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Absolute Stability Criteria for Systems with Sector or Norm Bounded Nonlinearities and Uncertain Delay

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***Abstract*—Finding conditions for absolute stability of a system containing a linear part and a scalar nonlinear sector restricted function is a classical Lur'e problem. Most of the corresponding results are based on the frequency domain or Lyapunov functions methods which are applied to systems with a time-invariant or periodic linear block. This paper develops a new approach to stability analysis of the problem based on a direct analysis of the corresponding integral Volterra equation about the input of the nonlinear block. The obtained sufficient stability criterion is applicable to non-autonomous systems with arbitrary time-varying delay in the feedback. The approach is extended to general time-varying systems including a linear block and norm bounded vector nonlinear terms with uncertain time-varying delays. The obtained delay-independent stability conditions are formulated in the terms of the transition matrix of the linear part and the norms of the nonlinear terms. The systems are indicated for which the obtained criteria are not only sufficient but also necessary for any delay function. The obtained results are applied to stability analysis of some systems previously studied in the literature; in all cases less conservative stability bounds are found.**

I.INTRODUCTION

The classical Lur'e problem is to find conditions for absolute stability of a control system consisting of a linear block and a nonlinear feedback contained within a prescribed sector [1]. Over the last few decades there has appeared an extensive literature devoted to the problem and its generalization. Most of the known results are obtained by the frequency domain or Lyapunov functions methods and relate to systems with a time-invariant or periodic linear block (e.g., [2]-[10]). The Lyapunov method enables, in principle, to tackle arbitrary time-varying systems; however, finding the Lyapunov function for such systems is, generally, a difficult problem.

In paper [11] sufficient stability conditions for the Lur'e problem which are equally applied to time-invariant and time-varying systems are found. The results are based on a direct analysis of the corresponding integral Volterra equation about the input of the nonlinear block $\sigma(t)$. In this paper we extend this approach to systems with delay in the feedback. Namely, we assume that the corresponding output is of the form $\varphi = \varphi(\sigma(t-\tau(t)),t)$ where the function $\tau(t)$ is piecewise continuous, nonnegative and bounded for $t \in [0,\infty)$. The corresponding integral equation becomes

$$\sigma(t) = f(t) + \int_0^t w(t,s)\varphi(\sigma(s-\tau(s)),s)\,\mathrm{d}s, \quad (1)$$

where $w(t,s)$ is the transfer function of the linear block. Note that no other information on the linear block is employed, so the last may be described by time-varying ordinary or partial differential equations with or without delay, integral equations, etc.

The piecewise continuous scalar valued function $f(t)$ describes a solution in the absence of the feedback for nonzero initial conditions and, perhaps, external perturbation disappearing at infinity. We assume that the linear block is exponentially stable, so

$$|f(t)| \to 0 \text{ as } t \to \infty. \quad (2)$$

The function $\varphi(\sigma,t)$ belongs to the class $\Phi(K_1,K_2)$, i.e. satisfies the inequality

$$K_1\sigma^2 \le \varphi(\sigma,t)\sigma \le K_2\sigma^2, \sigma \in (-\infty,\infty) \quad (3)$$

We assume that with a given initial function $\sigma(t)$ for $t<0$, the solution $\sigma(t)$ of equation (1) is continuable on $[0,\infty)$.

Definition. System (1) is called absolutely stable in the class $\Phi(K_1,K_2)$ if for any functions $f(t)$, $\varphi(\sigma,t)$, satisfying conditions (2), (3), and any piecewise continuous nonnegative bounded for $t \in [0,\infty)$ function $\tau(t)$, the corresponding solution $\sigma(t)$ of (1.1) satisfies the condition

$$|\sigma(t)| \to 0 \text{ as } t \to \infty. \quad (4)$$

If condition (4) is not fulfilled for some $\varphi(\sigma,t)$, $\tau(t)$ and $f(t)$ from the indicated classes, the system is referred to as unstable.

