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A distributed Nyquist criterion for heterogeneous networks

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Abstract— **We derive a distributed Nyquist type criterion that can certify scalable robust stability for linearly interconnected heterogeneous dynamical systems. The result holds for linear SISO dynamical systems with bidirectional links between them. Unlike previous results of this kind, we allow for otherwise arbitrary interconnection topologies (i.e linear systems on arbitrary underlying undirected graphs). Each agent is required to satisfy a local test that involves only a knowledge of its own dynamic and those of its neighbours; a new agent introduces only an additional such condition hence the stability certificates scale with the network size.**

I. INTRODUCTION

Analysis and decentralized control of interconnected dynamical systems has traditionally received considerable attention by the control community. Typical examples of early work include the dissipativity approach in [1] and [2] and the use of vector Lyapunov functions in [3]. A renewed interest in the recent years has led to work on decentralized optimal control design, such as the spatially invariant case in [4] and the relaxation to a more general setting using LMI techniques in [5] and [6].

Nevertheless in many applications including data networks, flocking phenomena, financial markets decentralization is not sufficient: scalability is also a major property that needs to be maintained without often being feasible to implement sophisticated control strategies. By this we mean that we want on the one hand stability guarantees for the entire network with each agent satisfying a rule that involves only local information, but at the same time we require that network stability is preserved even when the network is modified with the addition/removal of agents.

As we are looking for decentralized stability certificates that hold for arbitrary interconnections, the degree of conservatism will inevitably be based on an interplay between possible structure in the interconnections or certain homogeneity assumption in the participating dynamics. This kind of structure has played a substantial role in most of the scalable stability results available so far.

For example, in [1] and [2] a generalized dissipativity description with quadratic supply rates is adopted for participating agents. By appropriate conditions on the interconnection matrix the summation of individual storage functions becomes a common Lyapunov function for the interconnected system. Note, however, that such interconnection constraints can be rather conservative for arbitrary networks unless these constraints appear naturally within the interconnection protocol obeyed by the system. Decentralized stability conditions of Internet congestion control protocols as described in [7] and references therein, are an example of an application where scalability is important. These protocols, however, impose a very special interconnection structure that simplifies the analysis [8].

Interconnection constraints are being relaxed in the paper by deriving stability conditions that involve not only the dynamics of an individual agent, but also those of its neighbours i.e. only cycles of length two are taken into account. Scalability follows from the fact that a new agent introduces only an additional such decentralized condition.

Our main result, roughly speaking, requires that a certain convexification of each "loop gain" (i.e. product of neighbouring dynamics) satisfies a common Nyquist-like condition. This is an extension to arbitrary underlying graphs of ideas in [9] where a bipartite interconnection structure has been exploited to derive decentralized delay stability conditions for congestion control models in data networks. The motivation for working in the frequency domain in this way goes far beyond the possibility of developing an appealing graphical test to verify stability: converting the stability certification to a spectral inclusion problem of a complex matrix allows one to exploit the internal structure present in the system through the use of the numerical range together with tools from convex and complex analysis. The S-hull, a relaxed notion of convexity in the complex plane, is used as a crucial tool in this direction since it enables to define numerical range type spectral bounds with only the frequency responses of participating agents, in a distributed way. Note that the linear analysis is rather more involved than a mere dissipativity argument as it requires taking the square root of frequency response functions. It is thus intriguing to contemplate what any nonlinear generalization of this theory might look like.

Finally, even though we are not considering controller design with performance criteria, robust stability certificates in such networks (where scalability is important) can have substantial contribution in their design. This is because one can see whether part of the network is being unnecessarily too conservative in its response. For example, an important contribution along these lines has been a theoretical justification in [10] that TCP is being too sluggish at high bandwidths, with a proposal for modification in [11].

The paper is structured as follows. In section II we prove

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the main lemmas on which the stability results are going to be based. We define the S-hull and show how it can be used to bound the Numerical Range of matrices with a structure relevant to our analysis. In section III we give the stability results, for bipartite graphs first and then for arbitrary graphs. Examples of potential application, such as Internet and consensus protocols, are finally outlined.

II. Preliminary Results

A. Notation

The field of real and complex numbers are denoted by $\mathbb{R}$, $\mathbb{C}$ respectively. $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$ are the m by n matrices with elements in the corresponding fields. $\mathbb{R}_+$ is the set of non-negative reals and $\overline{\mathbb{C}}_+$ the closed right half plane. $\sigma(M)$ denotes the spectrum of a square matrix M, $\rho(M)$ its spectral radius, $|M|$ the elementwise absolute value of the matrix i.e. $|[M_{ij}]| := [|M_{ij}|]$, $M_{i\bullet}$ is the ith row and $M_{\bullet j}$ the jth column. $Co(S)$ denotes the convex hull of a set S and $\mathrm{diag}(x_i)$ the matrix with elements $x_1, x_2, \ldots$ on the leading diagonal and zeros elsewhere. We denote the square of a set $P \subset \mathbb{C}$ as the set of the squares of its elements i.e. $P^2 = \{p^2 : p \in P\}$.

