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Output synchronization control of Euler-Lagrange systems with nonlinear damping terms

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Abstract— **A coordinated synchronization control scheme to synchronize two or more Euler-Lagrange systems with nonlinear damping in a leader-follower configuration is presented. The scheme is based on position measurements only, and no mathematical model of the leader is required. Observers are designed to estimate the velocity and acceleration of the systems, and the scheme yields semi-global ultimately bounded closed-loop errors for the output synchronization problem. The control scheme is valid for systems with nonlinear damping.**

I. INTRODUCTION

Synchronization of two systems in a leader-follower configuration can be considered as a tracking control problem where the reference is a physical object with dynamics that is subject to disturbances and actuator limitations (e.g. robot arm, ship, satellite, underwater vehicle). As opposed to tracking a theoretical and ideal reference path, the actual states of the reference object can diverge from its ideal path due to disturbances, unmodeled dynamics, actuator limitations, poor control design or actuator failure. Under these constraints we cannot guarantee that the reference object tracks its desired path perfectly, and knowledge of the desired path of the reference may thus not be enough to assure synchronization in the leader-follower system. In particular, any two physical systems that is not identical in their design will experience different impacts from environmental forces such as wind, drag, current, terrain or waves. This difference may possibly lead to critical situations when employing simple tracking controllers to predefined reference paths where the coordination of the two systems is only done at the path planning stage, and not through active control. The output synchronization control scheme with only position measurements of the physical reference is also different from the output tracking problem in that the velocity and acceleration of the reference is unknown, and must be estimated based only on the position measurements. Output synchronization control is an important aspect in applications such as formation control of vehicles and teleoperation.

Synchronization is found both as a natural phenomenon in nature like in the flashing of fireflies, choruses of crickets and musical dancing, as well as the controlled synchronization of a pacemaker or a transmitter-receiver system. Synchronization has recently attracted an increasing interest from researchers within physics, dynamical systems, circuit theory, and more lately control theory through [1]. The synchronization control problem can be seen as making a set of physical objects cooperate in their states. [2] and [3] have expanded on traditional tracking methods with predefined paths, and introduced a feedback from the actual position of a object (subject to disturbances) to the other objects through a path parametrization variable. All objects have predefined paths with individual tracking controllers requiring mathematical models and control availability, and the objects synchronize in terms of progression along the path. Thus, disturbances affecting tracking performance along the path is canceled, but cross-track errors due to any difference in disturbances are not. [2] used a coordinated approach with a leader and a follower, while [3] allowed for a cooperative approach where all objects mutually coordinate to the reference. Both a coordinated and a cooperative synchronization control scheme where the objects are synchronized in their states were presented in [4] and [5], and applied to robot control. Based on these results, a synchronization scheme for ship rendezvous control at sea for underway replenishment was presented in [6] with experimental results in [7]. There is no need for a predefined path or a dynamic model for the leader in the coordinated schemes of [5] and [6], and the coordination of the objects is achieved using a controller that synchronizes the position and velocity of each follower system to the leader based on position measurements only. This places all the control responsibility on the followers, and permits coordinated motion between a leader and a follower in situations where the control design and mathematical model of the leader is unknown or unavailable. Disturbances affecting the objects differently are inherently canceled through the synchronization. For a view on the output tracking problem from a synchronization perspective, see [8].

Passivity-based tracking control of Euler-Lagrange systems through energy-shaping and damping injection has been thoroughly elaborated in [9] for state-feedback systems, while a nonlinear dynamic output feedback control approach for a class of Euler-Lagrange systems was suggested in [10]. The nonlinear damping was injected without velocity measurements using a dynamic extension technique and a dissipation propagation condition. The results were extended in [11] and [12] for systems with input constraints. [13] suggested a tracking controller with a velocity observer for a class of mechanical systems with nonlinear damping terms, and [14] proposed an output tracking observer-controller scheme to estimate the velocity by imposing a monotone

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damping condition on the nonlinearities in the unmeasured states. A systematic nonlinear observer design procedure for a class of Euler-Lagrange systems can be found in [15], while a passivity-based tracking controller and velocity observer design using the sliding surface of [16] for robots can be found in [17].

This paper expands the results of [5] and [6] to the synchronization of Euler-Lagrange systems with nonlinear damping terms. In addition, application of the sliding surface approach of [16] introduces a stable passivity-based filtering of the unknown states of the leader system. It also expands the results of [11] and [13] to the output synchronization problem where the time derivatives of the reference are unknown. The leader-follower output synchronization closed-loop error dynamics is shown to be semi-globally ultimately bounded with an arbitrarily small bound.

The Euler-Lagrange system with the necessary properties are presented in Section II, while the output synchronization control scheme with controller and observers are presented in section III. Stability is addressed in Section IV, and simulations are presented in Section V. Conclusions and future work are presented in Section VI.

II. Preliminaries

In this paper we consider systems described by Euler-Lagrange equations of the form

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}(\mathbf{x},\dot{\mathbf{x}})}{\partial \dot{\mathbf{x}}}\right)-\frac{\partial \mathscr{L}(\mathbf{x},\dot{\mathbf{x}})}{\partial \mathbf{x}}+\frac{\partial F(\dot{\mathbf{x}})}{\partial \dot{\mathbf{x}}}=\tau \tag{1}$$

where $\mathbf{x}\in\mathbb{R}^n$ are generalized coordinates assumed measurable, and $\tau\in\mathbb{R}^n$ are generalized forces acting on the system. $\mathscr{L}(\mathbf{x},\dot{\mathbf{x}})=T(\mathbf{x},\dot{\mathbf{x}})-V(\mathbf{x})$ is the Lagrangian function of potential energy $V(\mathbf{x})$ and kinetic energy

$$T(\mathbf{x},\dot{\mathbf{x}})=\frac{1}{2}\dot{\mathbf{x}}^T\mathbf{I}(\mathbf{x})\dot{\mathbf{x}},\qquad \mathbf{I}(\mathbf{x})=\mathbf{I}^T(\mathbf{x})>0 \tag{2}$$

The frictional forces in the system is derived from the scalar dissipative function $F(\dot{\mathbf{x}})$ defined from the rate of energy E dissipating from the system as ([18])

$$\frac{dE}{dt}=-F(\dot{\mathbf{x}})=\frac{1}{n+1}\sum_{i=1}^{n}c_i|\dot{x}_i|^{n+1} \tag{3}$$

where F is a power function and in general a function of the velocity and c_i, $i=1,\ldots n$ are positive damping coefficients. For $n=1$ this is known as Rayleigh's dissipation function. Considering only fully actuated Euler-Lagrange systems with dynamics

$$\mathbf{I}(\mathbf{x})\ddot{\mathbf{x}}+\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\dot{\mathbf{x}}+\mathbf{d}(\mathbf{x},\dot{\mathbf{x}})+\mathbf{g}(\mathbf{x})=\tau \tag{4}$$

where $\mathbf{I}(\mathbf{x})$ is the positive definite inertia matrix (including added mass effects), $\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\dot{\mathbf{x}}$ is the vector of Coriolis and centrifugal forces and $\mathbf{g}(\mathbf{x})=\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}$. The dissipative forces $\frac{\partial F(\dot{\mathbf{x}})}{\partial \dot{\mathbf{x}}}=\mathbf{d}(\mathbf{x},\dot{\mathbf{x}})=\mathbf{D}(\mathbf{x},\dot{\mathbf{x}})\dot{\mathbf{x}}$ is dry friction for $n=0$ in $\frac{\partial F(\dot{\mathbf{x}})}{\partial \dot{\mathbf{x}}_i}=c_i|\dot{x}_i|^{n-1}\dot{x}_i$, linear viscous friction or Newtonian damping for $n=1$, and quadratic damping for $n=2$. Further, we assume that the following well known properties holds for $x,\ y,\ z\in\mathbb{R}^n$ ([9])

