Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Precise Solution of Stability Problem for Linear Plane Switched Systems

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Abstract— **We consider second-order switched dynamical systems consisting of a family of subsystems. The problem is to find conditions guaranteeing exponential stability of the system for any switching sequence. The most of the results on the problem are obtained by the Lyapunov function method which provides sufficient conditions for system stability. The known necessary and sufficient conditions are too complicated and can hardly be used for actual check of stability. The checkable precise stability results are found for particular second-order systems with two subsystems. In this paper simple explicit necessary and sufficient conditions for exponential stability of a general second-order system with an arbitrary number of subsystems are obtained. It is shown that the boundary of a stability region correspond to either infinitely fast switching (chattering) or a periodic switching law. For the last case, a precise upper bound for the number of switching points is found.**

I.INTRODUCTION

We consider the system

$$\dot{x} = A_i x\,, \tag{1}$$

$$x \in R^2\,, \quad A_i \in \Omega = \{A_1,\dots,A_n\}\,, \quad A_i = [a^i_{pk}]^2_{p,k=1}\,.$$

All the matrices A_i are stable, i.e. the real parts of their eigenvalues are negative. This implies that for all *i*,

$$a^i_{11} + a^i_{22} < 0 \quad \text{and} \quad a^i_{11}a^i_{22} - a^i_{12}a^i_{21} > 0\,. \tag{2}$$

System (1) is called exponentially stable if there exist positive constants C and ν such that for any switching sequence, the solution of (1) satisfies the inequality

$$r(t) \le Cr(0)\exp(-\nu t) \quad \text{for } t > 0\,, \tag{3}$$

where $r(t) = \|x(t)\|$ is the Euclidean norm of $x(t)$. If this condition is not fulfilled for some switching sequence, the system is referred to as unstable.

There exists an extensive literature devoted to stability analysis of system (1) [1]. The most of the results are obtained by the Lyapunov unction method which provides sufficient conditions for system stability. For $n=2$, necessary and sufficient conditions for the existence of a quadratic Lyapunov function were found in [2,3]; however, as is shown in [4], such a function may not exist even when the system is exponentially stable. The necessary and sufficient stability conditions obtained in [5] are too complicated and can hardly be used for an actual check of stability. So far checkable conditions are obtained only for special systems (1) with $n=2$.

II. PRELIMINARY RESULTS

Setting $x_1 = r\cos\varphi,\ x_2 = -r\sin\varphi$ and $R = \ln r$, we reduce (1) to the form

$$\begin{aligned}\dot{R} &= F^i_1(\varphi),\\ \dot{\varphi} &= F^i_2(\varphi)\end{aligned} \tag{4}$$

where

$$F^i_1(\varphi) = a^i_{11}\cos^2\varphi - (a^i_{12} + a^i_{21})\cos\varphi\sin\varphi + a^i_{22}\sin^2\varphi\,,$$
$$F^i_2(\varphi) = (a^i_{22} - a^i_{11})\cos\varphi\sin\varphi + a^i_{12}\sin^2\varphi - a^i_{21}\cos^2\varphi.$$

Note that $F^i_1(\varphi)$ and $F^i_2(\varphi)$ are π-periodic in φ.

The following theorem provides sufficient condition for instability of system (1).

Theorem 1. If for some p,q and $\varphi_0 \in [0,\pi)$,

$$\begin{aligned}&F^q_1(\varphi_0)F^p_2(\varphi_0) - F^p_1(\varphi_0)F^q_2(\varphi_0) = \delta \ge 0,\\ &F^p_2(\varphi_0)F^q_2(\varphi_0) < 0,\end{aligned} \tag{5}$$

then system (1) is unstable.

Proof. Without loss of generality we assume that $F^p_2(\varphi_0) > 0$ and $F^q_2(\varphi) < 0$. Let $R(t),\varphi(t)$ be the solution of (4) with

$$\begin{aligned}&R(0) = R_0,\ \varphi(0) = \varphi_0, i = p \ \text{ for } t \in [0,\varepsilon],\\ &i = q \ \text{ for } t \in (\varepsilon, T].\end{aligned} \tag{6}$$

Since $F^p_2(\varphi_0) > 0$ and $F^q_2(\varphi) < 0$, then for sufficiently small ε and T, $\varphi(t)$ increases for $t \in [0,\varepsilon]$ and decreases

Manuscript received March 7, 2005.
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for $t \in (\varepsilon, T]$. We determine $T = T(\varepsilon)$ by the condition $\varphi(T) = \varphi_0$. Denote by $t_1(\varphi) \in [0, \varepsilon]$ and $t_2(\varphi) \in [\varepsilon, T]$ the corresponding inverse functions,

$$\tau(\varphi) = t_2(\varphi) - t_1(\varphi), \ \rho(\varphi) = R(t_2(\varphi)) - R(t_1(\varphi)). \quad (7)$$