The Numerical Range of a matrix $M \in \mathbb{C}^{n\times n}$ is the set $N(M) := \{v^*Mv : v \in \mathbb{C}^n, v^*v = 1\}$. The property $\sigma(M) \subset N(M)$ is frequently used in this paper (see e.g. [12] [13] for a more detailed discussion of the properties of the Numerical Range).$\mathscr{H}_\infty$ is the set of proper transfer functions analytic and bounded in $\overline{\mathbb{C}}_+$. $\mathscr{C}_0$ is the class of functions continuous in $j\mathbb{R} \cup \{\infty\}$ and $\mathscr{A}_0 := \mathscr{H}_\infty \cap \mathscr{C}_0$.

B. The S-hull and the Numerical Range

Definition 1 (S-hull): Let $P \subset \mathbb{C}$. The S-hull of set P is defined as

$$S(P) := (Co(\sqrt{P}))^2 \qquad \text{where} \quad \sqrt{P} := \{x : x^2 \in P\}$$

Given a set P it is always true that $0 \in S(P)$ and[1] $S(P) \supseteq Co\{P \cup 0\}$. The discrepancy between the S-hull and the corresponding Convex Hull in the last inclusion is generally small relative to the size of P (see example in fig. 1). The S-hull is a map that plays a crucial role in this paper because it relates the numerical range of a product of matrices with a particular structure, which is relevant to our analysis, to the nonzero elements of those matrices. The following lemma is a key result in this context and gives an alternative definition for the S-hull.

Lemma 1: Let $g_i \in \mathbb{C}$ for $i = 1, 2, \ldots, n$. Then

$$\left\{ \frac{(\sum_i a_i^* g_i)(\sum_i a_i g_i)}{(\sum_i |a_i|)^2} : a_i \in \mathbb{C}, i = 1, \ldots, n \right\} = S(\{g_i^2 : i = 1, \ldots, n\}) \quad (1)$$

Proof: Main Observation:

$$(Co(\{\pm g_i : i = 1, \ldots, n\}))^2 = \left(\left\{ \frac{\sum_i \lambda_i g_i}{\sum_i |\lambda_i|} : \lambda_i \in \mathbb{R}, i = 1, \ldots, n \right\} \right)^2 \quad (2)$$

[1]The statement $0 \in S(P)$ follows easily from the fact that $Co(\sqrt{P})$ is symmetric about the origin. $S(P) \supseteq Co\{P \cup 0\}$ is true since $S(P)$ is a convex set (Theorem 3 in the appendix) and $S(P)$ always includes P and 0.

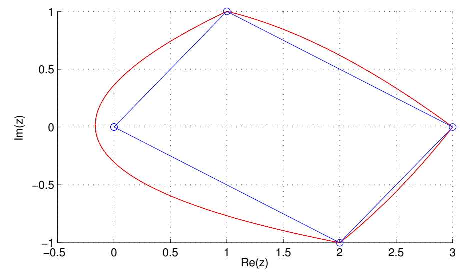


Fig. 1. Comparing the S-hull and Convex Hull of a set that includes the origin.

Let $a = [\, a_1 \quad a_2 \quad \ldots \quad a_n \,]^T, g = [\, g_1 \quad g_2 \quad \ldots \quad g_n \,]^T$.

$$\begin{aligned}
\text{Then} \quad & \left(\sum_i a_i^* g_i\right)\left(\sum_i a_i g_i\right) = \\
&= (\Re(a) - j\Im(a))^T g \;.\; (\Re(a) + j\Im(a))^T g \\
&= (\Re(a^T)g)^2 + (\Im(a^T)g)^2 \\
&\in \left(\sum_i |\Re(a_i)|\right)^2 (Co(\{\pm g_i : i = 1, \ldots, n\}))^2 + \\
&\quad \left(\sum_i |\Im(a_i)|\right)^2 (Co(\{\pm g_i : i = 1, \ldots, n\}))^2 \quad \text{(using (2))} \\
&\subset \left[\left(\sum_i |\Re(a_i)|\right)^2 + \left(\sum_i |\Im(a_i)|\right)^2 \right] . \\
&\qquad Co\left((Co(\{\pm g_i : i = 1, \ldots, n\}))^2 \right)
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{(\sum_i a_i^* g_i)(\sum_i a_i g_i)}{(\sum_i |a_i|)^2} &\in \underbrace{\left(\frac{(\sum_i |\Re(a_i)|)^2 + (\sum_i |\Im(a_i)|)^2}{\left(\sum_i \sqrt{[\Re(a_i)]^2 + [\Im(a_i)]^2}\right)^2} \right)}_{\leq 1} . \\
&\qquad Co\left((Co(\{\pm g_i : i = 1, \ldots, n\}))^2 \right) \\
&\subset Co\left((Co(\{\pm g_i : i = 1, \ldots, n\}))^2 \right) \quad (3)
\end{aligned}$$