Putting $\varphi_1(\sigma,t) = \varphi(\sigma,t) - K_1\sigma - K\sigma$, $K = (K_2 - K_1)/2$ and returning to the previous notation, we reduce (3) to the form

$$-K\sigma^2 \le \varphi(\sigma,t)\sigma \le K\sigma^2, \ \sigma \in (-\infty,\infty). \quad (5)$$

Thus, we replace the class $\Phi(K_1,K_2)$ by $\Phi(-K,K)$; thus we assume that the transfer function $w(t,s)$ in (1) is changed correspondingly.

In Section 2 a value K_* is found such that for $K < K_*$, the system is absolutely stable independent on the delay function $\tau(t)$ (Theorem 1). For some linear parts (including,

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in particular, the autonomous ones) the value K_0 is found such that the system is unstable in the class $\Phi(-K_0, K_0)$ for any $\tau(t)$ (Theorem 2).

In Section 3 systems with a nonnegative transfer function are considered. It is shown (Theorem 3) that asymptotic stability for $\varphi(\sigma,t) = K\sigma$ guarantees absolute stability of the system in the class $\Phi(-K,K)$. Thus, such systems for arbitrary delay $\tau(t)$ in the feedback, satisfy the Aizerman conjecture [12] (note that the known results of such kind [13-15] relate to time-invariant systems). Under some additional condition, a precise upper bound for the stability sector is found (Theorem 4).

In Section 5 applications of the obtained results to some systems are discussed. It is shown that a closed-loop system consisting of any number of first order time-varying links and arbitrary delay in the feedback satisfies the Aizerman conjecture in the class $\Phi(-K,K)$. The efficiency of the derived stability criterion for systems with vector nonlinearities is illustrated by examples which were studied in the literature using different stability conditions.

In Section 4 the developed approach is extended to systems consisting of a linear block and norm bounded vector nonlinear terms with uncertain time-varying delays. An explicit delay-independent sufficient stability condition, expressed in the terms of the transition matrix of its linear part and bounds for the norms, is obtained (Theorem 5). This condition turns out to be also necessary (Theorem 6) when the matrix of the linear part is symmetric and time-invariant. We show that the last system satisfies the Aizerman conjecture for any delays in the nonlinear terms.

II. ABSOLUTE STABILITY AND INSTABILITY CRITERIA

Suppose that the linear block is exponentially stable, then

$$|w(t,s)| \le C\exp[-\Delta(t-s)], \qquad (6)$$

where the constants *C* and $\Delta > 0$ are independent on *t* and *s*.

Let us put

$$W(t) = \int_0^t |w(t,s)|\,\mathrm{d}s, \quad W_+(t_k) = \sup W(t) \text{ for } t \ge t_k,$$

$$W_\infty = \overline{\lim_{t\to\infty}}\, W(t) = \lim_{t_k\to\infty} W_+(t_k). \qquad (7)$$

Here W_∞ is the upper limit of $W(t)$ as $t \to \infty$; it coincides with the conventional limit when the last exists. This is certainly the case when the linear block is time-invariant. Really, here $w(t,s) = w(t-s)$, so, setting $t - s = z$, we obtain

$$W(t) = \int_0^t |w(z)|\,\mathrm{d}z. \qquad (8)$$

The function $W(t)$ in (8) increases monotonically and, therefore, tends to a limit.

The following theorem establishes a sufficient condition for absolute stability of system (1), (5).

Theorem 1. If

$$K < K_* = 1/W_\infty, \qquad (9)$$

the system is absolutely stable in the class $\Phi(-K,K)$.

