Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Comparison of Hysteresis Current Controllers using Nonsmooth Lyapunov Functions

Alessandro Cavini*, Carlo Rossi*, Andrea Tilli*

Abstract— **Multivariable hysteresis current controllers for three-phases inductive loads fed by means of voltage source inverters are considered in this paper. The stability of three control strategies presented in literature ([1], [2] and [3]) is rigorously analyzed by means of nonsmooth Lyapunov function and keeping into account the switching nature of the considered solutions. The main aim of this analysis is to enlighten the different robustness properties guaranteed by the aforementioned solutions with respect to typical disturbances for this kind of systems.**

I. INTRODUCTION

Hysteresis current control for Voltage Source Inverters (VSIs) is a common solution in apparatus as ac motor drives, active filters, UPSs, and so on. This technique yields a number of advantages in comparison with other current control approaches that make the Hysteresis Current Controllers (HCCs) the preferable solution for current control in high demanding applications where accuracy, wide bandwidth and robustness are required.

Because of their effectiveness, a common solution in the design of HCCs are the Multivariable Hysteresis Current Controllers (MHCCs). Differences among this type of controllers proposed in literature are mainly related to the adopted reference frame and the control algorithm. A multivariable controller has the objective to keep the current error, computed in vector form, inside a multidimensional tolerance region (TR) of suitable form. Because of its simple implementation by means of inexpensive hardware, the usual choice for TRs is a polygon. Hexagons of different orientations in the natural fixed abc frame are proposed in [1] and in [3], a rectangle in the dq rotating frame is adopted in [4] and a square in the $\alpha\beta$ fixed reference frame is taken into account in [2] and [5].

Although in literature HCC is so widely considered the issue of the system stability is usually roughly discussed or, in the most of cases, neglected. For example in [1], [2] and [3] the stability analysis is not deduced in a general framework and is based on simple "physical" considerations.

In the last years many efforts have been spent in the development of stability investigation tools for the class of switched systems which HCCs belong to. This systems present equations with discontinuous right-hand side and, with respect to the classical techniques, one cannot define a solution, and even less discuss existence of equilibria and stability. A deep study of the stability for discontinuous dynamics was made by Peleties and DeCarlo in [6] where the work focuses on the stabilization of the so called m-switched systems through Lyapunov-like functions. The main limit consists in dynamics that has to be piecewise linear and does not take into account other terms as disturbances. A more general work in the field of Lyapunov methods is the one by Branicky [7]. In this work the author introduces analysis tools for switched and hybrid systems as the multiple Lyapunov functions for analyzing Lyapunov stability.

Among the various Lyapunov methods for the study of switched systems stability, in [8] and [9] the authors developed nonsmooth Lyapunov stability theory for a wide class of nonsmooth Lipschitz continuous Lyapunov functions. The theory developed is based on Filippov's differential inclusions and Clarke's generalized gradient. The tools introduced are quite interesting and are helpful for the design of the manipulator controllers presented in [9] and [10].

The aim of this paper is to compare the stability properties of the HCCs presented in [1], [2] and [3]. The stability analysis is performed exploiting the tools developed in [8], [9] and for every controller is presented an Admissible Disturbance Set (ADS); if the disturbance belongs to the ADS of the controller taken into account the asymptotic stability of the comprehensive system is ensured.

The paper is organized as follows. Section II presents the system model and the nomenclature used throughout the paper. In Section III the analysis is performed. In particular, in Subsection III-A the main assumptions and considerations, used to make uniform the stability analysis and comparable the results, are reported. In Subsections III-B, III-C and III-D, the three solutions are considered and in Subsections III-E the results are compared and discussed.

II. SYSTEM MODEL

In order to compare the stability of the three controllers [1], [2] and [3] the same load is assumed. The basic circuit taken into account for the three-phase VSI with balanced load is shown in the first picture of Fig.1 while the second one shows the related space-vectors representation. The corresponding values of control voltages are reported in Table I. System equations in the three-phase abc fixed reference frame are:

$$\mathbf{v_k}_{abc} = \mathbf{R}\mathbf{i}_{abc} + \mathbf{L}\frac{d\mathbf{i}_{abc}}{dt} + \mathbf{e}_{abc} \quad (1)$$

$$x_a + x_b + x_c = 0 \quad (2)$$

where $\mathbf{x}$ is any vector in (1). Definition of vectors and matrices are the same presented in [2].

*The authors are with CASY-Center for Research on Complex Automated Systems "G. Evangelisti", DEIS-Department of Electronics, Computer and System Sciences, University of Bologna, Viale Risorgimento n.2, 40136 Bologna, ITALY, Tel. +39 051 20 93069, Fax. +39 051 20 93073, E-mail: {acavini, crossi, atilli}@deis.unibo.it

0-7803-9568-9/05/$20.00

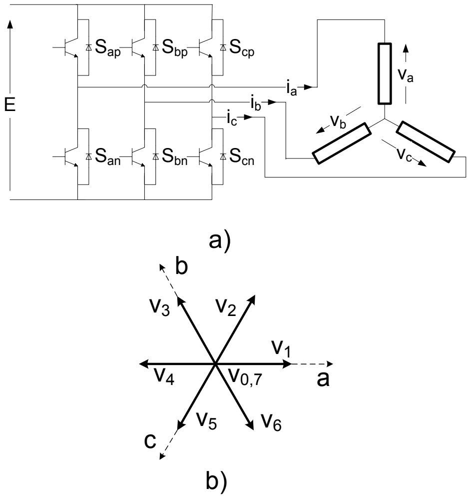

Fig. 1. a) Basic circuit of three-phase VSI with balanced load. b) Space-vector representation.