Using the fact that $(Co(\{\pm g_i : i = 1, \ldots, n\}))^2$ is a convex set (proved in Theorem 3 in the appendix), (3) gives the following inclusion for Lemma 1.

$$\frac{(\sum_i a_i^* g_i)(\sum_i a_i g_i)}{(\sum_i |a_i|)^2} \subset (Co(\{\pm g_i : i = 1, \ldots, n\}))^2$$

The inclusion in the reverse direction is obvious from (2), since we can always choose real a_i's so as to match the λ_i's in (2) . ■

The following is the main lemma in this section that gives a bound for the numerical range of a product of matrices with a specific structure. As it will be discussed in detail in section III, this structure can characterize the return ratio of any symmetric linear interconnection of dynamical systems in a bipartite graph. The importance of this bound is that it is the convex hull of S-hulls of products of elements which are adjacent in a graph theoretic sense. This graph theoretic interpretation is an important part of the problem formulation given in section III-A.

Lemma 2: Let $R \in \mathbb{C}^{m\times n}$ satisfy $\rho(|R|^T|R|) \leq 1$, and $G = \mathrm{diag}(g_1,\dots,g_n)$, $F = \mathrm{diag}(f_1,\dots,f_m)$, $g_i, f_j \in \mathbb{C}\ \forall i,j$ then

$$N(G^{1/2}R^*FRG^{1/2}) \subset Co\left(\{f_i S(\{g_k : R_{ik} \neq 0\}) : i = 1,\dots,m\}\right)$$

where $G^{1/2} = \mathrm{diag}(\sqrt{g_1},\dots,\sqrt{g_n})$, and either of the square roots can be used.

Proof:

$$\rho(|R|^T|R|) \leq 1 \Rightarrow \quad v^*|R|^T|R|v \leq 1\ \forall v \in \mathbb{C}^m \text{ s.t. } v^*v = 1$$

$$\text{since} \quad \rho(|R|^T|R|) = \||R|\|_2^2 = \sup_{v\in\mathbb{C}^n, v\neq 0} \frac{\||R|v\|_2^2}{\|v\|_2^2}$$

expanding $|R|^T|R|$ we get

$$\sum_i (|v_1|R_{i1}| + v_2|R_{i2}| + \dots|)^2 \leq 1\ \forall v \in \mathbb{C}^n \text{ s.t. } v^*v = 1$$

And since this is true for all such v

$$\sum_i (|v_1 R_{i1}| + |v_2 R_{i2}| + \dots)^2 \leq 1\ \forall v \in \mathbb{C}^n \text{ s.t. } v^*v = 1 \tag{4}$$

Considering now the definition of the Numerical Range note that

$$v^*G^{1/2}R^*FRG^{1/2}v = \sum_{k=1}^m f_k(v^*G^{1/2}R^*_{k\bullet}R_{k\bullet}G^{1/2}v)$$

$$= \sum_{k=1}^m f_k \left(\sum_{j=1}^n v_j^* R_{kj}^* \sqrt{g_j}\right)\left(\sum_{j=1}^n v_j R_{kj}\sqrt{g_j}\right) \tag{5}$$

$$= \sum_{k=1}^m \left(\sum_{j=1}^n |v_j R_{kj}|\right)^2 f_k \frac{\left(\sum_{j=1}^n v_j^* R_{kj}^* \sqrt{g_j}\right)\left(\sum_{j=1}^n v_j R_{kj}\sqrt{g_j}\right)}{\left(\sum_{j=1}^n |v_j R_{kj}|\right)^2}$$

$$\in \sum_{k=1}^m \left(\sum_{j=1}^n |v_j R_{kj}|\right)^2 \cdot$$

$$Co\left\{ f_k \frac{\left(\sum_{j=1}^n v_j^* R_{kj}^* \sqrt{g_j}\right)\left(\sum_{j=1}^n v_j R_{kj}\sqrt{g_j}\right)}{\left(\sum_{j=1}^n |v_j R_{kj}|\right)^2} : k = 1,\dots,m \right\}$$

$$\subset \sum_{k=1}^m \left(\sum_{j=1}^n |v_j R_{kj}|\right)^2 \cdot \qquad \text{(using Lemma 1)}$$

$$Co\left(\left\{f_k S(\{g_j : R_{kj} \neq 0\}) : k = 1,\dots,m\right\}\right) \tag{6}$$