Proof. Let $\sigma(t)$ be a solution of equation (1). First let us show that for any $t_1 \ge 0$, there exists $t_m \ge t_1$ such that $|\sigma(t_m)| \ge |\sigma(t)|$ for $t \in [t_1, \infty)$. Really, otherwise, there is a sequence $t_1, t_2, ...;\ t_k \to \infty$ as $k \to \infty$, such that $|\sigma(t)| \le |\sigma(t_k)|$ for $t \in [t_1, t_k]$. Then from (1) and (5) we have

$$|\sigma(t_k)| \le R(t_k,t_1) + K\int_{t_1}^{t_k} |w(t_k,s)||\sigma(s-\tau(s))|\,\mathrm{d}s \le R(t_k,t_1) + KW(t_k)|\sigma(t_k)| \qquad (10)$$

where

$$R(t_k,t_1) = |f(t_k)| + K\int_0^{t_1} |w(t_k,s)||\sigma(s-\tau(s))|\,\mathrm{d}s.$$

Observing that $W(t_k) \le W_+(t_k)$, $W_+(t_k) \to W_\infty$, $R(t_k,t_1) \to 0$ as $t_k \to \infty$ and, by (9), $KW_\infty < 1$, we find that inequality (10) cannot hold for large k. The contradiction obtained shows that there exists a sequence t_m, $m = 1,2,...$ such that $t_m \to \infty$ as $m \to \infty$ and $|\sigma(t_m)| \ge |\sigma(t)|$ for $t \in [t_m, \infty)$. Evidently, $|\sigma(t_m)| \ge |\sigma(t_{m+1})| \ge 0$, therefore; there exists $\sigma_\infty = \lim |\sigma(t_m)|$ as $t_m \to \infty$. Let us prove that $\sigma_\infty = 0$.

By assumption, $\tau(t) \le h$ for some h. Assuming $t_m - t_i \ge h$, analogously (10) we find

$$|\sigma(t_m)| \le R(t_m,t_i) + K\int_{t_i}^{t_m} |w(t_m,s)||\sigma(s-\tau(s))|\,\mathrm{d}s \le R(t_m,t_i) + KW(t_m)|\sigma(t_i)|. \qquad (11)$$

Since the sequence $|\sigma(t_m)|$, $m = 1,2,...$ is convergent, then for any $\varepsilon > 0$, there exists such *i* that $|\sigma(t_i| - |\sigma(t_m| < \varepsilon$ for all $m > i$. Therefore, from (11) we find

$$|\sigma(t_m)|[1 - K\,W(t_m)] < R(t_m,t_i) + \varepsilon K\,W(t_m) \qquad (12)$$

Since $R(t_m,t_i) < \varepsilon$ for large $t_m - t_i$, $\lim KW_+(t_m) = KW_\infty < 1$ as $t_k \to \infty$ and $W(t_m) \le W_+(t_m)$, then $KW(t_m) < 1$ for large m. Therefore, inequality (12) is true only if $|\sigma(t_m)| \to 0$ as $m \to \infty$, i.e. $|\sigma(t)| \to 0$ as $t \to \infty$. □

Let us now obtain a condition guaranteeing instability of the system. To this end, we put

$$W^0(t) = \int_0^t w(t,s)\,\mathrm{d}s. \tag{13}$$

Suppose there exists

$$W_0 = \lim_{t\to\infty} W^0(t) \neq 0. \tag{14}$$

Theorem 2. If

$$K \geq K_0 = 1/|W_0|, \tag{15}$$

then system (1),(2) is unstable.

Proof. Let us put

$$f_0(t) = 1 - W^0(t)/W_0,\ \varphi_0(\sigma) = K_0\sigma\,\mathrm{sgn}\,W_0, \quad \sigma(t) \equiv 1 \text{ for } t < 0. \tag{16}$$

In view of (14) and (16), $|f_0(t)| \to 0$ as $t \to \infty$; by (15), $\varphi_0(\sigma) \in \Phi(-K,K)$. By a direct substitution one can check that $\sigma(t) \equiv 1$ is the corresponding solution of (1). Since it does not satisfy condition (4), the system is unstable. □

Let K_b be the value of the constant K such that the system is stable in the class $\Phi(-K,K)$ for $K < K_b$ and unstable for $K \geq K_b$. Then from Theorems 1 and 3 it follows that K_b satisfies the inequality

$$1/W_\infty \leq K_b \leq 1/|W_0|. \tag{17}$$

III. SYSTEMS WITH SIGN-CONSTANT TRANSFER FUNCTION

Suppose now that the transfer function $w(t,s)$ is sign-constant. Without loss of generality, we assume that

$$w(t,s) \geq 0 \quad \text{for} \quad t \geq s \geq 0. \tag{18}$$

Theorem 3. System (1), (18) is absolutely stable in the class $\Phi(-K,K)$ if it is stable for $\varphi = K\sigma(t-\tau(t))$.