TABLE I

CONTROL COMMANDS (S_{ip}) AND RELATED VOLTAGE VECTORS.

S_{ap}, S_{bp}, S_{cp}	$\mathbf{v_k}$	v_a, v_b, v_c	v_α, v_β
$(0,0,0)$	$\mathbf{v_0}$	$(0,0,0)$	(0,0)
$(1,0,0)$	$\mathbf{v_1}$	$(\frac{2}{3}E, \frac{-1}{3}E, \frac{-1}{3}E)$	$(\sqrt{\frac{2}{3}}E, 0)$
$(1,1,0)$	$\mathbf{v_2}$	$(\frac{1}{3}E, \frac{1}{3}E, \frac{-2}{3}E)$	$(\frac{1}{2}\sqrt{\frac{2}{3}}E, \frac{\sqrt{3}}{2}\sqrt{\frac{2}{3}}E)$
$(0,1,0)$	$\mathbf{v_3}$	$(\frac{-1}{3}E, \frac{2}{3}E, \frac{-1}{3}E)$	$(\frac{-1}{2}\sqrt{\frac{2}{3}}E, \frac{\sqrt{3}}{2}\sqrt{\frac{2}{3}}E)$
$(0,1,1)$	$\mathbf{v_4}$	$(\frac{-2}{3}E, \frac{1}{3}E, \frac{1}{3}E)$	$(-\sqrt{\frac{2}{3}}E, 0)$
$(0,0,1)$	$\mathbf{v_5}$	$(\frac{-1}{3}E, \frac{-1}{3}E, \frac{2}{3}E)$	$(\frac{-1}{2}\sqrt{\frac{2}{3}}E, \frac{-\sqrt{3}}{2}\sqrt{\frac{2}{3}}E)$
$(1,0,1)$	$\mathbf{v_6}$	$(\frac{1}{3}E, \frac{-2}{3}E, \frac{1}{3}E)$	$(\frac{1}{2}\sqrt{\frac{2}{3}}E, \frac{-\sqrt{3}}{2}\sqrt{\frac{2}{3}}E)$
$(1,1,1)$	$\mathbf{v_7}$	$(0,0,0)$	(0,0)

After defining the current error $\mathbf{i_e} = \mathbf{i} - \mathbf{i_r}$, where $\mathbf{i_r}$ is the current reference, by means of some algebraic manipulations (1) can be rewritten in the error form as follows:

$$\mathbf{L}\frac{d\mathbf{i_e}_{abc}}{dt} = \mathbf{v_k}_{abc} + \mathbf{e_0}_{abc} \tag{3}$$

where the term

$$\mathbf{e_0}_{abc} = -\mathbf{L}\frac{d\mathbf{i_r}_{abc}}{dt} - \mathbf{R}\mathbf{i_r}_{abc} - \mathbf{e}_{abc}$$

collects all the exogenous and endogenous disturbance acting on the system and, since it is always stabilizing, the term $-\mathbf{R}\mathbf{i_e}$ is neglected. The subscript $\mathbf{k} = 0, \ldots, 7$ in the voltage vectors $\mathbf{v_k}$ specifies that only eight control commands are available, as reported in Table I.

Since constraint (2) introduces a linear dependency on the three-phase quantities in (3), a two-dimensional representation can be adopted. The standard transformation matrix to the equivalent system in two-phase $\alpha\beta$ fixed frame is:

$${}^{\alpha\beta}\mathbf{T}_{abc} = \sqrt{\tfrac{2}{3}}\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix}$$

The resulting two-phase model is:

$$\frac{d\mathbf{i_e}_{\alpha\beta}}{dt} = \frac{1}{L+M}(\mathbf{v_k}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta}) \tag{4}$$

It is worth to mention that equation (1), and therefore also the error dynamics (4), is a general model that can describe the dynamics of a number of apparatus as electrical motor, active filter and so on.

III. STABILITY ANALYSIS

A. Preliminaries

The main control specification of the three controllers presented in [1], [2] and [3] is to bring the current error inside the respective TRs and, once within, to keep it inside in spite of the presence of the disturbance term. The three different strategies [1], [2], [3] impose different voltage vector according to the value of $\mathbf{i_e}$ as enlighten in Fig. 2. In some regions two or more voltage vectors can be selected, this feature will be discussed later. The capability of the controllers to meet the aforementioned requirement is largely demonstrated in the related articles. However the system stability discussion is based on simple "physical" considerations and is not arranged in a classical framework. The objective of this paper is to study rigorously the stability of the systems by means of the Lyapunov methods introduced in [8] and [9]. In particular, the analysis aims to find, for each control strategy [1], [2], [3], sharp (possibly, the largest) bounds on the disturbance, $\mathbf{e_0}$, which guarantee that the respective TR, see Fig. 2, is an attractive invariant. This information can be use to compare the robustness properties of the different strategies; but this comparison is formally not exhaustive since Lyapunov analysis gives only sufficient conditions for the stability, depending on selected Lyapunov function. To overcome this issue practically, a uniform choice of the Lyapunov function is imposed for each controller. Keeping in mind the previous discussion, the analysis is performed with the following assumptions and considerations:

1) In order to simplify the analysis, without impairing its effectiveness, the TRs of the controllers are reduced to the origin of the error plane choosing an hysteresis band equal to zero. In this case when the origin is reached an infinity switching frequency is assumed to keep within the error in the TR.
2) Throughout the paper the nomenclature defined in [8] is assumed. For example: $K[f](\mathbf{x}, t)$ represents the set obtained by means of the convex closure of the limiting values of the vector field $\dot{\mathbf{x}} = f(\mathbf{x}, t)$ in progressively smaller neighborhoods around the Filipov's solution of the system; $\partial V(\mathbf{x}, t)$ represents the Clarke's generalized gradient of the candidate Lyapunov function $V : R^n \to R$ in the point $(\mathbf{x}, t)$; $\tilde{\dot{V}}(\mathbf{x}, t)$ is the set obtained by means of the *Chain Rule* stated in [8].
3) In order to demonstrate the asymptotic stability of the TR the *Theorem 3.1* presented in [8] is taken into account. For the sake of completeness the theorem is reported below.
Theorem 3.1: Let $\dot{\mathbf{x}} = f(\mathbf{x}, t)$ be essentially locally bounded and

$$\mathbf{0} \in K[f](\mathbf{0}, t) \tag{5}$$

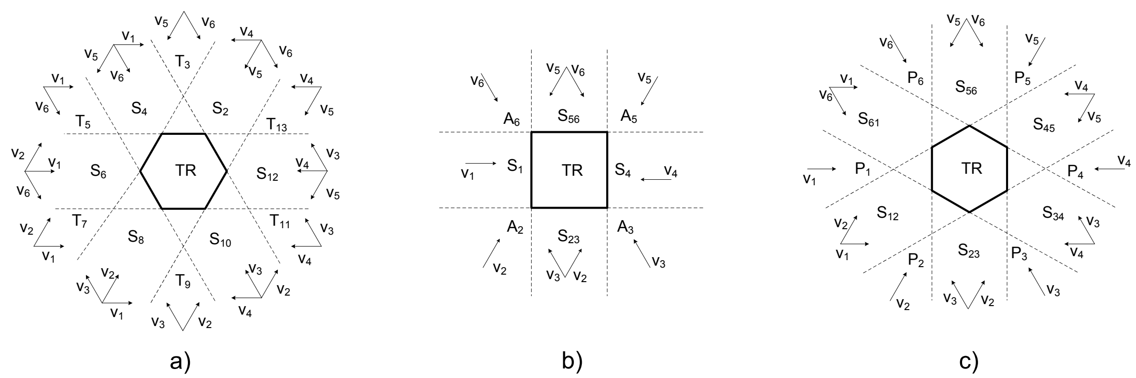

Fig. 2. TRs of the controllers [1], [2] and [3] respectively.

in a region $Q \supset \{\mathbf{x} \in R^n | \, \|\mathbf{x}\| < r\} \times \{t | t_0 \leq t < \infty\}$. Also let $V : R^n \times R$ be regular function satisfying:

$$V(\mathbf{0}, t) = 0 \tag{6}$$

and

$$0 < V_1(\|\mathbf{x}\|) \leq V(\mathbf{x}, t) \leq V_2(\|\mathbf{x}\|) \quad \text{for } \mathbf{x} \neq \mathbf{0} \tag{7}$$

in Q for some V_1, $V_2 \in$ class $\mathcal{K}$. Then if there exists a class $\mathcal{K}$ function $\omega(\cdot)$ in Q with the property

$$\dot{\tilde{V}} \leq -\omega(\mathbf{x}) < 0 \tag{8}$$

the solution $\mathbf{x}(t) \equiv \mathbf{0}$ is uniformly asymptotically stable (condition trivially satisfied if $\dot{\tilde{V}}$ is the empty set).

4) In order to make uniform the stability analysis and consistently compare the robustness results, the Lyapunov functions used for each controller are a measure of the distance from the adopted TR, i.e. their level curves have the same shape of the considered TR; therefore nonsmooth functions are used.
5) The disturbance functions $\mathbf{e}_{\mathbf{0}\alpha\beta}(t)$ of equation (4) is assumed continuous and bounded.

B. Stability Analysis of Controller [1]

The controller taken into account presents the TR depicted in the first picture of Fig. 2. Outside the region the space of the current error is split in triangular zones, viz the zone marked with T_i with i odd number, and strips that are the

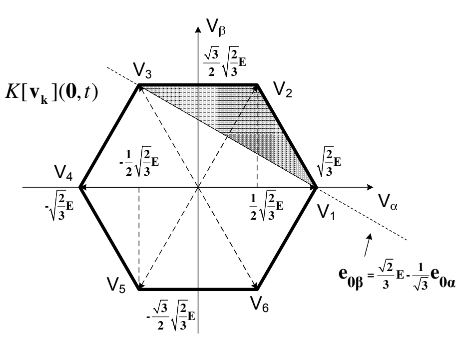

Fig. 3. Set $K[\mathbf{v_k}](\mathbf{0}, t)$

regions labeled with S_i, i even one. However requirement 1) reduce the region into the origin of the plane and the space is therefore rearranged as the picture a) of Fig 4.