From (4) we note that the convex hull in (6) is multiplied by a real number in $[0,1]$. We hence deduce that

$$\{v^*G^{1/2}R^*FRG^{1/2}v : v \in \mathbb{C}^n,\ v^*v = 1\} \subset Co\left(\left\{f_k S(\{g_j : R_{kj} \neq 0\}) : k = 1,\dots,m\right\}\right)$$

■

Remark 1: The inclusion in Lemma 2 is actually an equality if we consider the union of all Numerical Ranges such that the matrix $|R|$ satisfies the spectral radius bound and has a specified sparsity (Theorem 4 in the Appendix).

III. MAIN RESULTS

A. Problem formulation

We consider single input single output linear time invariant dynamical systems with transfer functions in $\mathscr{A}_0$. These are linearly interconnected i.e. the input to each system is a linear combination of the outputs from other systems. We consider a directed graph representation of the interconnected system and use the notation below.

$G = (V, E, A)$ is a weighted directed graph[2], where $V = \{v_1,\dots,v_n\}$ is the set of nodes, $E \subseteq V \times V$ the set of directed edges and $A = [a_{ij}]$ a weighted adjacency matrix. Directed edges are denoted as $e_{ij} = (v_i, v_j)$, such that e_{ij} is defined to be incident to node v_j. The adjacency matrix $A \in \mathbb{R}^{n\times n}$ satisfies $a_{ji} \neq 0 \Leftrightarrow e_{ij} \in E$. The in-neighbours of a node v_i are defined as $N_i^{in} = \{v_j \in V : (v_j, v_i) \in E\}$ and its in-degree as $|N_i^{in}|$. Similarly the out-neighbours are defined as $N_i^{out} = \{v_j \in V : (v_i, v_j) \in E\}$ and the out-degree as $|N_i^{out}|$. In the digraph representation of the network each dynamical element corresponds to a node of the graph. Furthermore in a network of n dynamic agents, each with scalar input $u_i(t)$, scalar output $y_i(t)$ and transfer function $g_i(s)$, the input and output vectors, $u(t) = [u_1(t),\dots,u_n(t)]^T$ and $y(t) = [y_1(t),\dots,y_n(t)]^T$ respectively, satisfy the relation $u(t) = Ay(t)$, where A is the adjacency matrix of the graph.

The section is structured as follows. We consider first symmetric (the adjacency matrix A is symmetric) bipartite graphs. These are easy to analyze due to the block antidiagonal structure of the adjacency matrix, or in graph theoretic terms, all cycles have even length. The results are then extended to arbitrary graphs by an appropriate transformation of the problem.

B. Stability conditions

An interconnected system with n dynamic agents with transfer functions $g_1(s),\dots,g_n(s)$ respectively and adjacency matrix A, can be represented with the block diagram in figure 2(a), where $G(s) = \mathrm{diag}(g_i(s))$.
In a bipartite graph there exist two disjoint sets of dynamics $\{h_1(s),\dots,h_m(s)\}$ and $\{f_1(s),\dots,f_n(s)\}$ such that dynamics from one set are connected directly only to dynamics from the other set. This means that the interconnection is defined by

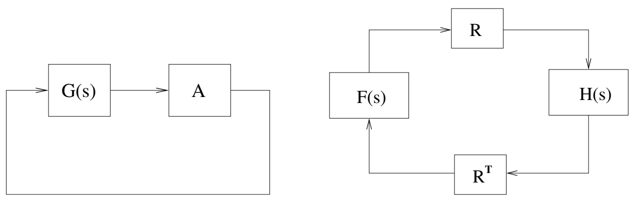
$$y(s) = G(s)u(s), \quad u(s) = Ay(s),$$

$$G(s) = \begin{bmatrix} F(s) & 0 \\ 0 & H(s) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & R^T \\ R & 0 \end{bmatrix} \tag{7}$$

where $F(s) = \mathrm{diag}(f_1(s),\dots,f_n(s))$, $H = \mathrm{diag}(h_1(s),\dots,h_m(s))$, $R \in \mathbb{R}^{m\times n}$. An alternative block diagram representation in this case is shown in figure 2(b).

