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Robust Stability Analysis of Single-Parameter Dependent Descriptor Systems

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Abstract— **Exact robust stability analysis is considered for time-invariant descriptor systems whose coefficient matrices are affine functions of an uncertain parameter. A necessary and sufficient condition for robust stability is given as a parameterized linear matrix inequality (LMI) whose variable matrices are polynomial matrices of the parameter, where an upper bound of the degree of the polynomial matrices with respect to the parameter is derived explicitly. Then, it is shown that the parameterized LMI is reduced to a parameter-independent and finite-dimensional LMI, which implies that robust stability analysis of this class of systems can be recast exactly as a finite-dimensional convex feasibility problem.**

I. Introduction

Robust stability analysis for uncertain systems is one of the fundamental problems in systems and control. The analysis depends on a way to represent uncertainty of the systems. A typical way is to describe the uncertainty as parameters in the coefficient matrices of state space models, which is consistent with the situation that physical parameters of the systems appear in these matrices.

Recently, several new results have been obtained in this area (see, e.g., [1], [2], [3]), which focus on *exact* robust stability analysis of parameter-dependent state space models. For example, in [1], it is shown that robust stability of parameter-dependent state space models whose coefficient matrices are *affine* functions of the parameters is equivalent to the existence of a polynomially parameter-dependent quadratic (PPDQ) Lyapunov function. In [2], an upper bound of the degree of such a PPDQ Lyapunov function is obtained for a subset of the models, i.e., the models with *single* uncertain parameter. On the other hand, in [3], it is shown that such an upper bound can be derived even for *multi*-parameter case.

Notice here that these papers give a complete answer only for parameter-dependent state space models whose coefficient matrices are *affine* functions of parameters, which seems a fairly restrictive class, though their approach could be extended to a *polynomial* function case in a straightforward way. Since physical models contain their physical parameters as rational functions in the coefficient matrices of the state space models in general, there exists an important question: Can we derive a similar robust stability condition for parameter-dependent systems whose coefficient matrices are *rational* functions of uncertain parameters?

In this paper, we consider *exact* robust stability analysis of parameter-dependent *descriptor systems* whose coefficient matrices are *affine* functions of a parameter. Note that parameter-dependent *state space models* whose coefficient matrices are *rational* functions of a parameter can be rewritten as this class of descriptor systems [4]. Thus, the result of the present paper gives a complete solution of the question above for the single parameter case.

The tool we use in this paper is LMI conditions for stability analysis of descriptor systems without uncertainty [5], [6], [7]. In particular, a strict LMI condition, i.e., an LMI condition which consists only of strict inequalities is derived in [7], which we extensively use in this paper since a tolerance ensured by strict inequalities are useful in showing *necessity* of the condition. By replacing the variable matrices of the existing LMI condition with polynomially parameter-dependent matrices, we immediately state a necessary and sufficient condition for robust stability.

Then, the questions we consider in this paper are as follows: (i) Can we give an explicit upper bound of the degree of the polynomial matrix variables in the parameter-dependent LMI condition with respect to the parameter? (ii) Can we rewrite the parameter-dependent LMI condition as a parameter-*independent* and *finite* dimensional LMI feasibility problem? In the main body of this paper, we actually present complete answers to these questions for the single parameter case.

Note that these questions have been considered by [2] for *state space models*, and this paper follows a similar discussion. However, the strict LMI condition for stability analysis of descriptor systems contains not only a variable corresponding to a Lyapunov function of exponential modes but also an auxiliary variable corresponding to impulsive and/or static modes, which brings a technicality to the problems considered in this paper.

This paper is organized as follows. In Section II, we define stability of descriptor systems, and state the problem formulation we consider in this paper. We also remark generality of the formulation with an example. In Section III, we present the main result, that is, exact robust stability condition based on polynomially parameter-dependent Lyapunov equations. Subsequently, we show that the condition can be

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reduced to a parameter-independent and finite-dimensional LMI feasibility problem. A simple numerical example is given in Section IV. We make some concluding remarks in Section V.

II. Problem Statement

A. Descriptor Systems

Let us consider a descriptor system of the form

$$E\dot{x} = Ax \tag{1}$$

where $x \in \mathbb{R}^n$ is descriptor vector, $E, A \in \mathbb{R}^{n\times n}$, and $\operatorname{rank} E = r \leq n$. The stability of the system is defined as follows.

Definition 1: System (1) is stable if it is regular (i.e., $\det(sE - A) \not\equiv 0$) and has neither impulsive modes nor unstable finite modes.

When we define the stability of descriptor systems as the above, it is shown that stability analysis of descriptor systems can be reduced to an LMI feasibility problem [7].

Lemma 1: A system (1) is stable if and only if there exists a positive definite matrix $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and a matrix $S \in \mathbb{R}^{(n-r)\times(n-r)}$ satisfying

$$A(PE^{\mathrm{T}} + VSU^{\mathrm{T}}) + (EP + US^{\mathrm{T}}V^{\mathrm{T}})A^{\mathrm{T}} < 0 \tag{2}$$

where $V \in \mathbb{R}^{n\times(n-r)}$ is a full column rank matrix such that $EV = 0$, and $U \in \mathbb{R}^{n\times(n-r)}$ is a full column rank matrix such that $E^{\mathrm{T}}U = 0$.

