinate free approach*. Springer Verlag, London, 2000.

[3] R. K. Bryton and J. K. Moser, "A theory of nonlinear networks (part 1)," *Quarterly Of Applied Mathematics*, vol. XXII, no. 1, pp. 1–33, april 1964.

[4] D. Jeltsema, R. Ortega, and J. M. A. Scherpen, "On passivity and power-balance inequalities of nonlinear *RLC* circuits," *IEEE Trans. Circ. Syst.*, vol. 50, no. 9, pp. 1174–1179, september 2003.

[5] D. Jeltsema and J. M. A. Scherpen, "On mechanical mixed potential, content and co-content," *Proceedings of the European Control Conference (ACC)*, pp. 73–78, 2003.

[6] R. Ortega, D. Jeltsema, and J. M. A. Scherpen, "Power shaping: A new paradigm for stabilization of nonlinear rlc circuits," *IEEE Trans. Aut. Cont.*, vol. 48, no. 10, pp. 1762–1767, october 2003.

[7] D. Jeltsema and J. M. A. Scherpen, "Tuning of passivity-preserving controllers for switched-mode power converters," *IEEE Trans. Aut. Cont.*, vol. 49, no. 8, pp. 1333–1344, 2004.

[8] S. Arimoto and T. Nakayama, "Another language for describing motions of mechatronics systems: a nonlinear position-dependent circuit theory," *IEEE/ASME Trans. Mechatronics*, vol. 1, no. 2, pp. 168–180, 1996.

[9] D. Jeltsema and J. M. A. Scherpen, "A power-based perspective mechanical systems," *accepted to CDC/ECC 05, Seville, Spain*, 2005.

[10] R. Ortega, A. Loria, P. J. Nicklasson, and H. Sira-Ramirez, *Passivity-based Control of Euler-Lagrange Systems*. Springer, London, 1998.

[11] J. Scherpen, D. Jeltsema, and J. Klaassens, "Lagrangian modeling of switching electrical networks," *Systems & Control Letters*, vol. 48, no. 5, pp. 365–374, april 2003.