Theorem 1 (Stability of bipartite graphs): The interconnection of linear dynamical systems described in (7) where $\rho(|R|^T|R|) \leq 1$ and $f_i(s),\ h_j(s) \in \mathscr{A}_0$ for $i = 1,\dots,n$, $j =$

[2] from now on the word graph or digraph will refer to a weighted directed graph.



(a) The underlying graph can have arbitrary structure.

(b) The underlying graph is bipartite.

Fig. 2. Block diagram representations of interconnected systems.

$1,\dots,m$ is asymptotically stable if

$$1 \notin Co\left(\{f_k(j\omega)S(\{h_j(j\omega) : R_{jk} \neq 0\}) : \omega \in \mathbb{R}_+, k = 1,\dots,n\}\right) \quad (8)$$

$$\text{OR} \quad 1 \notin Co\left(\{h_k(j\omega)S(\{f_j(j\omega) : R_{kj} \neq 0\}) : \omega \in \mathbb{R}_+, k = 1,\dots,m\}\right) \quad (9)$$

Proof: From the block diagram in figure 2(b) the return ratio of the interconnected system is $L(s) = H(s)RF(s)R^T$. Using the multivariable Nyquist criterion [14] the closed loop system is stable if the eigenloci of the return ratio do not encircle the 1 point. A sufficient condition is the existence of a convex set that does not include 1, but includes $\sigma(L(j\omega))$ $\forall\omega \in \mathbb{R}_+$. We generate such a set by noting that $L(j\omega)$ and $H(j\omega)^{1/2}RF(j\omega)R^TH(j\omega)^{1/2}$ have the same non-zero eigenvalues. Zero eigenvalues are not a problem since these are always included by the bounding set we will consider. The Numerical Range is always a bound for the spectrum of a matrix. Hence condition (8) follows directly from Lemma 2. Similarly the sufficiency of (9) can be proved by bounding the spectrum of $F(j\omega)^{1/2}R^TH(j\omega)RF(j\omega)^{1/2}$. ∎

Remark 2: The convex hull conditions (8), (9) can easily be given decentralized interpretations by means of hyperplane arguments. Note first that

$$Co\left(\{h_k(j\omega)S(\{f_j(j\omega) : R_{kj} \neq 0\}) : \omega \in \mathbb{R}_+,\ k = 1,\dots,m\}\right) = Co\left(\{S(\{h_k(j\omega)f_j(j\omega) : R_{kj} \neq 0\}) : \omega \in \mathbb{R}_+, k = 1,\dots,m\}\right)$$

Using a duality argument the convex hull of S-hulls condition is equivalent to each of the S-hulls not intersecting a globally specified hyperplane through the point 1 since the S-hulls will necessarily lie on the same side of the hyperplane (this is because the S-hull of a set always includes the point 0). Decentralization is a result of the fact that the domain of each of the S-hulls depends only on a given dynamic and its neighbours. Scalability follows from the fact that a new agent introduces only an additional hyperplane condition for an S-hull.

Remark 3: Given relations $y = |R|x$ and $q = |R|^T p$ the 2-norm bound on R can be achieved by appropriate scaling. This is because the return ratio $L(s) = H(s)R^TF(s)R$ is similar to

$$\mathrm{diag}(y_j)H(s)\hat{R}F(s)\mathrm{diag}\left(\frac{q_i}{x_i}\right)\hat{R}^T\mathrm{diag}\left(\frac{1}{p_j}\right)$$

$$\text{where} \quad \hat{R} = \mathrm{diag}\left(\sqrt{\frac{p_j}{y_j}}\right)R\,\mathrm{diag}\left(\sqrt{\frac{x_i}{q_i}}\right)$$

The spectral radius bound on $\hat{R}$ is valid since

$$\rho(|\hat{R}|^T|\hat{R}|) = \rho\left(\mathrm{diag}\left(\frac{1}{q_i}\right)|R|^T\mathrm{diag}\left(\frac{p_j}{y_j}\right)|R|\mathrm{diag}(x_i)\right)$$
$$\leq \left\|\mathrm{diag}\left(\frac{1}{q_i}\right)|R|^T\mathrm{diag}(p_j)\right\|_\infty \left\|\mathrm{diag}\left(\frac{1}{y_j}\right)|R|\mathrm{diag}(x_i)\right\|_\infty \leq 1$$

where the last inequality follows using the fact that for a matrix $M \in \mathbb{C}^{m\times n}$ the induced infinity norm satisfies $\|M\|_\infty = \max_i \sum_j |M_{ij}|$. In the case all elements of $|R|$ are in $[0,1]$ it is sufficient to take p, x as vectors with all elements being equal to 1 and y, q vectors of in-degrees i.e. $y_j = |N_j^{in}|$ where v_j is the node associated with the dynamic with transfer function $h_j(s)$, similarly $q_i = |N_i^{in}|$ where v_i is node associated with the dynamic with transfer function $f_i(s)$.