In order to demonstrate the asymptotic stability of the origin the requirements of *Theorem 3.1* must be met. Assume $\mathbf{x} = \mathbf{i}_{\mathbf{e}\alpha\beta}$ as state vector and $f(\mathbf{x}, t) = \frac{1}{L+M}(\mathbf{v}_{\mathbf{k}\alpha\beta}(\mathbf{i_e}) + \mathbf{e}_{\mathbf{0}\alpha\beta})$ as vector field that, keeping in mind requirement 5) about the disturbance, is essentially locally bounded. Taken into account properties 2), 5), 7) and their proof of *Theorem 1* stated in [9] the condition (5) is satisfied if

$$\mathbf{e}_{\mathbf{0}\alpha\beta} \in K[\mathbf{v_k}](\mathbf{0}, t)$$

set sketched in Fig. 3.

Assume the following candidate nonsmooth Lyapunov function:

$$V(\mathbf{i_e}) = \begin{cases} \frac{\sqrt{3}}{2}(-i_{e\alpha} - \frac{1}{\sqrt{3}} i_{e\beta}) & \text{in } T_7 \\ -i_{e\beta} & \text{in } T_9 \\ \frac{\sqrt{3}}{2}(i_{e\alpha} - \frac{1}{\sqrt{3}} i_{e\beta}) & \text{in } T_{11} \\ \frac{\sqrt{3}}{2}(i_{e\alpha} + \frac{1}{\sqrt{3}} i_{e\beta}) & \text{in } T_{13} \\ i_{e\beta} & \text{in } T_3 \\ \frac{\sqrt{3}}{2}(-i_{e\alpha} + \frac{1}{\sqrt{3}} i_{e\beta}) & \text{in } T_5 \end{cases}$$

The design of the Lyapunov function aim to define a function with level set of the shape of TR boundary, viz an hexagon with the orientation of picture a) in Fig. 2. Such function is a regular one and satisfy requirement (6). Because of the equivalence of the norms in an Euclidian space, once requirement (7) is satisfied for a norm, then is satisfied for all norms. Take into account the $\|\cdot\|_2$, and assume $V_1(\phi) = \frac{\sqrt{3}}{2}\phi; V_2(\phi) = \phi$ then requirement (7) is satisfied and therefore is satisfied for all $\|\cdot\|$.

Finally, in order to demonstrate the uniform asymptotic stability of the origin, it is necessary to meet requirement (8). The analysis is performed region by region.

1) Region T_3: in this case the controller could applies whether vectors $\mathbf{v_5}$ or vector $\mathbf{v_6}$:

$$K[f](\mathbf{i_e}, t) = \frac{1}{L+M}(\mathbf{v}_{\mathbf{5}\div\mathbf{6}\alpha\beta} + \mathbf{e}_{\mathbf{0}\alpha\beta})$$

$$\partial V = \begin{bmatrix} \frac{\partial}{\partial i_{e\alpha}}(i_{e\beta}) \\ \frac{\partial}{\partial i_{e\beta}}(i_{e\beta}) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

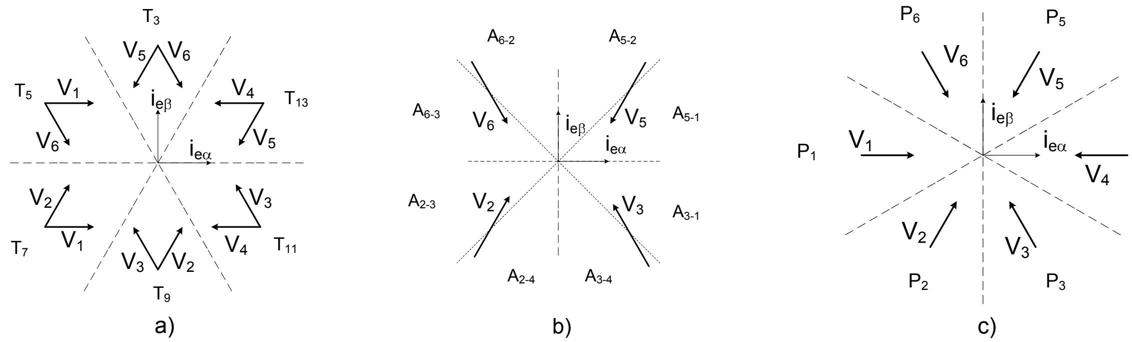

Fig. 4. Reduced TRs of the controllers [1], [2] and [3] respectively.

$$\dot{\tilde{V}} = \bigcap_{\xi \in \partial V(\mathbf{i_e},t)} \xi^T K[f](\mathbf{i_e},t) = \tfrac{1}{L+M}\left(-\tfrac{\sqrt{3}}{2}\sqrt{\tfrac{2}{3}}E + e_{0\beta}\right)$$

and therefore:

$$e_{0\beta} \leq \tfrac{\sqrt{3}}{2}\sqrt{\tfrac{2}{3}}E - \varepsilon \Rightarrow \dot{\tilde{V}} \leq -\tfrac{\varepsilon}{L+M}. \tag{9}$$

where ε is a strictly positive constant.