Proof. Let $\sigma_0(t)$ be the solution of the equation

$$\sigma_0(t) = f_0(t) + K\int_0^t w(t,s)\sigma_0(s-\tau(s))\,\mathrm{d}s, \tag{19}$$

where

$$\sigma_0(t) = |\sigma(t)| \quad \text{for } t < 0, \qquad f_0(t) = |f(t)| + \exp(-t) \ \text{ for } t \geq 0. \tag{20}$$

Clearly, $\sigma_0(t-\tau(s)) > |\sigma(t-\tau(s))| > 0$ for sufficiently small $t > 0$. Let us show that this inequality cannot break as t increases. Really, let $\sigma_0(t_1 - \tau(t_1)) = \sigma(t_1 - \tau(t_1))$ for some t_1, then, subtracting (1.1) from (3.2), we find

$$0 = |f(t_1)| - f(t_1) + \exp(-t_1) + \int_0^{t_1} w(t_1,s)[K\sigma_0(s-\tau(s)) - \varphi(\sigma(s-\tau(s),s)]\,\mathrm{d}s, \tag{21}$$

which is impossible, because the right-hand side of (21) is positive ($K\sigma_0 > \varphi(\sigma)$ for $\sigma_0 > |\sigma|$). `

If $\sigma_0(t_1) = -\sigma(t_1)$, then, summing (19) and (1), we find

$$0 = |f(t_1)| + f(t_1) + \exp(-t_1) + \int_0^{t_1} w(t_1,s)[K\sigma_0(s-\tau(s)) + \varphi(\sigma(s-\tau(s),s)]\,\mathrm{d}s,$$

where the right-hand side is also positive.

The obtained contradiction shows that $\sigma_0(t) > |\sigma(t)|$ for $t > 0$ and, therefore, $\sigma(t) \to 0$ as $t \to \infty$. □

Suppose, moreover, that limit (14) exists.

Theorem 4. For absolute stability of system (1), (18), it is necessary and sufficient that

$$K < 1/W_\infty. \tag{22}$$

In fact, by (18), $W(t) = W^0(t)$, $W_\infty = W_0$, so Theorem 4 follows directly from inequality (17).

IV. SYSTEMS WITH VECTOR NONLINEARITIES

The above approach can be extended to a linear system with arbitrary time-varying delays and nonlinear vector perturbations:

$$\dot{x}(t) + A(t)x(t) = \sum_{i=1}^{k} f_i(x(t-\tau_i(t),t), \tag{23}$$

where $x \in R^n$ and $\tau_i(t)$ are piecewise continuous functions ($\tau_i(t) \leq h$ for $t \in [0,\infty)$ where h is an arbitrary constant). The nonlinear perturbations are norm-bounded, i.e.

$$\|f_i(x,t)\| \leq \alpha_i(t)\|x\|,\ i = 1,\ldots,k, \tag{24}$$

where $\alpha_i(t)$ are continuous bounded functions, $\|a\|$ denotes the Euclidean norm of a vector a.

The initial condition is given by

$$x(t) = \varphi(t) \ \text{ for } t \in [-h,0], \tag{25}$$

where $\varphi(t)$ is a continuous function.

We assume that for any admissible $\varphi(t), \tau_i(t)$ and $f_i(x,t)$, the corresponding solution $x(t)$ is continuable on $[0,\infty)$.

We call system (23) absolutely stable if for any choice of the functions $\varphi(t), \tau_i(t)$ and $f_i(x,t)$ satisfying the above conditions, the corresponding solution

$$x(t) \to 0 \text{ as } t \to \infty . \qquad (26)$$

The system is called unstable if (26) fails for some admissible $\varphi(t), \tau_i(t)$ and $f_i(x,t)$.