Remark 4: In the special case where there are no dynamics associated with one of the disjoint sets of vertices in the bipartite graph, i.e. $H(s) = k_i,\ k_i \leq 1$, the stability condition in Theorem 1 reduces to

$$1 \notin Co(\{f_k(j\omega) : \omega \in \mathbb{R}_+, k = 1,\dots,n\} \cup 0)$$

Remark 5: Theorem 1 also holds when R is a transfer matrix $R(s)$ analytic in the closed right half plane such that it can be factorized as $R^T(s) = \mathrm{diag}(g_i(s))R^T(-s)$ and $R(j\omega)$ satisfies the spectral radius bound $\rho(|R(j\omega)|^T|R(j\omega)|) \leq 1$. Conditions (8), (9) then hold but with $f_k(j\omega)$ replaced by $f_k(j\omega)g_k(j\omega)$. This is particularly useful for some networks, such as the Internet, where $R(s)$ can be a transfer matrix of the propagation/return delays (see [10]).

We now consider general symmetric graphs i.e. the interconnection is defined by

$$y(s) = G(s)u(s), \quad u(s) = Ay(s) \quad (10)$$

where $G(s) = \mathrm{diag}(g_1(s),\dots,g_n(s))$ and $A = A^T \in \mathbb{R}^{n\times n}$ without any restrictions in its sparsity.

Theorem 2 (Stability of arbitrary graphs): The interconnection of linear dynamical systems described in (10), where $\rho(|A|^T|A|) \leq 1$ and $g_i(s) \in \mathscr{A}_0$, for $i = 1,\dots,n$ is asymptotically stable if

$$1 \notin Co\left(\{g_k(j\omega)S(\{g_j(j\omega) : A_{kj} \neq 0\}) : \omega \in \mathbb{R}_+,\ k = 1,\dots,n\}\right)$$

Proof: The return ratio in this case is $G(s)A$ and we use the fact $1 \notin \sigma(G(j\omega)AG(j\omega)A) \Rightarrow 1 \notin \sigma(G(j\omega)A)$. This is true since for a matrix $M \in \mathbb{C}^{k\times k}$ $\sigma(M^2) = (\sigma(M))^2$. Moreover, given a convex set P s.t. $0 \in P$ and $P \supset \sigma(G(j\omega)AG(j\omega)A)$ $\forall\omega \in \mathbb{R}_+$ then $1 \notin P$ implies $1 \notin kP$ $\forall k \in [0,1]$ and hence $1 \notin \sigma(kG(j\omega)A)$ $\forall k \in [0,1], \omega \in \mathbb{R}_+$. Therefore $\{\sigma(G(j\omega)A) : \omega \in \mathbb{R}_+\}$ does not encircle the point $+1$. It is hence sufficient for stability to bound the spectrum of $G(j\omega)AG(j\omega)A$ by a convex set for all $\omega \in \mathbb{R}_+$. A return ratio $G(s)AG(s)A$ may be interpreted as that of a bipartite graph with $F(s) = H(s) = \mathrm{diag}(g_1(s),\dots,g_n(s))$

and $R = R^T = A$, where F(s), H(s), R are as defined in (7). Theorem 2 follows then from Theorem 1. ■

Remark 6: A decentralized interpretation of the stability condition can be given as explained in Remark 2. As in Remark 3, the 2-norm of matrix $|A|$ in the case all its elements are in $[0,1]$ can be bounded by an in-degree scaling of the inputs of the dynamics.

Remark 7: In the case $A \geq 0$ it is shown in [15] that a result analogous to that in Remark 4 applies i.e. the interconnected system is stable if
$\rho(A) \leq 1$ and $1 \notin Co(\{g_k(j\omega) : \omega \in \mathbb{R}_+, k = 1, \ldots, n\} \cup 0)$

Remark 8: A being skew symmetric i.e. $A = -A^T$ corresponds to the case where all cycles of length 2 are in negative feedback. In this case the same stability conditions hold but with respect to the point -1 instead of $+1$.

Remark 9: In comparison with Theorem 1, the price paid for guaranteeing stability of arbitrary graphs is that the stability condition requires all dynamics to appear in the domain of an S-hull whereas, in Theorem 1 only one of the two sets of the bipartition of dynamics has to appear in the S-hull. More specifically, if Theorem 2 is applied to the bipartite case, both (8) and (9) are required to hold, whereas Theorem 1 needs only either of them. Even though we are eventually considering the convex hull of the S-hulls, the conservatism lies in the fact that the S-hull of a set always includes the corresponding convex hull.