- P1 The Coriolis matrix satisfies $\mathbf{C}(\mathbf{x},\mathbf{y})\mathbf{z}=\mathbf{C}(\mathbf{x},\mathbf{z})\mathbf{y}$.
- P2 The inertia matrix $\mathbf{I}(\mathbf{x})$ is positive definite, differentiable in $\mathbf{x}$ and satisfies $\mathbf{y}^T\left(\dot{\mathbf{I}}(\mathbf{x})-2\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\right)\mathbf{y}=0$.

Throughout this paper, the minimum and maximum eigenvalue of a positive definite matrix $\mathbf{M}$ will be denoted as $\mathbf{M}_m$ and $\mathbf{M}_M$, respectively. The norm of a vector $\mathbf{x}$ is defined as

$$\|\mathbf{x}\|=\sqrt{\mathbf{x}^T\mathbf{x}} \tag{5}$$

and the induced norm of a matrix $\mathbf{M}$ is

$$\|\mathbf{M}\|=\max_{\|\mathbf{x}\|=1}\|\mathbf{M}\mathbf{x}\| \tag{6}$$

Additional assumptions are made on the bounds of the system matrices and dissipative term as in [19]:

- P3 The norm of $\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})$ satisfies $\|\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\|\le C_M\|\dot{\mathbf{x}}\|$
- P4 The inertia matrix satisfy $0<\mathbf{I}_m\le\mathbf{I}(\mathbf{x})\le\mathbf{I}_M<\infty$.
- P5 The dissipative vector $\mathbf{d}(\mathbf{x},\dot{\mathbf{x}})$ is continuously differentiable in $\mathbf{x}$ and $\dot{\mathbf{x}}$ and satisfies

$$\exists\delta,\quad \forall\mathbf{x},\dot{\mathbf{x}},\mathbf{y},\quad \mathbf{y}^T\frac{\partial\mathbf{d}(\mathbf{x},\dot{\mathbf{x}})}{\partial\dot{\mathbf{x}}}\mathbf{y}\ge\delta\mathbf{y}^T\mathbf{y} \tag{7}$$

for a positive constant $\delta>0$, and $\frac{\partial\mathbf{d}(\mathbf{x},\dot{\mathbf{x}})}{\partial\dot{\mathbf{x}}}$ is bounded for all $\mathbf{x}$ and for $\dot{\mathbf{x}}$ bounded.

Under these assumptions we will propose a synchronization scheme for the dynamics in Eq. (4) in Section III. The following results will be useful in the stability analysis of Section IV:

Definition 1: ([20]) The solutions of $\dot{x}=f(t,x)$ are **uniformly ultimately bounded** if there exits positive constants b and c such that for every $a\in(0,c)$ there is $T=T(a,b)$ such that

$$\|x(t_0)\|\le a\ \Rightarrow\ \|x(t)\|\le b,\quad \forall t\ge t_0+T \tag{8}$$

with ultimate bound b. If this holds for an arbitrarily large a, then it is **globally uniformly ultimately bounded**.

Lemma 1: ([5]) Consider the following function $g:\mathbb{R}\to\mathbb{R}$

$$g(y)=\alpha_0-\alpha_1 y+\alpha_2 y^2,\qquad y\in\mathbb{R}^+ \tag{9}$$

where $\alpha_i>0, i=0,1,2$. Then $g(y)<0$ if $y_1<y<y_2$, where

$$y_1=\frac{\alpha_1-\sqrt{\alpha_1^2-4\alpha_2\alpha_0}}{2\alpha_2}\qquad y_2=\frac{\alpha_1+\sqrt{\alpha_1^2-4\alpha_2\alpha_0}}{2\alpha_2} \tag{10}$$

with $y_1,y_2>0$.

Proposition 1: ([5]) Let $x(t)\in\mathbb{R}^n$ be the solution of the differential equation $\dot{x}=f(t,x(t))$ where $f(t,x(t))$ is Lipschitz and under initial conditions $x(t_0)=x_0$, and assume that there exists a function $V(x(t),t)$ that satisfies

$$\begin{aligned}P_m\|x(t)\|^2\le V(x(t),t)\le P_M\|x(t)\|^2 \\ \dot{V}(x(t),t)\le\|x(t)\|\cdot g(\|x(t)\|)<0,\quad \forall y_1<\|x(t)\|<y_2\end{aligned} \tag{11}$$

with P_m and P_M positive constants, $g(\cdot)$ as in (9) and y_1,y_2 as in (10). If $y_2>\sqrt{P_m^{-1}P_M}\,y_1$ then $x(t)$ is **locally uniformly ultimately bounded**.

This implies that a system is uniformly ultimately bounded if it has a Lyapunov function whose time derivative is negative in an annulus of a certain width around the origin ([5]).

III. Synchronization scheme

The synchronization control scheme synchronizes the states $\mathbf{x}$ and $\dot{\mathbf{x}}$ of an Euler-Lagrange system to the states $\mathbf{y}_d$ and $\dot{\mathbf{y}}_d$ of a reference system. The reference system is assumed to be a physical Euler-Lagrange system with an unknown dynamic model with the position vector $\mathbf{y}_d$ as the only measured output. The synchronization scheme adopts a leader-follower strategy, where the position $\mathbf{x}$ and velocity $\dot{\mathbf{x}}$ of the follower is synchronized to the position $\mathbf{y}_d$ and velocity $\dot{\mathbf{y}}_d$ of the leader. The synchronization objective is to find state observers and a feedback control law $\tau(\mathbf{x},\mathbf{y}_d)$ such that the synchronization error defined as

$$\mathbf{e}(t)=\begin{bmatrix}\mathbf{e}\\ \dot{\mathbf{e}}\end{bmatrix}=\begin{bmatrix}\mathbf{x}-\mathbf{y}_d\\ \dot{\mathbf{x}}-\dot{\mathbf{y}}_d\end{bmatrix} \tag{12}$$

is uniformly ultimately bounded as defined in Definition 1. The reference motion in the synchronization problem is given by the leader system where only the position $\mathbf{y}_d$ is known. The unknown reference states $\dot{\mathbf{y}}_d$ and $\ddot{\mathbf{y}}_d$ can be passively filtered by restricting the position error $\mathbf{e}$ to lie on a sliding surface ([16])

$$\dot{\mathbf{e}}+\mathbf{\Lambda}\mathbf{e}=0 \tag{13}$$

where $\mathbf{\Lambda}$ is a constant matrix whose eigenvalues are strictly in the right half complex plane (e.g. of positive elements). This is achieved by replacing the unknown reference states $\dot{\mathbf{y}}_d$ and $\ddot{\mathbf{y}}_d$ with a virtual reference trajectory