Note that recently a significant attention has been devoted to stability analysis of systems with an uncertain delay. A number of stability conditions for such systems were developed using various approaches (see, e.g. papers [19]-[22] and survey [23]). The *delay-dependent* conditions contain a prescribed upper bound for the uncertain delays, the *delay-independent* ones relate to systems for which such a bound can be arbitrarily large. Below we obtain delay-independent conditions for absolute stability of system (23), (24).

Denote by $W(t,s)$ $(W(t,t) = I$ where I is the unit matrix) the transition matrix of the equation

$$\dot{x}(t) + A(t)x(t) = 0 . \qquad (27)$$

Let us put

$$w(t,s) = \sum_{i=1}^{k} \alpha_i(s) \|W(t,s)\| \|x(s - \tau_i(s))\| ,$$

$$v(t,s) = \sum_{i=1}^{k} \alpha_i(s) \|W(t,s)\| , \qquad V(t) = \int_0^t v(t,s)\, \mathrm{d}s , \quad (28)$$

where $\|A\|$ denotes the induced norm of the matrix A which is equal to the largest eigenvalue of the matrix $(A^T A)^{1/2}$.

Suppose that 27) is exponentially stable, then

$$\|w(t,s)\| \le C \exp[-\Delta(t-s)] , \qquad (29)$$

where the constants C and $\Delta > 0$ are independent on t and s.

The following theorem provides a sufficient stability condition for system (23), (24),(29).

Denote by V_∞ the upper limit of $V(t)$.

Theorem 5. If

$$V_\infty < 1 , \qquad (30)$$

then the system is absolutely stable.

Proof. The solution $x(t)$ of equation (23) satisfies the relation

$$x(t) = W(t,0)x(0) + \int_0^t W(t,s) \sum_{i=1}^{k} f_i(x(s - \tau_i(s), s))\, \mathrm{d}s . \quad (31)$$

From (31), (24) and (28) it follows that

$$\|x(t)\| \le \|W(t,0)x(0)\| + \int_0^t w(t,s)\, \mathrm{d}s . \qquad (32)$$

The further proof is analogous to that of Theorem 1. □

Let us indicate a class of systems for which stability condition (30) is not only sufficient but is necessary as well.

Consider the system

$$\dot{x}(t) + Ax(t) = \sum_{i=1}^{k} f_i((x(t - \tau_i(t), t) , \qquad (33)$$

where A is a constant symmetric matrix ($A = A^T$) and the functions $f_i(x,t), i = 1,..,n$ satisfy inequalities (24) with constant α_i.

In view of the symmetry of the matrix A, its eigenvalues $\lambda_i, i = 1,...,n$ are real [24]. Since, by supposition, equation (27) is exponentially stable, the smallest eigenvalue, $\lambda_1 > 0$.

The following theorem shows that here stability condition (30) is necessary for any delays $\tau_i(t)$.

Theorem 6. For absolute stability of system (33), it is necessary and sufficient that

$$\sum_{i=1}^{k} \alpha_i < \lambda_1 . \qquad (34)$$

Proof. Since the matrix A_0 is constant, then

$$W(t,s) = \exp[-(t-s)A_0] . \qquad (35)$$

By (35), the eigenvalues of the matrix $W(t,s)$, $\mu_i = \exp[-\lambda_i(t-s)],\ i = 1,...,n$. Since A is symmetric, $W(t,s)$ is symmetric as well, so

$$\|W(t,s)\| = \max_i |\mu_i| = \mu_1 = \exp[-\lambda_1(t-s)] . \qquad (36)$$

Then from (28), and (36) we have

$$v(t,s) = \sum_{i=1}^{k} \alpha_i \exp[-\lambda_1(t-s)] , \quad V_\infty = \frac{1}{\lambda_1} \sum_{i=1}^{k} \alpha_i . \qquad (37)$$