IV. Examples

An important illustrative example of the applicability of the more general results developed in section III is stability analysis of models for Internet congestion control protocols. Such protocols fit within the graph theoretic setting of this paper, in the sense that TCP imposes an underlying bipartite graph. This is the case because users/sources communicate directly only with links/routers. This implies that Theorem 1 can be applied to derive decentralized local robust stability conditions in the presence of dynamics at both users and routers (see [9]). In addition, systems of interacting agents such as consensus protocols (e.g. [16], [17], [18]) could be analysed in the case of bidirectional links between agents, despite the potential presence of heterogeneous agent dynamics.

As a final comment, it should be noted that the stability conditions presented in the paper are tight in the sense that there exist configurations for which they are also necessary (e.g. a delayed integrator connected to a static agent). On the other hand, there exist dynamics and topologies for which they are only sufficient. The point that needs to be emphasized here is that our main contribution is that we have derived a novel methodology for designing networks such that stability can be guaranteed for *heterogeneous* systems on *arbitrary underlying graphs*. Any conservatism in specific configurations is the price paid for maintaining stability in arbitrary networks while relaxing structure in the underlying graph (Remark 9)[3]. Successful, non conservative, applications of special cases of these results (Remarks 4, 7) to data network protocol design problems [7] is an indication of the fertile ground lying ahead.

[3]Extensions to MIMO systems is likely to increase such conservatism unless there is some internal structure associated with these systems.

V. Conclusions

We have shown a generalization of the Nyquist criterion that enables one derive scalable decentralized conditions for the stability of a network of interconnected linear dynamical systems. These conditions require that each dynamic agent knows only the dynamics of its neighbours and hence are independent of the size of the network and the way it is interconnected. If there is some structure in the interconnections, e.g. a bipartite graph, the stability conditions can be reduced to less conservative certificates. Possible applications of these results include Internet congestion control and consensus protocols. Our primary aim has been to demonstrate that it could provide a basis for analyzing many classes of networks where scalability to the network size and topology is a primary issue.

Appendix

The following lemma is used to prove the fact that the S-hull is a convex set (Theorem 3).

Lemma 3: Let $a, b \in \mathbb{C}$ and $l_1 : [0,1] \to \mathbb{C}$, $l_1(t) = ta + (1-t)b$. Then

$$\pm\sqrt{l_1(t)} \in Co\{0, \pm\sqrt{a}, \pm\sqrt{b}\} \quad \forall t \in [0,1]$$

Proof: Without loss of generality assume

$$a = r_a e^{j\theta_a},\ b = r_b e^{j\theta_b}, \quad \pi \geq \theta_b \geq \theta_a \geq 0,\ r_a, r_b \geq 0 \quad (11)$$

We will in fact prove a stronger statement. This is

$$\sqrt{l_1(t)} \in Co\{0, \sqrt{a}, \sqrt{b}\} \quad \forall t \in [0,1] \quad (12)$$

where for

$$g \in \mathbb{C},\ g = re^{j\theta},\ r \geq 0,\ \theta \in [0, 2\pi] \quad \sqrt{g} := \sqrt{r}e^{j\theta/2}$$

Let $l_2 : [0,1] \to \mathbb{C}$, $l_2(t) = t\sqrt{a} + (1-t)\sqrt{b}$. (12) is equivalent to $l_1(t) \in (Co\{0, \sqrt{a}, \sqrt{b}\})^2 \quad \forall t \in [0,1]$.
We prove this by showing that the origin and any point $(l_2(t_2))^2$ lie on opposite sides of the track of the line $l_1(t)$ (see figure 3). This statement can be stated in cross product form as

$$[(b-a) \times ((l_2(t))^2 - a)] \ . \ [(b-a) \times (0-a)] \leq 0 \qquad \forall t \in [0,1] \quad (13)$$

where for a, b as in (11) $\quad b \times a := r_a r_b \sin(\theta_b - \theta_a)$

$$\begin{aligned}
&\text{Now } (b-a) \times ((l_2(t))^2 - a) = \\
&= (b-a) \times (t^2 a + (1-t)^2 b + 2t(1-t)\sqrt{ab} - a) \\
&= (b-a) \times (t^2 + (1-t)^2 - 1) + 2t(1-t)(b-a) \times \sqrt{ab} \\
&= 2t(t-1) b \times a + 2t(1-t)\sqrt{r_a r_b}(r_b e^{j\theta_b} - r_a e^{j\theta_a}) \times e^{j\frac{\theta_a+\theta_b}{2}} \\
&= 2t(t-1) r_b r_a \left[\sin(\theta_b - \theta_a) - \frac{r_b + r_a}{\sqrt{r_b r_a}} \sin\left(\frac{\theta_b - \theta_a}{2}\right)\right] \\
&= 4t \underbrace{(t-1)}_{\leq 0} r_b r_a \sin\left(\frac{\theta_b - \theta_a}{2}\right) \left[\underbrace{\cos\left(\frac{\theta_b - \theta_a}{2}\right)}_{0 \leq . \leq 1} - \underbrace{\frac{r_b + r_a}{2\sqrt{r_b r_a}}}_{\geq 1}\right]
\end{aligned}$$