$$\mathbf{y}_r=\mathbf{y}_d-\mathbf{\Lambda}\int_0^t\mathbf{e}\,dt,\qquad \dot{\mathbf{y}}_r=\dot{\mathbf{y}}_d-\mathbf{\Lambda}\mathbf{e},\qquad \ddot{\mathbf{y}}_r=\ddot{\mathbf{y}}_d-\mathbf{\Lambda}\dot{\mathbf{e}} \tag{14}$$

and define

$$\mathbf{s}=\dot{\mathbf{x}}-\dot{\mathbf{x}}_r=\dot{\mathbf{e}}+\mathbf{\Lambda}\mathbf{e} \tag{15}$$

as a measure of tracking. The vector $\mathbf{s}$ conveys information about boundedness and convergence of $\mathbf{x}$ and $\dot{\mathbf{x}}$, and the definition can be seen as a stable first-order differential equation in $\mathbf{e}$ with $\mathbf{s}$ as an input. For bounded initial conditions, boundedness of $\mathbf{s}$ will imply boundedness of $\mathbf{e}$ and $\dot{\mathbf{e}}$ (and thus for $\mathbf{x}$ and $\dot{\mathbf{x}}$). The formal definition of $\mathbf{y}_r$ in (14) is equivalent to adding an internal feedback loop in the controller, but the integral term $\int_0^t\mathbf{e}\,dt$ will not be used explicitly in the controller.

Remark 1: Note that the uniform ultimate boundedness of the synchronization scheme is valid for $\mathbf{\Lambda}\geq 0$.

A. State feedback synchronization

Rewriting the dynamics of (4) using (15) gives

$$\begin{aligned}\mathbf{I}(\mathbf{x})\dot{\mathbf{s}} &= -\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\mathbf{s}-\mathbf{d}(\mathbf{x},\dot{\mathbf{x}})+\mathbf{d}(\mathbf{x},\dot{\mathbf{y}}_r)+\tau \\ &\quad -\mathbf{I}(\mathbf{x})\ddot{\mathbf{x}}_r-\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\dot{\mathbf{y}}_r-\mathbf{d}(\mathbf{x},\dot{\mathbf{y}}_r)-\mathbf{g}(\mathbf{x})\end{aligned} \tag{16}$$

and using the Lyapunov function

$$V(t)=\frac{1}{2}\mathbf{s}^T\mathbf{I}(\mathbf{x})\mathbf{s}+\frac{1}{2}\mathbf{e}^T\mathbf{K}_p\mathbf{e},\qquad \mathbf{K}_p=\mathbf{K}_p^T>0 \tag{17}$$

we see that choosing the control law

$$\tau=\mathbf{I}(\mathbf{x})\ddot{\mathbf{y}}_r+\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\dot{\mathbf{y}}_r+\mathbf{d}(\mathbf{x},\dot{\mathbf{y}}_r)+\mathbf{g}(\mathbf{x})-\mathbf{K}_p\mathbf{e}-\mathbf{K}_d\mathbf{s} \tag{18}$$

gives

$$\dot{V}(t)=-\mathbf{s}^T\left(\mathbf{d}(\mathbf{x},\dot{\mathbf{x}})-\mathbf{d}(\mathbf{x},\dot{\mathbf{y}}_r)+\mathbf{K}_d\mathbf{s}\right)-\mathbf{e}^T\mathbf{K}_p\mathbf{\Lambda}\mathbf{e} \tag{19}$$

The Mean Value Theorem, see for instance [20], and Property P5 implies that

$$\mathbf{d}(\mathbf{x},\dot{\mathbf{x}})-\mathbf{d}(\mathbf{x},\dot{\mathbf{y}}_r)=\left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{z})}{\partial\mathbf{z}}\right|_{\mathbf{z}=\xi}\mathbf{s} \tag{20}$$

where ξ is on the line segment between $\dot{\mathbf{x}}$ and $\dot{\mathbf{y}}_r$. Similarily,

$$\mathbf{d}(\mathbf{x},\dot{\mathbf{x}})-\mathbf{d}\left(\mathbf{x},\hat{\dot{\mathbf{x}}}\right) = \left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{v})}{\partial\mathbf{v}}\right|_{\mathbf{v}=\zeta}\tilde{\dot{\mathbf{x}}} \tag{21}$$

$$\mathbf{d}\left(\mathbf{x},\hat{\dot{\mathbf{y}}}_r\right)-\mathbf{d}(\mathbf{x},\dot{\mathbf{y}}_r) = \left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{w})}{\partial\mathbf{w}}\right|_{\mathbf{w}=\kappa}\tilde{\dot{\mathbf{y}}}_r \tag{22}$$

where ζ is on the line segment between $\dot{\mathbf{x}}$ and $\hat{\dot{\mathbf{x}}}$, and κ is on the line segment between $\hat{\dot{\mathbf{y}}}_r$ and $\dot{\mathbf{y}}_r$. Now (19) becomes

$$\dot{V}(t)=-\mathbf{s}^T\left(\left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{z})}{\partial\mathbf{z}}\right|_{\mathbf{z}=\xi}+\mathbf{K}_d\right)\mathbf{s}-\mathbf{e}^T\mathbf{K}_p\mathbf{\Lambda}\mathbf{e} \tag{23}$$

Since $V(t)$ is positive definite, and $\dot{V}(t)$ is negative definite it follows that the equilibrium $(\mathbf{e},\mathbf{s})=(\mathbf{0},\mathbf{0})$ is globally exponentially stable (GES), and from convergence of $\mathbf{s}\to\mathbf{0}$ and $\mathbf{e}\to\mathbf{0}$ that $\dot{\mathbf{e}}\to\mathbf{0}$.

B. Output synchronization

The control law (18) cannot be implemented when only the states $\mathbf{x}$, $\mathbf{e}$ are measured. Instead we design a control law that depends on estimated values for the states $\dot{\mathbf{x}}$, $\dot{\mathbf{y}}_r$, $\ddot{\mathbf{y}}_r$, $\mathbf{s}$ as

$$\tau=\mathbf{I}(\mathbf{x})\hat{\ddot{\mathbf{y}}}_r+\mathbf{C}\left(\mathbf{x},\hat{\dot{\mathbf{x}}}\right)\hat{\dot{\mathbf{y}}}_r+\mathbf{d}\left(\mathbf{x},\hat{\dot{\mathbf{y}}}_r\right)+\mathbf{g}(\mathbf{x})-\mathbf{K}_p\mathbf{e}-\mathbf{K}_d\hat{\mathbf{s}} \tag{24}$$

We design a full-state nonlinear observer to estimate $\hat{\mathbf{e}}$ and $\hat{\mathbf{s}}$ as

$$\frac{d}{dt}\hat{\mathbf{e}} = \hat{\mathbf{s}}-\mathbf{\Lambda}\hat{\mathbf{e}}+L_{\mathbf{e}1}\tilde{\mathbf{e}} \tag{25}$$