2) Region T_{13}: the controller applies vector $\mathbf{v_4}$ or $\mathbf{v_5}$.

$$K[f](\mathbf{i_e},t) = \tfrac{1}{L+M}(\mathbf{v_{4\div5}}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta})$$

$$\partial V = \begin{bmatrix} \frac{\partial}{\partial i_{e\alpha}}\left(\frac{\sqrt{3}}{2}(i_{e\alpha} + \frac{1}{\sqrt{3}}i_{e\beta})\right) \\ \frac{\partial}{\partial i_{e\beta}}\left(\frac{\sqrt{3}}{2}(i_{e\alpha} + \frac{1}{\sqrt{3}}i_{e\beta})\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\dot{\tilde{V}} = \tfrac{1}{L+M}\left(-\tfrac{\sqrt{3}}{2}\sqrt{\tfrac{2}{3}}E + \tfrac{\sqrt{3}}{2}e_{0\alpha} + \tfrac{1}{2}e_{0\beta}\right)$$

Therefore:

$$e_{0\beta} \leq -\sqrt{3}e_{0\alpha} + \sqrt{2}E - 2\varepsilon \Rightarrow \dot{\tilde{V}} \leq -\tfrac{\varepsilon}{L+M} \tag{10}$$

3) Boundary between region T_3 and region T_{13}: in the set of null Lebesgue measure that is the boundary between T_3 and T_{13} the controller pass trough the application of one of the vectors $\{\mathbf{v_5}, \mathbf{v_6}\}$ to one of the couple $\{\mathbf{v_4}, \mathbf{v_5}\}$ or viceversa.

Subcase $\mathbf{v_4} - \mathbf{v_6}$:

$$K[f](\mathbf{i_e},t) = \overline{co}\left\{(\mathbf{v_4}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta}), (\mathbf{v_6}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta})\right\} =$$
$$= \tfrac{1}{L+M}\begin{bmatrix} \left(\frac{3\chi}{2} - 1\right)\sqrt{\frac{2}{3}}E + e_{0\alpha} \\ -\chi\frac{\sqrt{3}}{2}\sqrt{\frac{2}{3}}E + e_{0\beta} \end{bmatrix},\ \chi \in [0,1]$$

$$\partial V = \overline{co}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}\right\} = \begin{bmatrix} \frac{\sqrt{3}}{2}\lambda \\ 1 - \frac{\lambda}{2} \end{bmatrix},\ \lambda \in [0,1]$$

$$\dot{\tilde{V}} = \bigcap_{\xi \in \partial V(\mathbf{i_e},t)} \xi^T K[f](\mathbf{i_e},t) =$$
$$= \bigcap_{\lambda \in [0,1]} \tfrac{1}{L+M}\left(\tfrac{\sqrt{3}}{2}\lambda\left(\left(\tfrac{3\chi}{2} - 1\right)\sqrt{\tfrac{2}{3}}E + e_{0\alpha}\right) + \right.$$
$$\left. + \left(1 - \tfrac{\lambda}{2}\right)\left(-\chi\tfrac{\sqrt{3}}{2}\sqrt{\tfrac{2}{3}}E + e_{0\beta}\right)\right)$$

When the intersection over all the values of λ is not empty the set of the *Chain Rule* reduces to the point:

$$\dot{\tilde{V}}_{\lambda=\frac{1}{2}} = \tfrac{1}{L+M}\left(-\tfrac{\sqrt{3}}{4}\sqrt{\tfrac{2}{3}}E + \tfrac{\sqrt{3}}{4}e_{0\alpha} + \tfrac{3}{4}e_{0\beta}\right)$$

$$e_{0\beta} \leq -\tfrac{1}{\sqrt{3}}e_{0\alpha} + \tfrac{\sqrt{2}}{3}E - \tfrac{4}{3}\varepsilon \Rightarrow \dot{\tilde{V}} \leq -\tfrac{\varepsilon}{L+M} \tag{11}$$

to ensure the asymptotic stability of the system the shaded region in Fig. 3 has to be discarded. In this manner the condition $\dot{\tilde{V}} \leq -\frac{\varepsilon}{L+M}$ is ensured.

Subcases $\mathbf{v_4} - \mathbf{v_5}$, $\mathbf{v_5} - \mathbf{v_5}$, $\mathbf{v_5} - \mathbf{v_6}$: the analysis is performed following the same procedure of the above subcase. The conditions found on the disturbance, when $\dot{\tilde{V}}$ is not empty, are the same of (9) and (10) and substantially do not introduce other stability conditions for the disturbance.

The stability analysis of the other regions and their boundary leads to similar conditions about the disturbance.

Now it is possible to argue that when controller [1] is employed and when the switching between two vectors selects only adjacent vectors (avoiding cases as *Subcase* $\mathbf{v_4} - \mathbf{v_6}$) the asymptotic stability of the comprehensive system is guaranteed if the disturbance belongs to the interior of the largest hexagon depicted in Fig. 5, a) that is the ADS. On the other hand, if also the selection of not-adjacent vectors is taken into account, the conditions on the boundaries among the various regions as (11) lead to an ADS that consists of the interior of the smallest hexagon reported in Fig. 5, a).

C. Stability Analysis of Controller [2]

In the study of this controller the following preliminary has to be pointed out. The current error space is characterized by a quadratic TR (picture b) of Fig. 2)) and, taking into account requirement 1), reduces to the regions depicted in b), Fig. 4: regions A_{5-i} with $i \in \{1, 2\}$ are related to region A_5 of Fig. 2), regions A_{6-i}, $i \in \{2, 3\}$ are the corresponding of region A_6, and so on. In the rearrangement of the current error space the boundary between region A_6 and region S_1 reaches the boundary between S_1 and A_2 involving the suppression of region S_1. Hence in the following stability analysis the vector $\mathbf{v_1}$ will not be taken into account. The same holds for region S_4 and vector $\mathbf{v_4}$.