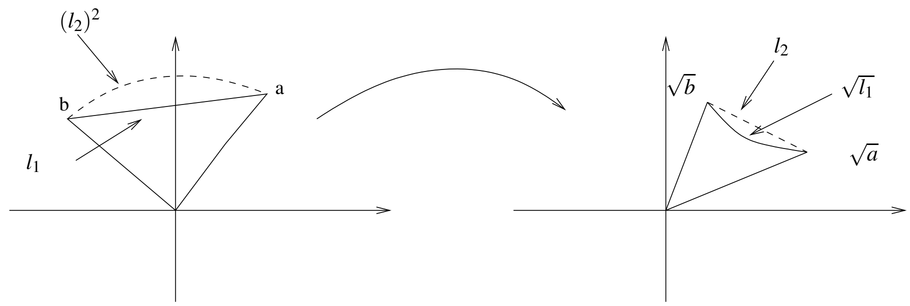


Fig. 3. The root of the straight line joining $a, b \in \mathbb{C}$ and the square of the line joining $\sqrt{a}, \sqrt{b}$.

$$\text{So} \quad (b-a) \times ((l_2(t))^2 - a) = k_1(t) \sin\left(\frac{\theta_b - \theta_a}{2}\right)$$
$$\text{where} \quad k_1(t) \geq 0 \quad \forall t \in [0,1]$$

and also it is apparent that

$$(b-a) \times (0-a) = -k_2 \sin(\theta_b - \theta_a) \quad \text{where } k_2 \geq 0$$

Hence (13) is true. ■

Theorem 3 (Convexity of S-hull): Let $A \subset \mathbb{C}$. $S(A)$ is convex where

$$S(A) = (Co(\{\pm\sqrt{g} : g \in A\}))^2$$

Proof: Let $a, b \in S(A)$. Then

$$\pm\sqrt{a}, \pm\sqrt{b} \in Co(\{\pm\sqrt{g} : g \in A\})$$
$$\text{hence} \quad Co(\pm\sqrt{a}, \pm\sqrt{b}) \subset Co(\{\pm\sqrt{g} : g \in A\}) \qquad (14)$$

In order to prove convexity of $S(A)$ we need to show that the line joining any two points in $S(A)$ lies in $S(A)$ i.e.

$$ta + (1-t)b \in S(A) \quad \text{for all} \quad a, b \in S(A), \quad t \in [0,1]$$

This is equivalent to showing that

$$\pm\sqrt{ta + (1-t)b} \in Co(\{\pm\sqrt{g} : g \in A\}) \ \forall\, a, b \in S(A),\ t \in [0,1]$$

And from (14) it is sufficient to show that

$$\pm\sqrt{ta + (1-t)b} \in Co(\pm\sqrt{a}, \pm\sqrt{b}) \ \forall\, a, b \in \mathbb{C},\ t \in [0,1]$$

This follows from Lemma 3. ■

Theorem 4 (S-hull gives tight bound for Numer. Range): Let $\hat{R} \in \mathbb{C}^{m\times n}$ and $G = \text{diag}(g_1, \ldots, g_n)$, $F = \text{diag}(f_1, \ldots, f_m)$, $g_i, f_j \in \mathbb{C}\ \forall i, j$ then

$$\left\{ \bigcup_R N(G^{1/2} R^* F R G^{1/2}) \text{ s.t. } R \in \mathbb{C}^{m\times n}, \right.$$
$$\left. \rho(|R|^T |R|) \leq 1, R_{ij} = 0 \Leftrightarrow \hat{R}_{ij} = 0 \right\}$$
$$= Co\left(\left\{ f_i S(\{g_k : \hat{R}_{ik} \neq 0\}) : i = 1, \ldots, m \right\}\right)$$
$$= Co\left(\left\{ (Co(\{\pm\sqrt{f_i g_k} : \hat{R}_{ik} \neq 0\}))^2 : i = 1, \ldots, m \right\}\right)$$

Proof: See [19] . ■

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