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{s}} &= -\mathbf{I}^{-1}(\mathbf{x})\left[\mathbf{C}\left(\mathbf{x},\hat{\dot{\mathbf{x}}}\right)\hat{\mathbf{s}}+\mathbf{d}\left(\mathbf{x},\hat{\dot{\mathbf{x}}}\right)-\mathbf{d}\left(\mathbf{x},\hat{\dot{\mathbf{y}}}_r\right)\right.\\ &\qquad\left.+\mathbf{K}_p\hat{\mathbf{e}}+\mathbf{K}_d\hat{\mathbf{s}}\right]+L_{\mathbf{e}2}\tilde{\mathbf{e}}\end{aligned} \tag{26}$$

where $\tilde{\mathbf{e}}=\mathbf{e}-\hat{\mathbf{e}}$. Similarly an observer for the states $\hat{\mathbf{x}}$ and $\hat{\dot{\mathbf{x}}}$ is

$$\frac{d}{dt}\hat{\mathbf{x}} = \hat{\dot{\mathbf{x}}}+L_{\mathbf{x}1}\tilde{\mathbf{x}} \tag{27}$$

$$\begin{aligned}\frac{d}{dt}\hat{\dot{\mathbf{x}}} &= -\mathbf{I}^{-1}(\mathbf{x})\left[\mathbf{C}\left(\mathbf{x},\hat{\dot{\mathbf{x}}}\right)\hat{\mathbf{s}}+\mathbf{d}\left(\mathbf{x},\hat{\dot{\mathbf{x}}}\right)-\mathbf{d}\left(\mathbf{x},\hat{\dot{\mathbf{y}}}_r\right)\right.\\ &\qquad\left.+\mathbf{K}_p\mathbf{e}+\mathbf{K}_d\hat{\mathbf{s}}\right]+L_{\mathbf{x}2}\tilde{\mathbf{x}}\end{aligned} \tag{28}$$

where $\tilde{\mathbf{x}}=\mathbf{x}-\hat{\mathbf{x}}$. The virtual reference states $\dot{\mathbf{y}}_r$ and $\ddot{\mathbf{y}}_r$ in the control law (24) can now be algebraically found through

$$\hat{\dot{\mathbf{y}}}_r=\hat{\dot{\mathbf{x}}}-\hat{\mathbf{s}},\qquad \hat{\ddot{\mathbf{y}}}_r=-\left(\mathbf{I}^{-1}(\mathbf{x})\mathbf{K}_p+L_{\mathbf{e}2}\right)\tilde{\mathbf{e}}+L_{\mathbf{x}2}\tilde{\mathbf{x}} \tag{29}$$

Remark 2: Note that through the definition of the state observer in (28), the virtual reference acceleration $\ddot{\mathbf{y}}_r$ will be present as a non-vanishing disturbance in the controller-observer scheme, and thus the origin of the closed-loop error space is no longer an equilibrium. The closed-loop error is therefore at best ultimately bounded by some function of the virtual reference acceleration $\ddot{\mathbf{y}}_r$.

IV. Stability

A. Closed-loop error dynamics

The closed-loop error dynamics of the system (4) and the controller (24) is

$$\begin{aligned}\mathbf{I}(\mathbf{x})\dot{\mathbf{s}}+\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\mathbf{s}+\mathbf{d}(\mathbf{x},\dot{\mathbf{x}})+\mathbf{K}_p\mathbf{e}+\mathbf{K}_d\mathbf{s} \;=&\; \qquad (30)\\ \mathbf{I}(\mathbf{x})\frac{d}{dt}\left(\widetilde{\mathbf{s}}-\widetilde{\dot{\mathbf{x}}}\right)+\mathbf{C}\left(\mathbf{x},\widehat{\dot{\mathbf{x}}}\right)\widehat{\dot{\mathbf{y}}}_r-\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\dot{\mathbf{y}}_r+\mathbf{d}\left(\mathbf{x},\widehat{\dot{\mathbf{y}}}_r\right)+\mathbf{K}_d\widetilde{\mathbf{s}}\end{aligned}$$

where $\widetilde{\mathbf{s}}=\mathbf{s}-\widehat{\mathbf{s}}$ and $\widetilde{\dot{\mathbf{x}}}=\dot{\mathbf{x}}-\widehat{\dot{\mathbf{x}}}$ and we have used the fact that $\widehat{\ddot{\mathbf{y}}}_r-\ddot{\mathbf{y}}_r=\frac{d}{dt}\left(\widetilde{\mathbf{s}}-\widetilde{\mathbf{e}}\right)$. Using Property P1 and that $\widehat{\dot{\mathbf{x}}}=\dot{\mathbf{x}}-\widetilde{\dot{\mathbf{x}}}$ we can rewrite

$$\begin{aligned}\mathbf{C}\left(\mathbf{x},\widehat{\dot{\mathbf{x}}}\right)\widehat{\dot{\mathbf{y}}}_r-\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\dot{\mathbf{y}}_r=&\\ \mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\left(\widetilde{\mathbf{s}}-2\widetilde{\dot{\mathbf{x}}}\right)+\mathbf{C}\left(\mathbf{x},\widetilde{\dot{\mathbf{x}}}\right)\left(\widetilde{\dot{\mathbf{x}}}+\mathbf{s}-\widetilde{\mathbf{s}}\right) \qquad (31)\end{aligned}$$

By using (20) and (22) in (30) we get the closed-loop synchronization error dynamics

$$\begin{aligned}\mathbf{I}(\mathbf{x})\dot{\mathbf{s}}+\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\mathbf{s}+\left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{z})}{\partial\mathbf{z}}\right|_{\mathbf{z}=\xi}\mathbf{s}+\mathbf{K}_p\mathbf{e}+\mathbf{K}_d\mathbf{s} \;=&\\ \mathbf{I}(\mathbf{x})\frac{d}{dt}\left(\widetilde{\mathbf{s}}-\widetilde{\dot{\mathbf{x}}}\right)+\mathbf{K}_d\widetilde{\mathbf{s}}+\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\left(\widetilde{\mathbf{s}}-2\widetilde{\dot{\mathbf{x}}}\right) \qquad (32)&\\ +\mathbf{C}\left(\mathbf{x},\widetilde{\dot{\mathbf{x}}}\right)\left(\widetilde{\dot{\mathbf{x}}}+\mathbf{s}-\widetilde{\mathbf{s}}\right)+\left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{w})}{\partial\mathbf{w}}\right|_{\mathbf{w}=\kappa}\left(\widetilde{\mathbf{s}}-\widetilde{\dot{\mathbf{x}}}\right)&\end{aligned}$$

and through (21) in the observers (25) and (28) we get the estimation error dynamics as