The analysis procedure follows the same steps of the previous one. Assume the state vector and the vector field as defined in III-B. Again the field taken into account is essentially locally bounded. Condition (5) is satisfied when the $\mathbf{e_0}_{\alpha\beta}$ belongs to the set $K[\mathbf{v_k}](\mathbf{0},t)$ that, taking into account the previous considerations about vectors $\mathbf{v_1}$ and $\mathbf{v_4}$,

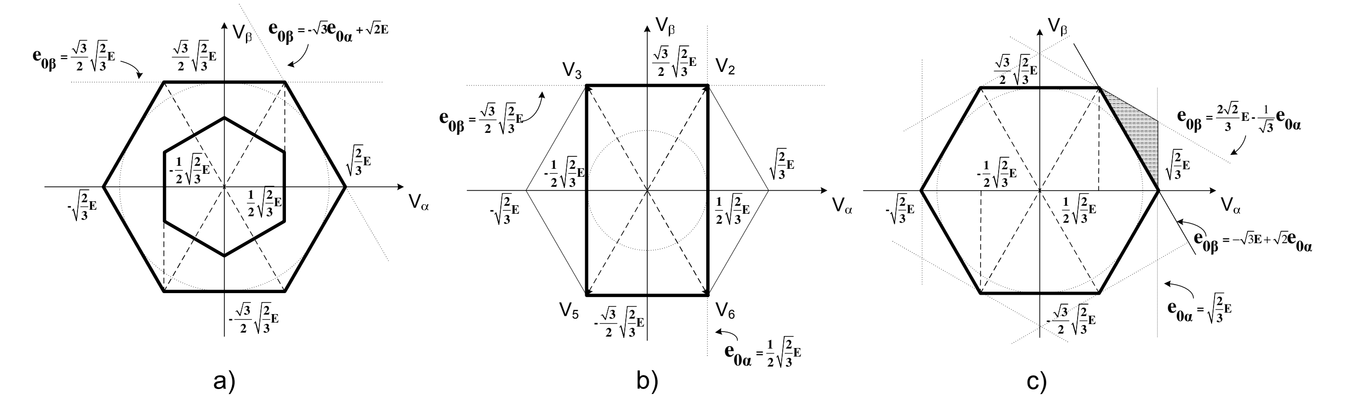

Fig. 5. ADS of the controllers [1], [2] and [3], respectively.

assume the shape of the bold region of picture b), Fig. 5. The following candidate Lyapunov function is taken into account:

$$V(\mathbf{i_e}) = \|\mathbf{i_e}\|_\infty = \begin{cases} (i_{e\alpha}, 0)^T & \text{in } A_{j-1},\ j \in \{3,5\} \\ (0, i_{e\beta})^T & \text{in } A_{j-2},\ j \in \{5,6\} \\ (-i_{e\alpha}, 0)^T & \text{in } A_{j-3},\ j \in \{2,6\} \\ (0, -i_{e\beta})^T & \text{in } A_{j-4},\ j \in \{2,3\} \end{cases}$$

This Lyapunov function splits further the four regions of the current error in eight subregions as depicted in Fig. 4, has level sets of the TR shape, viz squares, is a max function [9] and therefore a regular one and satisfy requirement (6). Condition (7) is satisfied by means of the functions $V_1(\phi) = \frac{1}{\sqrt{2}}\phi$; $V_2(\phi) = \phi$ and with $\|\cdot\| = \|\cdot\|_2$. Then the requirement (8) of *Theorem 3.1* has to be satisfied to prove uniform asymptotic stability. As in previous subsection the analysis is performed region by region.

1) Region A_{5-1}: the following computations hold:

$$K[f](\mathbf{i_e}, t) = \tfrac{1}{L+M}(\mathbf{v_5}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta})$$
$$\partial V = \partial\|\mathbf{i_e}\|_\infty = \begin{bmatrix} \frac{\partial}{\partial i_{e\alpha}}(i_{e\alpha}) \\ \frac{\partial}{\partial i_{e\beta}}(i_{e\alpha}) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\dot{\tilde{V}} = \tfrac{1}{L+M}\left(-\tfrac{1}{2}\sqrt{\tfrac{2}{3}}E + e_{0\alpha}\right)$$
$$e_{0\alpha} \le \tfrac{1}{2}\sqrt{\tfrac{2}{3}}E - \varepsilon \Rightarrow \dot{\tilde{V}} \le -\tfrac{\varepsilon}{L+M}. \quad (12)$$

2) Region A_{5-2}: in order to guarantee asymptotic stability here the disturbance has to satisfy the condition:

$$K[f](\mathbf{i_e}, t) = \tfrac{1}{L+M}(\mathbf{v_5}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta})$$
$$\partial V = \begin{bmatrix} \frac{\partial}{\partial i_{e\alpha}}(i_{e\beta}) \\ \frac{\partial}{\partial i_{e\beta}}(i_{e\beta}) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\dot{\tilde{V}} = \tfrac{1}{L+M}\left(-\tfrac{\sqrt{3}}{2}\sqrt{\tfrac{2}{3}}E + e_{0\beta}\right)$$
$$e_{0\beta} \le \tfrac{\sqrt{3}}{2}\sqrt{\tfrac{2}{3}}E - \varepsilon \Rightarrow \dot{\tilde{V}} \le -\tfrac{\varepsilon}{L+M}. \quad (13)$$