From (8) one has

$$\frac{d\tau}{d\varphi} = \frac{1}{F_2^q(\varphi)} - \frac{1}{F_2^p(\varphi)},$$

$$\frac{d\rho}{d\varphi} = \frac{F_1^q(\varphi)}{F_2^q(\varphi)} - \frac{F_1^p(\varphi)}{F_2^p(\varphi)},$$

whence

$$\frac{d\rho}{d\tau} = \Phi_{pq}(\tau),$$
$$\Phi(\tau) = \frac{F_1^q(\varphi(\tau))F_2^p(\varphi(\tau)) - F_1^p(\varphi(\tau))F_2^q(\varphi(\tau))}{F_2^p(\varphi(\tau)) - F_2^q(\varphi(\tau))} \quad (8)$$

where $\varphi(\tau)$ is determined by first equality (7).

Observing that $\rho(0) = 0, \ \rho(T) = R(T) - R(0)$, we find

$$R(T) - R_0 = \int_0^T \Phi_{pq}(\tau)\, d\tau = \Phi_{pq}(\tau')T \quad (9)$$

where $\tau' \in [0, T]$.

Setting in (4) $i = p$ for $t \in [0, \varepsilon)$, $i = q$ for $t \in [\varepsilon, T)$, $i(t+T) = i(t)$, we find that the solution of the second equation, $\varphi(t)$, is T-periodic and, therefore,

$$R(kT) = kR(T) + R_0, \quad k=1,2,\ldots \quad (10)$$

From (2.6 and (10) it follows that the Lyapunov exponent of the function $r(t)$,

$$\lambda = \lim_{t\to\infty} \frac{\ln r(t)}{t} = \Phi_{pq}(\tau'). \quad (11)$$

Since $T, \tau' \to 0, \varphi(\tau') \to \varphi(0) = \varphi_0$ as $\varepsilon \to 0$, then, by (5), (8) and (11), $\lambda > 0$ for $\delta > 0$ and sufficiently small ε. If $\delta = 0$, $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$ and, therefore, the value ν in (3) can be made arbitrarily small. Thus, according to the above definition, in the both cases the system is unstable. ⌐

III. MAIN RESULTS

Let us establish necessary and sufficient conditions for exponential stability of system (1).

Taking into account Theorem 1, in what follows we assume that for any $\varphi \in [0, \pi)$ and $p, q \in [1, \ldots, n]$,

$$F_1^q(\varphi_0)F_2^p(\varphi_0) - F_1^p(\varphi_0)F_2^q(\varphi_0) < 0 \text{ when } F_2^p(\varphi_0)F_2^q(\varphi_0) < 0. \quad (12)$$

From (4) it follows that the phase trajectory $R(\varphi)$ satisfies the equation

$$\frac{dR}{d\varphi} = F_i(\varphi) \quad (13)$$

where

$$F_i(\varphi) = \frac{F_1^i(\varphi)}{F_2^i(\varphi)}.$$

Suppose that for $\varphi \in [0, \pi)$,

$$\min_i F_2^i(\varphi) < 0 \quad \text{and} \quad \max_i F_2^i(\varphi) > 0. \quad (14)$$

Then for any $\varphi \in [0, \pi)$, there exist

$$F^+(\varphi) = \max_i F_i(\varphi) \text{ for } F_2^i(\varphi) > 0 \text{ and}$$
$$F^-(\varphi) = \max_i F_-(\varphi) \text{ for } F_2^i(\varphi) < 0 \quad (15)$$

Let us put

$$\beta_+ = \int_0^\pi F^+(\varphi)\, d\varphi, \ \beta_- = \int_0^\pi F^-(\varphi)\, d\varphi. \quad (16)$$

Theorem 2. For exponential stability of system (1), (3.1), it is necessary and sufficient that

$$\beta_+ < 0 \text{ and } \beta_- < 0. \quad (17)$$

Proof. Let $R_+(\varphi)$ satisfies the equation

$$\frac{dR}{d\varphi} = F^+(\varphi), \ R(0) = R_0. \quad (18)$$

Denote by $i_+(\varphi)$ the piecewise constant integer valued function such that $F^+(\varphi) = F_{i_+(\varphi)}(\varphi)$**.** Clearly, $R_+(\varphi)$ is the phase trajectory of equation (4) with $i(t) = i_+(t) = i_+(\varphi_+(t))$ where $\varphi_+(t)$ is determined by second equation (4) with $i(\varphi) = i_+(\varphi)$. Since, by definition, $F_2^{i_+(\varphi)}(\varphi) > 0$, then $\dot{\varphi}_+(t) > 0$, so $\varphi_+(T) = \pi$ for some *T*.