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathbf{e}} &= \widetilde{\mathbf{s}}-(\mathbf{\Lambda}+L_{\mathbf{e}1})\widetilde{\mathbf{e}} \qquad (33)\\ \frac{d}{dt}\widetilde{\mathbf{s}} &= \mathbf{I}^{-1}(\mathbf{x})\left[\mathbf{C}\left(\mathbf{x},\widetilde{\dot{\mathbf{x}}}\right)\widetilde{\dot{\mathbf{x}}}+\left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{v})}{\partial\mathbf{v}}\right|_{\mathbf{v}=\zeta}\widetilde{\dot{\mathbf{x}}}-2\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\widetilde{\dot{\mathbf{x}}}\right.\\ &\qquad\left.-2\mathbf{K}_p\widetilde{\mathbf{e}}\;\right]-2L_{\mathbf{e}2}\widetilde{\mathbf{e}}+L_{\mathbf{x}2}\widetilde{\mathbf{x}}-\ddot{\eta}_r \qquad (34)\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathbf{x}} &= \widetilde{\dot{\mathbf{x}}}-L_{\mathbf{x}1}\widetilde{\mathbf{x}} \qquad (35)\\ \frac{d}{dt}\widetilde{\dot{\mathbf{x}}} &= \mathbf{I}^{-1}(\mathbf{x})\left[\mathbf{C}\left(\mathbf{x},\widetilde{\dot{\mathbf{x}}}\right)\widetilde{\dot{\mathbf{x}}}+\left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{v})}{\partial\mathbf{v}}\right|_{\mathbf{v}=\zeta}\widetilde{\dot{\mathbf{x}}}-2\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\widetilde{\dot{\mathbf{x}}}\right.\\ &\qquad\left.-\mathbf{K}_p\widetilde{\mathbf{e}}\;\right]-L_{\mathbf{e}2}\widetilde{\mathbf{e}} \qquad (36)\end{aligned}$$

We will in the following assume for simplicity that

$$L_1=L_{\mathbf{x}1}=L_{\mathbf{e}1},\qquad L_2=L_{\mathbf{x}2}=L_{\mathbf{e}2} \qquad (37)$$

and that the gain matrices $\mathbf{K}_p,\mathbf{K}_d,L_1,L_2$ are symmetric. Introducing a coordinate change through

$$\widetilde{\mathbf{y}}_d=\widetilde{\mathbf{e}}-\widetilde{\eta},\quad \dot{\widetilde{\mathbf{y}}}_r=\widetilde{\mathbf{s}}-\widetilde{\dot{\mathbf{x}}}-L_1\widetilde{\mathbf{y}}_d,\quad \dot{\widetilde{\mathbf{x}}}=\widetilde{\dot{\mathbf{x}}}-L_1\widetilde{\mathbf{x}} \qquad (38)$$

and defining

$$\bar{\mathbf{e}}=\mathbf{e}-\widetilde{\mathbf{y}}_d,\qquad \bar{\mathbf{s}}=\mathbf{s}-\dot{\widetilde{\mathbf{y}}}_r \qquad (39)$$

we can write the synchronization error dynamics of (30) as

$$\begin{aligned}\mathbf{I}(\mathbf{x})\dot{\bar{\mathbf{s}}}+\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\bar{\mathbf{s}}+\left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{z})}{\partial\mathbf{z}}\right|_{\mathbf{z}=\xi}\bar{\mathbf{s}}+\mathbf{K}_p\bar{\mathbf{e}}+\mathbf{K}_d\bar{\mathbf{s}} \;=&\\ \mathbf{I}(\mathbf{x})L_1\dot{\widetilde{\mathbf{y}}}_d-\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\left(\dot{\widetilde{\mathbf{x}}}+L_1(\widetilde{\mathbf{x}}-\widetilde{\mathbf{y}}_d)\right)-\mathbf{K}_p\widetilde{\mathbf{y}}_d&\\ +\mathbf{C}\left(\mathbf{x},\dot{\widetilde{\mathbf{x}}}+L_1\widetilde{\mathbf{x}}\right)(\bar{\mathbf{s}}-L_1\widetilde{\mathbf{y}}_d)+\mathbf{K}_d\left(\dot{\widetilde{\mathbf{x}}}+L_1(\widetilde{\mathbf{x}}+\widetilde{\mathbf{y}}_d)\right)&\\ +\left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{w})}{\partial\mathbf{w}}\right|_{\mathbf{w}=\kappa}\left(\dot{\widetilde{\mathbf{y}}}_d+\mathbf{\Lambda}(\widetilde{\mathbf{x}}+\widetilde{\mathbf{y}}_d)+L_1\widetilde{\mathbf{y}}_d\right)&\\ -\left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{z})}{\partial\mathbf{z}}\right|_{\mathbf{z}=\xi}\left(\dot{\widetilde{\mathbf{y}}}_d+\mathbf{\Lambda}(\widetilde{\mathbf{x}}+\widetilde{\mathbf{y}}_d)\right) \qquad (40)&\end{aligned}$$

and the estimation error dynamics of (25) and (28) as

$$\begin{aligned}\ddot{\widetilde{\mathbf{y}}}_d=& \qquad (41)\\ &-\mathbf{\Lambda}\left(\dot{\widetilde{\mathbf{x}}}+\dot{\widetilde{\mathbf{y}}}_d\right)-\mathbf{I}^{-1}(\mathbf{x})\mathbf{K}_p(\widetilde{\mathbf{x}}+\widetilde{\mathbf{y}}_d)-L_1\dot{\widetilde{\mathbf{y}}}_d-L_2\widetilde{\mathbf{y}}_d-\ddot{\mathbf{y}}_r\end{aligned}$$

and

$$\begin{aligned}\ddot{\widetilde{\mathbf{x}}}=&\\ &\mathbf{I}^{-1}(\mathbf{x})\left[\mathbf{C}\left(\mathbf{x},\dot{\widetilde{\mathbf{x}}}+L_1\widetilde{\mathbf{x}}\right)\left(\dot{\widetilde{\mathbf{x}}}+L_1\widetilde{\mathbf{x}}\right)-2\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\left(\dot{\widetilde{\mathbf{x}}}+L_1\widetilde{\mathbf{x}}\right)\right.\\ &\left.-\left.\frac{\partial\mathbf{d}(\mathbf{x},\mathbf{v})}{\partial\mathbf{v}}\right|_{\mathbf{v}=\zeta}\left(\dot{\widetilde{\mathbf{x}}}+L_1\widetilde{\mathbf{x}}\right)-\mathbf{K}_p(\widetilde{\mathbf{x}}+\widetilde{\mathbf{y}}_d)\right] \qquad (42)\\ &-L_2(\widetilde{\mathbf{x}}+\widetilde{\mathbf{y}}_d)-L_1\dot{\widetilde{\mathbf{x}}}\end{aligned}$$

Note the presence of the non-vanishing disturbance $\ddot{\mathbf{y}}_r$ of Remark 2 in (41). The change of coordinates takes us from the closed-loop error dynamics of

$$\mathbf{u}^T \;=\; \left[\mathbf{s}^T,\,\mathbf{e}^T,\,\widetilde{\mathbf{s}}^T,\;\widetilde{\mathbf{e}}^T,\widetilde{\dot{\mathbf{x}}}^T,\widetilde{\mathbf{x}}^T\right] \qquad (43)$$

through a transformation $\mathbf{z}=T\mathbf{u}$ to the closed-loop error dynamics of

$$\mathbf{z}^T \;=\; \left[\bar{\mathbf{s}}^T,\,\bar{\mathbf{e}}^T,\dot{\widetilde{\mathbf{y}}}_d^T,\,\widetilde{\mathbf{y}}_d^T,\,\dot{\widetilde{\mathbf{x}}}^T,\,\widetilde{\mathbf{x}}^T\right] \qquad (44)$$