3) Boundary between region A_{5-1} and A_{5-2}:

$$K[f](\mathbf{i_e}, t) = \tfrac{1}{L+M}(\mathbf{v_5}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta})$$
$$\partial V = \overline{co}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$$\dot{\tilde{V}} = \bigcap_{\lambda \in [0,1]} \tfrac{1}{L+M}\left(\lambda\left(-\tfrac{1}{2}\sqrt{\tfrac{2}{3}}E + e_{0\alpha} + \tfrac{\sqrt{3}}{2}\sqrt{\tfrac{2}{3}}E - e_{0\beta}\right) + -\tfrac{\sqrt{3}}{2}\sqrt{\tfrac{2}{3}}E + e_{0\beta}\right)$$

If the term multiplied by λ is not null the set $\dot{\tilde{V}}$ is empty, whilst if the term is null (13) is obtained again.

4) Boundary between region A_{5-2} and A_{6-2}:

$$K[f](\mathbf{i_e}, t) = \tfrac{1}{L+M}\,\overline{co}\left\{(\mathbf{v_5}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta}), (\mathbf{v_6}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta})\right\}$$
$$\partial V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\dot{\tilde{V}} = \tfrac{1}{L+M}\left(-\tfrac{\sqrt{3}}{2}\sqrt{\tfrac{2}{3}}E + e_{0\beta}\right)$$

and condition (13) holds again.

Similar conditions about the disturbance can be obtained in the other regions.

When controller [2] is adopted, the ADS which guarantee the asymptotic stability is the interior of the bold rectangular region depicted in b), Fig. 5.

D. Stability Analysis of Controller [3]

Condition (5) holds if $\mathbf{e_0}_{\alpha\beta}$ belongs to the hexagonal region of Fig. 3. The candidate Lyapunov function:

$$V(\mathbf{i_e}) = \begin{cases} -i_{e\alpha} & \text{in } P_1 \\ \frac{\sqrt{3}}{2}\left(-\frac{1}{\sqrt{3}}i_{e\alpha} - i_{e\beta}\right) & \text{in } P_2 \\ \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}i_{e\alpha} - i_{e\beta}\right) & \text{in } P_3 \\ i_{e\alpha} & \text{in } P_4 \\ \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}i_{e\alpha} + i_{e\beta}\right) & \text{in } P_5 \\ \frac{\sqrt{3}}{2}\left(-\frac{1}{\sqrt{3}}i_{e\alpha} + i_{e\beta}\right) & \text{in } P_6 \end{cases}$$

has hexagonal level sets with the orientation depicted in c), Fig. 2, is a regular function and satisfy requirement (6). Condition (7) holds with the functions $V_1(\phi) = \frac{\sqrt{3}}{2}\phi$; $V_2(\phi) = \phi$ with $\|\cdot\| = \|\cdot\|_2$. Now requirement (8) of *Theorem 3.1* has to be satisfied to derive the ADS.

1) Region P_4: the following condition holds:

$$K[f](\mathbf{i_e},t) = \frac{1}{L+M}(\mathbf{v_4}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta})$$
$$\partial V = \begin{bmatrix} \frac{\partial}{\partial i_{e\alpha}}\ (i_{e\alpha}) \\ \frac{\partial}{\partial i_{e\beta}}\ (i_{e\alpha}) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\dot{\tilde{V}} = \frac{1}{L+M}(-\sqrt{\tfrac{2}{3}}E + e_{0\alpha})$$
$$e_{0\alpha} \leq \sqrt{\tfrac{2}{3}}E - \varepsilon \Rightarrow \dot{\tilde{V}} \leq -\frac{\varepsilon}{L+M} \tag{14}$$

2) Region P_5:

$$K[f](\mathbf{i_e},t) = \frac{1}{L+M}(\mathbf{v_5}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta})$$
$$\partial V = \begin{bmatrix} \frac{\partial}{\partial i_{e\alpha}}\ \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}i_{e\alpha} + i_{e\beta}\right) \\ \frac{\partial}{\partial i_{e\beta}}\ \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}i_{e\alpha} + i_{e\beta}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$
$$\dot{\tilde{V}} = \frac{1}{L+M}(-\sqrt{\tfrac{2}{3}}E + \tfrac{1}{2}e_{0\alpha} + \tfrac{\sqrt{3}}{2}e_{0\beta})$$
$$e_{0\beta} \leq \frac{2\sqrt{2}}{3}E - \frac{1}{\sqrt{3}}e_{0\alpha} - \frac{2}{\sqrt{3}}\,\varepsilon \Rightarrow \dot{\tilde{V}} \leq -\frac{\varepsilon}{L+M} \tag{15}$$

3) Boundary between region P_4 and P_5:

$$K[f](\mathbf{i_e},t) = \frac{1}{L+M}\,\overline{co}\left\{(\mathbf{v_4}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta}), (\mathbf{v_5}_{\alpha\beta} + \mathbf{e_0}_{\alpha\beta})\right\}$$
$$\partial V = \overline{co}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}\right\}$$
$$\dot{\tilde{V}} = \bigcap_{\lambda\in[0,1]} \frac{1}{L+M}\left(\left(1-\tfrac{\lambda}{2}\right)\left(-\left(1-\tfrac{\chi}{2}\right)\sqrt{\tfrac{2}{3}}E + e_{0\alpha}\right) + \right.$$
$$\left. + \tfrac{\sqrt{3}}{2}\lambda\left(-\chi\tfrac{\sqrt{3}}{2}\sqrt{\tfrac{2}{3}}E + e_{0\beta}\right)\right)$$