Thus, inequalities (30) and (34) are identical and consequently the condition (34) is sufficient. To prove its necessity, we consider the marginal system

$$\dot{x}(t) + Ax(t) = \left(\sum_{i=1}^{k} \alpha_i \right) x(t - \tau_i(t)) . \qquad (38)$$

Let x_1^0 be the eigenvector of the matrix A_0 corresponding to the eigenvalue λ_1. By a direct substitution, one can check that for $\sum_{i=1}^{k} \alpha_i = \lambda_1$, equation (38) admits the solution

$x(t) \equiv x_1^0$. Since the last does not satisfy (26), equation (38) is unstable, which proves the necessity of condition (34). □

V. DISCUSSION

Stability condition (9) can be applied to a wide range of systems with, generally, time-varying linear block and arbitrary delay $\tau(t)$ in the feedback. Under this condition, a system is absolutely stable in the class $\Phi(-W_\infty^{-1}+\varepsilon, W_\infty^{-1}-\varepsilon)$ where $\varepsilon > 0$ is an arbitrary small value. If limit (14) exists, the system is certainly unstable in the wider class $\Phi(-W_0^{-1}, W_0^{-1})$ (Theorem 2).

The Lur'e problem was first formulated for the system

$$\dot{x} = Ax + b\varphi(cx) \tag{39}$$

where $x \in R^n$, b and c are column and row vectors, correspondingly. The problem is reduced to (23) where $\varphi = \varphi(\sigma), \sigma = cx$ and $w(t,s) = w(t-s)$, because (39) is time-invariant.

In 1949 Aizerman conjectured [12] that system (25) is absolutely stable in the class $\varphi(\sigma) \in \Phi(K_1, K_2)$, provided that the linear system $\dot{x} = Ax + kbcx$ is stable for any $k \in [K_1, K_2]$. Subsequently counterexamples showed that this conjecture is, in general, false (the history of the Aizerman conjecture can be found in the book by Gil' [13]). So, the problem is to find classes of systems satisfying the Aizerman conjecture. The first result in this direction was obtained by Gil' [13] who proved that if in system (39) the transfer function is nonnegative, then its absolute stability in the class $\Phi(0, K)$ is guaranteed by stability of the system $\dot{x} = Ax + Kbcx$. Recently he extended this result to distributed and delay time-invariant systems [14, 15].

In paper [11] it was shown that stability of a time-variable system with a nonnegative transfer function in the class $\Psi(-K, K)$ is guaranteed by stability for $\varphi(\sigma,t) = K\sigma$. Theorem 3 of the present paper extends this result to systems with arbitrary delay $\tau(t)$ in the feedback. If in (14) the limit W_0 exists, the precise bound for the stability sector equals $K = 1/W_\infty$ (Theorem 4) for any delay $\tau(t)$. Note that at first sight the invariance of the stability sector on $\tau(t)$ looks surprising; however, this is due to the fact that for $K = 1/W_\infty$ and $f(t)$, determined by (16), equation (1) admits the 'unstable' solution $\sigma(t) \equiv 1$ for any $\tau(t)$.

Let the linear block be a closed-loop system consisting of n (generally, time-varying) links. Suppose that the individual transfer functions $w_i(t,s), i = 1,\ldots,n$ are sign-constant. For a sign-constant input, the output of each link is sign-constant as well, therefore, so is the transfer function $w(t,s)$ of the entire linear block.

Suppose, in particular, that the links are of the first order, i.e. the linear block is described by the equations

$$\begin{aligned} &\dot{x}_1 + a_1(t)x_i = 0, \\ &\dot{x}_i + a_i(t)x_i = k_i x_{i-1},\ i = 2,\ldots,n. \end{aligned} \tag{40}$$

There exists an extensive literature devoted to an analysis of feasibility of the Aizerman conjecture to closed-loop systems with first order time-invariant links and a feedback $\varphi(\sigma)$. Bergen and Williams proved [17] that systems of the third order satisfy this conjecture. Trukhan extended this result on systems with up to five stable links [18]. For an arbitrary number of links, the transfer function is positive, so stability in the class $\Phi(0, K)$ follows from Gil' theorem [13]. Let us show that the above findings enable us to essentially generalize these results in some respects.