B. Stability analysis

Assume that the signals $\dot{\mathbf{y}}_r$ and $\ddot{\mathbf{y}}_r$ are bounded such that

$$\begin{aligned}\sup_t\|\dot{\mathbf{y}}_r(t)\| &= V_M<\infty \qquad (45)\\ \sup_t\|\ddot{\mathbf{y}}_r(t)\| &= A_M<\infty \qquad (46)\end{aligned}$$

and introduce the scalar parameters

$$\lambda_0>0,\quad \mu_0>0,\quad \gamma_0>0,\quad \varepsilon_0>0 \qquad (47)$$

Consider the vector $\mathbf{z}\in\mathbb{R}^{6n}$ as defined in (44) and take the Lyapunov function

$$V(\mathbf{z})=\frac{1}{2}\mathbf{z}^T\mathbf{P}(\mathbf{z})\mathbf{z} \qquad (48)$$

where $\mathbf{P}(\mathbf{z})=\mathbf{P}(\mathbf{z})^T$ is given by

$$\mathbf{P}(\mathbf{z})=\begin{bmatrix}\mathbf{P}_1 & 0 & 0\\ 0 & \mathbf{P}_2 & 0\\ 0 & 0 & \mathbf{P}_3\end{bmatrix} \qquad (49)$$

where

$$\mathbf{P}_1 = \varepsilon_0 \begin{bmatrix} \mathbf{I}(\mathbf{x}) & \lambda_0 \mathbf{I}(\mathbf{x}) \\ \lambda_0 \mathbf{I}(\mathbf{x}) & \mathbf{K}_p + \lambda_0 \mathbf{K}_d \end{bmatrix} \quad (50)$$

$$\mathbf{P}_2 = \begin{bmatrix} \mathbf{1} & \mu(\widetilde{\mathbf{y}}_d)\mathbf{1} \\ \mu(\widetilde{\mathbf{y}}_d)\mathbf{1} & L_2 \end{bmatrix} \quad (51)$$

$$\mathbf{P}_3 = \begin{bmatrix} \mathbf{1} & \gamma(\widetilde{\mathbf{x}})\mathbf{1} \\ \gamma(\widetilde{\mathbf{x}})\mathbf{1} & L_2 \end{bmatrix} \quad (52)$$

where $\mathbf{1} \in \mathbb{R}^{6\times 6}$ is the identity matrix, ε_0, $\lambda_0 \in \mathbb{R}$ are positive constants to be determined and $\mu(\widetilde{\mathbf{y}}_d)$ and $\gamma(\widetilde{\mathbf{x}})$ are defined as

$$\mu(\widetilde{\mathbf{y}}_d) = \frac{\mu_0}{1+\|\widetilde{\mathbf{y}}_d\|}, \qquad \gamma(\widetilde{\mathbf{x}}) = \frac{\gamma_0}{1+\|\widetilde{\mathbf{x}}\|} \quad (53)$$

where μ_0, $\gamma_0 \in \mathbb{R}$ are positive constants to be determined and $\mu(\widetilde{\mathbf{y}}_d)$ and $\gamma(\widetilde{\mathbf{x}})$ are bounded such that

$$0 < \mu(\widetilde{\mathbf{y}}_d) \leq \mu_0, \qquad 0 < \gamma(\widetilde{\mathbf{x}}) \leq \gamma_0 \quad (54)$$

Sufficient conditions for positive definiteness of $\mathbf{P}(\mathbf{z})$ are

$$\mathbf{K}_{d,m} > \lambda_0 \mathbf{I}_M, \qquad L_{2,m} > \max\left\{\mu_0^2, \gamma_0^2\right\} \quad (55)$$

where $\mathbf{I}_M$ is the largest eigenvalue of $\mathbf{I}$. By choosing the minimum eigenvalues of the gain matrices $L_{1,m}, L_{2,m}, \mathbf{K}_{p,m}, \mathbf{K}_{d,m}$ to satisfy a set of lower bounds, and together with the boundedness of $\mu(\widetilde{\mathbf{y}}_d)$ and $\gamma(\widetilde{\mathbf{x}})$, this implies that there exists constants $\mathbf{P}_m$ and $\mathbf{P}_M$ such that

$$\frac{1}{2}\mathbf{P}_m \|\mathbf{y}\|^2 \leq V(\mathbf{z}) \leq \frac{1}{2}\mathbf{P}_M \|\mathbf{y}\|^2 \quad (56)$$

The time derivative of (48) along the error dynamics of (40 - 42) yields

$$\dot{V}(\mathbf{z}) = -\mathbf{z}^T \mathbf{Q}(\mathbf{z})\mathbf{z} + \beta(\mathbf{z}, \dot{\mathbf{x}}, \ddot{\mathbf{y}}_r) \quad (57)$$

From the definition in (53) and the bounds in (54) it follows that

$$\dot{\mu}\, \widetilde{\mathbf{y}}_d^T \dot{\widetilde{\mathbf{y}}}_d = -\mu \left(\frac{\widetilde{\mathbf{y}}_d^T \dot{\widetilde{\mathbf{y}}}_d}{1+\|\widetilde{\mathbf{y}}_d\|} \right) \dot{\widetilde{\mathbf{y}}}_d^T \widetilde{\mathbf{y}}_d \leq \mu_0 \left\|\dot{\widetilde{\mathbf{y}}}_d\right\|^2 \quad (58)$$

$$\dot{\gamma}\, \widetilde{\mathbf{x}}^T \dot{\widetilde{\mathbf{x}}} = -\gamma \left(\frac{\widetilde{\mathbf{x}}^T \dot{\widetilde{\mathbf{x}}}}{1+\|\widetilde{\mathbf{x}}\|} \right) \dot{\widetilde{\mathbf{x}}}^T \widetilde{\mathbf{x}} \leq \gamma_0 \left\|\dot{\widetilde{\mathbf{x}}}\right\|^2 \quad (59)$$

and by using Properties P3 and P4, and introducing the vector $\mathbf{z}_N$ as

$$\mathbf{z}_N^T = \left[\|\bar{\mathbf{s}}\|, \|\bar{\mathbf{e}}\|, \left\|\dot{\widetilde{\mathbf{y}}}_d\right\|, \|\widetilde{\mathbf{y}}_d\|, \left\|\dot{\widetilde{\mathbf{x}}}\right\|, \|\widetilde{\mathbf{x}}\| \right] \quad (60)$$

and the upper bounds

$$\left.\frac{\partial \mathbf{d}(\mathbf{x},\mathbf{z})}{\partial \mathbf{z}}\right|_{\mathbf{z}=\xi} \leq D_s, \qquad \left.\frac{\partial \mathbf{d}(\mathbf{x},\mathbf{v})}{\partial \mathbf{v}}\right|_{\mathbf{v}=\zeta} \leq D_n, \qquad \left.\frac{\partial \mathbf{d}(\mathbf{x},\mathbf{w})}{\partial \mathbf{w}}\right|_{\mathbf{w}=\kappa} \leq D_r \quad (61)$$

we can find an upper bound for (57) as

$$\dot{V}(\mathbf{z}) = \|\mathbf{z}_N\| \left(\alpha_0 - Q_{N,m}\|\mathbf{z}_N\| + \alpha_2 \|\mathbf{z}_N\|^2 \right) \quad (62)$$

where $Q_{N,m}$ is the minimum eigenvalue of the matrix $\mathbf{Q}_N = \mathbf{Q}_N^T$ given by (see the Appendix)