By (16) and (18), $R_+(\pi) - R_0 = \beta_+$. Taking into account periodicity of $F_+(\varphi)$, we find

$$\varphi_+(kT) = k\pi, \ R_+(kT) = R_0 + k\beta_+, \ k = 1,2,\ldots \quad (19)$$

Observing that $r_+(t) = \exp(R_+(t))$, we find that $r_+(t) \to \infty$ for $\beta_+ > 0$ and $r_+(t)$ is periodic for $\beta_+ = 0$.

Clearly, $x_1(t) = r_+(t)\cos\varphi_+(t)$, $x_2(t) = -r_+(t)\sin\varphi_+(t)$ is the solution of (1) with $x_1(0) = R_0$, $x_2(0) = 0$ and

$i = i_+(t)$. Therefore, for $\beta_+ \geq 0$, the system is not exponentially stable that proves necessity of condition (17). Let us prove its sufficiency.

Let $R(t), \varphi(t)$ be a solution of (4) under a switching signal $i(\varphi)$. Suppose first that the corresponding function $F_2^{i(\varphi)}(\varphi) \geq \alpha > 0$, then $\varphi(t)$ monotonically increases and $\varphi(t) \to \infty$ as $t \to \infty$. Without loss of generality we assume that $\varphi(0) = 0$ and $\dot{\varphi}(0) > 0$. The corresponding phase trajectory $R(\varphi)$ satisfies the equation

$$\frac{dR}{d\varphi} = F_{i(\varphi)}(\varphi) . \qquad (20)$$

By definition, $F^+(\varphi) \geq F_{i(\varphi)}(\varphi)$, so for $R(0) \leq R_+(0)$ one has

$$R(\varphi) \leq R_+(\varphi) \qquad (21)$$

and, therefore, under condition (17) $R(\varphi) \to -\infty$ as $\varphi \to \infty$.

The Lyapunov exponents of the function $r_+(t)$,

$$\lambda_+ = \lim_{t\to\infty} \frac{R_+(t)}{t} = \frac{\beta_+}{T} < 0. \qquad (22)$$

Observing that $\varphi_+(kT) = k\pi, \varphi(kT) > \alpha kT$ and taking into account (21) and (22), we find that the Lyapunov exponents of the function $r(t)$ is also negative.

Inequality (21) holds also when $\varphi(t)$ decreases ($F_2^i(\varphi(t)) < 0$) for some t. In fact, otherwise, the phase trajectory $R(\varphi)$ crosses $R_+(\varphi)$ at a point φ_0, which is impossible in view of inequality (3.1). The equality $\dot{\varphi} = 0$ for $\varphi = \varphi_0$ is also impossible, because it means that for some fixed *i*, (4) admits a solution $\varphi(t) \equiv \varphi_0, R = kt, k > 0$ what contradicts to the supposition on stability of the individual subsystems (1).

The case $\varphi(t) \to -\infty$ as $t \to \infty$ is reduced to that considered above by substitution $t \to -t$.

Suppose now that $\varphi(t)$ is bounded on $(0, \infty)$. The last, in particular, the case when conditions (14) are not fulfilled for some φ_1 and φ_2, respectively (so that, any solution of (4), $\varphi(t) \in (\varphi_1, \varphi_2)$, provided that $\varphi(0) \in (\varphi_1, \varphi_2)$). If $\varphi(t) \to C$ as $t \to \infty$, then for some fixed *i*, equation (4) admits a solution $\varphi(t) \equiv C$; since, by assumption, each individual subsystem is exponentially stable, then $F_2^i(C) < 0$, i.e. $r(t)$ exponentially tends to zero.