Evidently, the individual transfer functions of a link,

$$w_i(t,s) = \exp[-\int_s^t a_i(s)\,\mathrm{d}s],\ i = 1,\ldots,n, \tag{41}$$

is positive, hence, the transfer function of time-varying system (40) is positive as well. So, from Theorem 3 it follows that system (40) with the feedback $\varphi(\sigma(t-\tau(t)),t)$ is absolutely stable in the class $\Phi(-K, K)$, provided that it is stable for $\varphi = K\sigma(t-\tau(t),t)$. If in (14) the limit W_0 exists, then for any prescribed delay $\tau(t)$, the obtained bound for the stability sector coincides with the upper bound of the Hurwitz angle, i.e. $K_* = 1/W_\infty = K_2^0$ (Theorem 4).

For system (33) with a symmetric constant matrix A, necessary and sufficient stability condition is provided by inequality (34) which does not depend on the delays $\tau_i(t)$ (Theorem 5). Note that for system (33) with arbitrary constant matrix A, sufficient stability condition in the form (34) was found in [14, p.192].

To check the efficiency of stability criterion (30), let us consider some system which were previously studied in the literature via various criteria.

Example 1. Consider the system

$$\begin{aligned} &\dot{x}(t) + Ax(t) = f(x(t-\tau(t),t), \\ &x \in R^2,\ A_0 = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix},\ \|f(x,t)\| \le \alpha\|x\|. \end{aligned} \tag{41}$$

The transition matrix of the system $\dot{x}(t) + Ax(t) = 0$ is

$$W(t) = \begin{bmatrix} 2\exp(-2t) - \exp(-t) & 2\exp(-2t) - 2\exp(-t)) \\ -\exp(-2t) + \exp(-t) & -\exp(-2t) + 2\exp(-t) \end{bmatrix}.$$

By Theorem 5, system (41) is absolutely stable for arbitrary bounded $\tau(t)$, provided that

$$V_\infty = \alpha \int_0^\infty \|W(z)\| \, \mathrm{d}z < 1, \tag{42}$$

where $z = t - s$, $\|W(z)\|$ is the largest eigenvalue of the matrix $(W(z)^T W(z))^{1/2}$. The calculations offer $V_\infty = 1.9289$, so that system (41) is absolutely stable for $\alpha \le 0.5184$. This result substantially improves the known bounds, $\alpha \le 0.1458$, $\alpha \le 0.178$ and $\alpha \le 0.2389$, obtained by different criteria in papers [20] and [21].

Example 2. Consider the system

$$\dot{x}(t) + A_0 x(t) = \alpha A x(t - \tau(t)),$$
$$x \in R^2,\ A_0 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{43}$$

Here

$$W(z) = \begin{bmatrix} \exp(-2z) & -z\exp(-2z) \\ 0 & \exp(-2z) \end{bmatrix},$$
$$W(z)A = \begin{bmatrix} -z\exp(-2z) & \exp(-2z) \\ \exp(-2z) & 0 \end{bmatrix}.$$

Since the matrix $W(z)A$ is symmetric, $\|W(z)A\| = \max|\lambda_i|$, where $\lambda_i, i = 1,2$ are the eigenvalues of the matrix $W(z)A$. Condition (30), taking form (42) in this case, implies that system (43) is absolutely stable for $\alpha \le 1.5322$.

Note that system (43) with $\alpha = 1$ and constant delay was studied in [22] and [23] where it was shown that the system is absolutely stable if $\tau < 0.3624$ and $\tau < 0.4212$, respectively. In fact, the system is stable for arbitrary time-varying bounded delay $\tau(t)$ as follows from the above result.

Thus, in the considered examples the criterion presented in this paper provides less conservative stability bounds for the uncertain terms as compared with the known ones.

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