$$\mathbf{Q}_N = \begin{bmatrix} \mathbf{Q}_{11_N} & \mathbf{Q}_{12_N} & \mathbf{Q}_{13_N} \\ \mathbf{Q}_{12_N}^T & \mathbf{Q}_{22_N} & \mathbf{Q}_{23_N} \\ \mathbf{Q}_{13_N}^T & \mathbf{Q}_{23_N}^T & \mathbf{Q}_{33_N} \end{bmatrix} \quad (63)$$

and α_0 and α_2 are positive scalars given by

$$\alpha_0 = (1+\sqrt{\mu_0})\sqrt{A_M} \quad (64)$$

$$\begin{aligned} \alpha_2 = {} & \sqrt{8\,\mathbf{I}_m^{-1} C_M \Lambda_M}\left(\sqrt{\gamma_0} + \sqrt{L_{1,M}+\Lambda_M}\right) \\ & + \sqrt{\varepsilon_0 C_M}\left(1+\sqrt{\lambda_0}\right)\left(L_{1,M} + 2\sqrt{L_{1,M}}\right) \\ & + \mathbf{I}_m^{-1} C_M \Big(5 + \sqrt{\gamma_0 + 2L_{1,M}} + \sqrt{\gamma_0 L_{1,M} + L_{1,M}^2 + \gamma_0} \\ & + L_{1,M}\sqrt{\gamma_0} + \sqrt{8\,\gamma_0 L_{1,M}}\Big) \\ & + \varepsilon_0 \Big(\sqrt{C_M}\left(1+2\sqrt{L_{1,M}+\Lambda_M}\right) \\ & + \sqrt{\lambda_0 \Lambda_M (\mathbf{I}_M + C_M)}\Big) + \sqrt{\varepsilon_0 \lambda_0 C_M}\Big(2\left(1+\sqrt{L_{1,M}}\right) \\ & + 2\sqrt{\Lambda_M}\left(1+\sqrt{L_{1,M}}\right) + \sqrt{\mathbf{I}_M + C_M}\Big) \end{aligned} \quad (65)$$

By choosing the gains $\mathbf{K}_p, \mathbf{K}_d, L_1, L_2$ and the constants $\varepsilon_0, \lambda_0, \mu_0, \gamma_0$ such that $\mathbf{Q}_N$ is positive definite, we can treat the synchronization observer-controllers scheme as a perturbed system. Equation (48) together with (62) and Proposition 1 allows us to conclude local uniform ultimate boundedness of $\mathbf{z}_N$ and consequently of $\mathbf{z}$. Through the coordinate transformation we can conclude that the original state $\mathbf{u}$ in (43) is locally uniformly ultimately bounded. Moreover, since α_2 depends explicitly on $L_{1,M}$, we can make y_2 in Proposition 1 arbitrarily small by a proper choice of $L_{1,M}$, and thus the ultimate bound for $\mathbf{u}$ can be made arbitrarily small. Also note that the region of attraction is given by

$$B = \left\{ \mathbf{u} \in \mathbb{R}^{6n} \quad | \quad \|\mathbf{u}\| < \frac{y_2}{\|T\|}\sqrt{P_m^{-1} P_M} \right\} \quad (66)$$

Since the size of the region of attraction B is proportional to y_2, this region can be expanded by increasing y_2. Thus the closed-loop errors $\mathbf{u}$ are semi-globally uniformly ultimately bounded.

The proof partly relies on using the Mean Value Theorem and P5. This property does not hold for an important class of Euler-Lagrange systems including robotic manipulators where the friction terms in $\mathbf{d}(\mathbf{x}, \dot{\mathbf{x}})$ are described by

$$d_i = d_{ci}\,\mathrm{sgn}(\dot{x}_i) + d_{vi}\dot{x}_i \quad (67)$$

where d_{ci} and d_{vi} are positive coefficients for Coulomb and viscous friction, respectively. Here, d_i is not differentiable because of the discontinuous sgn-function. Fortunately, the simplicity of the robot friction term allows us to investigate $\mathbf{d}(\mathbf{x}, \dot{\mathbf{x}}) - \mathbf{d}\left(\mathbf{x}, \hat{\dot{\mathbf{x}}}\right)$ and similar terms directly, since an error on the form $\mathrm{sgn}(\mathbf{x}) - \mathrm{sgn}(\mathbf{y}_d)$ is either zero or of the same sign as $\mathrm{sgn}(\mathbf{x} - \mathbf{y}_d)$ ([13]).

V. SIMULATIONS

The synchronization controller-observer scheme were applied to the ship replenishment problem. The objective is to coordinate the behaviour of two surface ships to transfer supplies from the follower to the leader. The surface ship model from [21] given in the body frame

$$\mathbf{M}\dot{\nu} + \mathbf{C}(\nu)\,\nu + \mathbf{D}(\nu)\,\dot{\nu} = \tau_{\nu} \tag{68}$$

is a function of the body fixed velocities $\nu = [u, v, r]^T$ in surge, sway and yaw, respectively. The inertia matrix $\mathbf{M}$, Coriolis and centrifugal matrix $\mathbf{C}(\nu)$, and the nonlinear damping matrix $\mathbf{D}(\nu) = \mathbf{D} + \mathbf{D}_n(\nu)$ are defined as

$$\mathbf{M} = \begin{bmatrix} 25.8 & 0 & 0 \\ 0 & 33.8 & 1.0115 \\ 0 & 1.0115 & 2.76 \end{bmatrix}$$

$$\mathbf{C}(\nu) = \begin{bmatrix} 0 & 0 & -33.8v - 1.0115r \\ 0 & 0 & 25.8u \\ 33.8v + 1.0115r & -25.8u & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0.72 & 0 & 0 \\ 0 & 0.8896 & 7.25 \\ 0 & 0.0313 & 1.90 \end{bmatrix}$$

$$\mathbf{D}_n(\nu) = \begin{bmatrix} 1.33|u| + 5.87u^2 & 0 & 0 \\ 0 & 36.5|v| + 0.805|r| & 0.845|v| + 3.45|r| \\ 0 & 3.96|v| - 0.130|r| & 0.080|v| + 0.75|r| \end{bmatrix}$$

The model can be transformed to generalized coordinates $\mathbf{x} = [x, y, \psi]^T$, where x, y is position and ψ is heading, in a local earth-fixed coordinate frame as

$$\mathbf{I}(\mathbf{x})\,\ddot{\mathbf{x}} + \mathbf{C}'(\mathbf{x}, \dot{\mathbf{x}})\,\dot{\mathbf{x}} + \mathbf{D}'(\mathbf{x}, \dot{\mathbf{x}})\,\dot{\mathbf{x}} = \tau_{\mathbf{x}} \tag{69}$$

through the kinematic transformations

$$\begin{aligned} \mathbf{I}(\mathbf{x}) &= \mathbf{J}^{-T}(\mathbf{x})\,\mathbf{M}\,\mathbf{J}^{-1}(\mathbf{x}) \\ \mathbf{C}'(\mathbf{x}, \dot{\mathbf{x}}) &= \mathbf{J}^{-T}(\mathbf{x})\left[\mathbf{C}(\nu) - \mathbf{M}\,\mathbf{J}^{-1}(\mathbf{x})\,\dot{\mathbf{J}}(\mathbf{x})\right]\mathbf{J}^{-1}(\mathbf{x}) \\ \mathbf{D}'(\mathbf{x}, \dot{\mathbf{x}}) &= \mathbf{J}^{-T}(\mathbf{x})\,\mathbf{D}(\nu)\,\mathbf{J}^{-1}(\mathbf{x}) \end{aligned}$$