If $\varphi(t)$ has no limit at infinity, then there exists a constant C such that $\varphi(t_i) = C, i = 1,2,....$ and $t_i \to \infty$ as $i \to \infty$. Let us put

$$\lambda(t_i) = \frac{1}{t_i} \ln r(t_i) . \qquad (23)$$

Using the representations

$$\lambda(t_i) = \prod_{k=2}^{i} \frac{r(t_k)}{r(t_{k-1})}, \qquad t_i = \sum_{k=2}^{i} (t_k - t_{k-1}), \qquad (24)$$

we find that the Lyapunov exponent of $r(t)$,

$$\lambda = \lim_{i\to\infty} \frac{\sum_{k=2}^{i} \ln[r(t_k)/r(t_{k-1})]}{\sum_{k=2}^{i} (t_k - t_{k-1})} \leq \sup_k \frac{\ln[r(t_k)/r(t_{k-1})]}{t_k - t_{k-1}} =$$

$$\sup_k \frac{R(t_k) - R(t_{k-1})}{t_k - t_{k-1}}. \qquad (25)$$

Suppose first that on each interval (t_i, t_{i+1}), the function $\varphi(t)$ has only one extremum. Then, as is seen from the proof of Theorem 1 and inequality (3.1), $\lambda < 0$ and, therefore, the solution $r(t)$ is exponentially stable.

In the general case, an interval (t_i, t_{i+1}) can be represented as a union of subintervals (t_k^i, t_{k+1}^i) and (t_{k+2}^i, t_{k+3}^i), $k = 1,...,q(i)$ for which $\varphi(t)$ changes monotonically and $\varphi(t_k^i) = \varphi(t_{k+3}^i)$, $\varphi(t_{k+1}^i) = \varphi(t_{k+2}^i)$. The further proof is quite analogous. ↿

IV. CONCLUDING REMARKS

As follows from the above results, for a complete stability analysis of system (1), it is sufficient to check conditions (3.1) and (17). The boundary of the stability region in the space $\Omega = \{A_1,..., A_n\}$ is determined, respectively, by the equalities $\delta = 0$ and $\beta_+ = 0$ or $\beta_- = 0$.

For $\delta = 0$, as is seen from the proof of Theorem 1, the critical solution is characterized by infinitely fast switching (chattering) between two subsystems, A_p and A_q.

For $\beta_+ = 0$ or $\beta_- = 0$, the critical solutions of (4), $R_+(\varphi)$ and $R_-(\varphi)$, are π-periodic and, therefore, the corresponding solutions of (1) is of the kind $x(t + T/2) = -x(t)$. Let N be the total number of the respective switching points $t_k \in [0, T/2)$. The following theorem provides an upper bound for the value N depending on the number of subsystems.

Theorem 3. The value N satisfies the inequality

$$N \leq 2(n-1) . \qquad (26)$$

Proof. Let a boundary point is determined by the condition $\beta_+ = 0$. As is shown above, the switching points

$t_k \in [0, T/2)$ correspond to the points $\varphi_1, \ldots, \varphi_N \in [0, \pi)$ of discontinuity of the function $i_+(\varphi)$, which are determined by the condition

$$F_{i_+(\varphi)}(\varphi) = F^+(\varphi) \,. \tag{27}$$

Let us put

$$F_{pq}(\varphi) = F_1^p(\varphi) F_2^q(\varphi) - F_2^p(\varphi) F_1^q(\varphi) = \\ C_{pq} + A_{pq} \sin(2\varphi + \alpha_{pq}). \tag{28}$$

Clearly, $F_p(\varphi_k) = F_q(\varphi_k)$, so $F_{pq}(\varphi_k) = 0$. From (4.3) it is evident that the function $F_{pq}(\varphi)$ has no more than two zeros for $\varphi \in [0, \pi)$. Therefore, for $n = 2$, inequality (26) is true.

Let us show that for any $n \geq 2$, the value N increases no more than by two as the number of subsystems, n, increases by one.

Without loss of generality we assume that

$$F_1(0) \leq F_2(0) \leq \ldots \leq F_{n+1}(0) \,. \tag{29}$$

Let $F^+(\varphi, n) = F^+(\varphi)$ for n subsystems. Suppose that $F_{n+1}(\varphi)$ crosses $F^+(\varphi, n)$ at some points φ_1 and $\varphi_2 > \varphi_1$; let $F_{n+1}(\varphi_2) = F_k(\varphi_2)$. Since, by definition, $F^+(\varphi, n) = \max F_i(\varphi)$ for $i \leq n$ and $F_{n+1}(0) \leq F_k(0), k \leq n$, then $F_{n+1}(\varphi)$ crosses $F_k(\varphi)$ at some points $\varphi_3 \leq \varphi_1$. Therefore, $F_{n+1}(\varphi) < F_k(\varphi) \leq F^+(\varphi, n)$ for $\varphi \in (\varphi_2, \pi)$, so the number of the points $\varphi_i \in (\varphi_{n+1}, \pi)$ remains unchanged. î

Note that for $n > 2$, bound (26) improves the known result, $N \leq n(n-1)$, obtained in [5].

It can be shown that bound (26) is precise in the sense that for any n, system (1) can be constructed such that the corresponding value $N = 2(n-1)$.

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