When $\dot{\tilde{V}}$ is not empty the intersection over all the values of λ is the following point:

$$\dot{\tilde{V}}_{\lambda=\frac{1}{2}} = \frac{1}{L+M}\left(-\tfrac{3}{4}\sqrt{\tfrac{2}{3}}E + \tfrac{3}{4}e_{0\alpha} + \tfrac{\sqrt{3}}{4}e_{0\beta}\right)$$
$$e_{0\beta} \leq -\sqrt{3}e_{0\alpha} + \sqrt{2}E - \frac{4}{\sqrt{3}}\,\varepsilon \Rightarrow \dot{\tilde{V}} \leq -\frac{\varepsilon}{L+M} \tag{16}$$

to ensure the asymptotic stability of the system the shaded region in Fig. 5 c) has to be discarded. In this manner the condition $\dot{\tilde{V}} \leq -\frac{\varepsilon}{L+M}$ is ensured.

The stability analysis in the other regions follows in the same way. The resulting ADS of the controller [3] ensuring the asymptotic stability of the system is the interior of the bold hexagonal region depicted in Fig. 5, c).

E. Comparison of the three controllers

The ADSs obtained in previous subsections ensure rigorously the asymptotic stability of the systems. The stability considerations stated in [1], [2] and [3] were obtained by means of physical considerations and substantially define the inscribed circumferences depicted in Fig. 5 inside the admissible regions obtained in this paper. From this point of view this work represents an improvement because the stability analysis is made by means of rigorous Lyapunov methods and moreover the admissible regions for the disturbance are enlarged.

The introduced stability analysis allows to compare the performances of the various controllers. The controller [2] is characterized by the smallest ADS while the others two, [1] and [3], have ADSs of the same type. However it is worth to point out some differences among these results. The ADS of controller [1] is basically due to condition (5). Requirement $\dot{\tilde{V}} \leq -\omega(\mathbf{x}) < 0$, both in the case of continuous regions T_i and in boundaries among the regions, does not modify the region depicted in Fig. 3. The same can be stated for the controller [2]. This does not hold as long as controller [3] is taken into account. Indeed, the stability conditions computed in the regions P_i, as for example (14) and (15), lead to a larger hexagon with orientation equal to the one of the controller's TR. This hexagon is drawn in Fig. 5 c) using dashed line. Requirement (5) and stability conditions obtained in the boundaries reduce this region to a smaller one, viz the bold hexagon.

IV. Conclusions

Three multivariable hysteresis current control strategies for three-phases inductive load fed by voltage source inverter have been compared. Nonsmooth Lyapunov functions have been exploited to derive the stability and robustness properties of each solution. In particular, to obtain robustness results which could be practically compared each other, the choice of the Lyapunov functions has been suitably restricted. The presented results enlighten that the different strategies have quite different robustness property with respect to disturbances typically affecting this kind of system. This means that the different strategies have a different effectiveness in using the voltage source inverter capability.

References

[1] C. Rossi and A. Tonielli, "Robust current controller for three-phase inverter using finite-state automaton," *IEEE Trans. Ind. Electron.*, vol. 42, no. 2, pp. 169–178, Apr. 1995.

[2] A. Tilli and A. Tonielli, "Sequential design of hysteresis current controller for three-phase inverter," *IEEE Trans. Ind. Electron.*, vol. 45, no. 5, pp. 771–781, Oct. 1998.

[3] D. Amato, A. Tilli, and A. Tonielli, "Robust current controller for three-phase inverter using sequential logic design and time-hysteresis zone," in *Proc. ICARCV2000*, Singapore, Dec. 2000.

[4] M. P. Kazmierkowski and W. Sulkowski, "A novel vector control scheme for transistor pwm inverter-fed induction motor drive," *IEEE Trans. Ind. Electron.*, vol. 38, pp. 41–47, Feb. 1991.

[5] C.-T. Pan and T.-Y. Chang, "An improved hysteresis current controller for reducing switching frequency," *IEEE Trans. Power Electron.*, vol. 9, pp. 97–104, Jan. 1994.

[6] P. A. Peleties and R. A. DeCarlo, "Asymptotic stability of m-switched systems using lyapunov-like functions," in *Proc. IEEE ACC91*, Boston, June 1991.

[7] M. S. Branicky, "Multiple lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 43, no. 4, pp. 475–482, Apr. 1998.

[8] D. Shevitz and B. E. Paden, "Lyapunov stability theory of nonsmooth systems," *IEEE Trans. Automat. Contr.*, vol. 39, no. 9, pp. 1970–1974, Sept. 1994.

[9] B. E. Paden and S. S. Sastry, "A calculus for computing filipov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Trans. Circuits Syst.*, vol. 34, no. 1, pp. 73–82, Jan. 1987.

[10] P. R. Pagilla and M. Tomizuka, "Control of mechanical systems subject to unilateral constraints," in *Proc. IEEE CDC95*, New Orleans, LA, Dec. 1995.