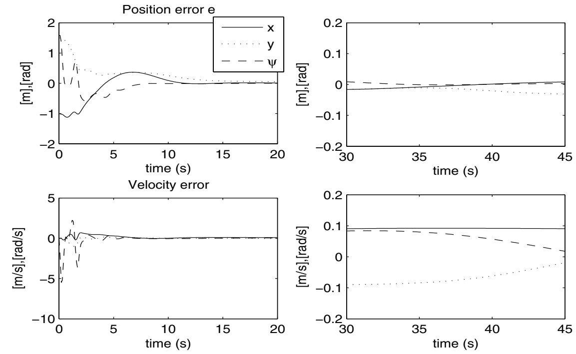

Fig. 1. Synchronization errors in position $\mathbf{e}$ (upper row) and velocity $\dot{\mathbf{e}}$ (lower row) in the initial phase (left) and after settling (right).

of [22] utilizing the kinematic relationship $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x})\,\nu$, where

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{70}$$

is a simple rotation around the z-axis in the earth-fixed frame. The transformed model of (69) is now in the form of (4), and the simulated synchronization errors in position $\mathbf{e}$ and velocity $\dot{\mathbf{e}}$ are shown in Fig. 1. Gains were chosen as $\mathbf{K}_p = \mathrm{diag}[10,\ 8,\ 20]$, $\mathbf{K}_d = \mathrm{diag}[50,\ 50,\ 9]$, $L_1 = \mathrm{diag}[10,\ 10,\ 10]$, $L_2 = \mathrm{diag}[4,\ 3,\ 40]$, and $\mathbf{\Lambda} = \mathrm{diag}[0.3,\ 0.3,\ 0.3]$ to illustrate the uniform ultimate boundedness of the scheme when the leader ship track a sine wave reference trajectory. The plots on the right illustrate the ultimate boundedness of the states due to the presence of a reference acceleration term $\ddot{\mathbf{y}}_r$ in the simulations. Initial conditions were chosen as $\mathbf{x}_0 = \left[-1,\ 1.5,\ \frac{\pi}{2}\right]^T$ for the follower position vector, and otherwise as zero for initial velocity and position estimates.

APPENDIX

$$\mathbf{Q}_{11_N} = \varepsilon_0 \begin{bmatrix} \mathbf{K}_{d,m} + \delta - \lambda_0 \mathbf{I}_M & \frac{1}{2}\lambda_0\left(\mathbf{C}_M V_M - \mathbf{I}_M V_M + D_s\right) \\ \frac{1}{2}\lambda_0\left(\mathbf{C}_M V_M - \mathbf{I}_M V_M + D_s\right) & \lambda_0 \mathbf{K}_{p,m} \end{bmatrix} \tag{A.1}$$

$$\mathbf{Q}_{12_N} = \frac{\varepsilon_0}{2} \begin{bmatrix} D_s - \mathbf{I}_M L_{1,M} - D_r & \mathbf{K}_{p,M} - \mathbf{K}_{d,M} L_{1,M} - \mathbf{C}_M V_M - D_r\left(\Lambda_M + L_{1,M}\right) + D_s \Lambda_M \\ \lambda_0\left(D_s - \mathbf{I}_M L_{1,M} - D_r\right) & \lambda_0\left(\mathbf{K}_{p,M} - \mathbf{K}_{d,M} L_{1,M} - \mathbf{C}_M V_M - D_r\left(\Lambda_M + L_{1,M}\right) + D_s \Lambda_M\right) \end{bmatrix} \tag{A.2}$$

$$\mathbf{Q}_{13_N} = \frac{\varepsilon_0}{2} \begin{bmatrix} \mathbf{C}_M V_M - \mathbf{K}_{d,M} & \mathbf{C}_M V_M L_{1,M} - \mathbf{K}_{d,M} L_{1,M} - D_r \Lambda_M + D_s \Lambda_M \\ \lambda_0\left(\mathbf{C}_M V_M - \mathbf{K}_{d,M}\right) & \lambda_0\left(\mathbf{C}_M V_M L_{1,M} - \mathbf{K}_{d,M} L_{1,M} - D_r \Lambda_M + D_s \Lambda_M\right) \end{bmatrix} \tag{A.3}$$

$$\mathbf{Q}_{22_N} = \begin{bmatrix} \left(\Lambda_m + L_{1,m}\right) - 2\mu_0 & \frac{1}{2}\left(\mathbf{I}_m^{-1}\mathbf{K}_{p,M} + \mu_0\left(\Lambda_M + L_{1,M}\right)\right) \\ \frac{1}{2}\left(\mathbf{I}_m^{-1}\mathbf{K}_{p,M} + \mu_0\left(\Lambda_M + L_{1,M}\right)\right) & \mu_0\left(\mathbf{I}_m^{-1}\mathbf{K}_{p,m} + L_{2,m}\right) \end{bmatrix} \tag{A.4}$$

$$\mathbf{Q}_{23_N} = \begin{bmatrix} \frac{1}{2}\Lambda_m & \frac{1}{2}\mathbf{I}_m^{-1}\mathbf{K}_{p,M} \\ \frac{1}{2}\left(\mu_0 \Lambda_M + \mathbf{I}_m^{-1}\mathbf{K}_{p,M} + L_{2,M}\right) & \frac{1}{2}\left(\left(\mu_0 + \gamma_0\right)\mathbf{I}_m^{-1}\mathbf{K}_{p,M} + \gamma_0 L_{2,M}\right) \end{bmatrix} \tag{A.5}$$

$$\mathbf{Q}_{33_N} = \begin{bmatrix} L_{1,m} - 2\gamma_0 + 2\mathbf{I}_m^{-1}\mathbf{C}_M V_M + \mathbf{I}_m^{-1} D_n & \frac{1}{2}\left(\mathbf{I}_m^{-1}\mathbf{K}_{p,M} + \gamma_0 L_{1,M}\right) + \mathbf{I}_m^{-1}\mathbf{C}_M V_M\left(L_{1,M} + \gamma_0\right) + \mathbf{I}_m^{-1} D_n\left(L_{1,M} + \gamma_0\right) \\ q_{56} & \gamma_0\left(\mathbf{I}_m^{-1}\mathbf{K}_{p,m} + L_{2,m} + 2\mathbf{I}_m^{-1}\mathbf{C}_M V_M L_{1,m} + \mathbf{I}_m^{-1} D_n L_{1,m}\right) \end{bmatrix} \tag{A.6}$$

VI. Conclusions

We have proposed an output synchronization control scheme for a class of Euler-Lagrange for applications where the reference is generated by a physical system. The leader-follower scheme is based on position measurements only, and no mathematical model of the leader is required. Disturbances affecting the systems differently are inherently canceled, and no control action of the leader is necessary to achieve synchronization of the systems in position and velocity. The class of systems includes systems with nonlinear damping terms such as robot manipulators, ships and underwater vehicles. The output synchronization scheme relies on nonlinear observers estimating the velocity and acceleration of the systems. We have shown that the closed-loop synchronization and observer errors are uniformly ultimately bounded. Future work aims at experimentally verifying the results.

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