Although a strict inequality is employed in this lemma, its equation version [8] exists, which is also used in this paper.

Lemma 2: A system (1) is stable if and only if there exists a positive definite matrix $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and a matrix $S \in \mathbb{R}^{(n-r)\times(n-r)}$ satisfying

$$A(PE^{\mathrm{T}} + VSU^{\mathrm{T}}) + (EP + US^{\mathrm{T}}V^{\mathrm{T}})A^{\mathrm{T}} + C^{\mathrm{T}}C = 0 \tag{3}$$

where the matrix C is selected so that

$$\operatorname{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \quad \forall s \in \mathbb{C} \tag{4}$$

$$\operatorname{rank}\begin{bmatrix} E & A \\ 0 & C \\ 0 & E \end{bmatrix} = n + r. \tag{5}$$

By using the singular value decomposition (SVD) of the matrix E, any descriptor system (1) can be reduced to

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} x \tag{6}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} = T^{\mathrm{T}}AR \tag{7}$$

where R is an orthogonal matrix and T is a nonsingular matrix both of which are determined by E. The following is a well-known fact regarding this SVD form.

Lemma 3: A system (1) is regular and has no impulsive mode if and only if A_{22} is nonsingular. When A_{22} is nonsingular, the system (1) is stable if and only if $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is Hurwitz.

B. Parameter-Dependent Descriptor Systems

In this paper, we consider a parameter dependent descriptor system whose coefficient matrices are affine functions of a parameter:

$$\begin{gathered} E\dot{x} = A_\rho x, \qquad A_\rho = A_0 + \rho A_1 \\ \rho \in [-1, 1], \quad E, A_0, A_1 \in \mathbb{R}^{n\times n} \end{gathered} \tag{8}$$

where ρ is a time invariant uncertain parameter. Our problem in this paper is to determine whether system (8) is stable for all ρ or not. Note that, since E is a constant matrix, we can simply say that $\operatorname{rank} E = r \leq n$ independently of ρ.

The class of the systems we consider here is fairly general in the following sense. We first remark that, when ρ is in a compact and connected region, we can take $\rho \in [-1, 1]$ without loss of generality. Furthermore, if the matrix E include a parameter, i.e., if a parameter-dependent system

$$\bar{E}_\rho\dot{\bar{x}} = \bar{A}_\rho\bar{x}, \quad \bar{E}_\rho = \bar{E}_0 + \rho\bar{E}_1, \quad \bar{A}_\rho = \bar{A}_0 + \rho\bar{A}_1 \tag{9}$$

is given, then it can be represented as

$$\begin{bmatrix} \bar{E}_0 & \bar{E}_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\bar{x}} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \bar{A}_\rho & 0 \\ \rho I_n & -I_n \end{bmatrix}\begin{bmatrix} \bar{x} \\ z \end{bmatrix} \tag{10}$$

where $z = \rho\bar{x}$, which meets the form of (8). Since we can show that stability of (10) is equivalent to that of (8) with a straightforward calculation, the class of the systems (8) covers any descriptor system whose coefficient matrices contain uncertain parameter as affine functions.

We further remark that the class of descriptor systems (8) contain the class of state space models whose coefficient matrices are rational functions of a parameter, as is pointed out in [4]. In fact, for example, a state space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 + 2\rho & 3\rho^2 \\ 5 & \dfrac{3 + 2\rho}{2 + \rho} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{11}$$

can be rewritten as

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} \\ &\quad = \begin{bmatrix} 1 + 2\rho & 0 & 3\rho & 0 \\ 5 & 0 & 0 & 3 + 2\rho \\ 0 & \rho & -1 & 0 \\ 0 & -1 & 0 & 2 + \rho \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{aligned} \tag{12}$$

which is a parameter-dependent descriptor system in an affine manner. That is, our result for descriptor systems can give an answer for state space models whose coefficient matrices are rational functions of a parameter.

III. Robust Stability Conditions

A. Parameterized LMI Condition

In this section, we derive a necessary and sufficient condition for robust stability of the descriptor system (8).

To this end, we employ a parameter-dependent (generalized) Lyapunov inequality. That is, we apply Lemma 1 to the system (8) replacing the variable matrices P, S with parameter-dependent matrices $P(\rho)$, $S(\rho)$. Then, we see that system (8) is robustly stable if and only if there exists parameter-dependent matrices $P(\rho)$, $S(\rho)$ such that

$$A_\rho(P(\rho)E^{\mathrm{T}} + VS(\rho)U^{\mathrm{T}}) + (EP + US(\rho)^{\mathrm{T}}V^{\mathrm{T}})A_\rho^{\mathrm{T}} < 0 \quad (13)$$

for all ρ. A key issue here is how to select the class of $P(\rho)$, $S(\rho)$. If it is too small, then the condition becomes a sufficient condition of robust stability. If it is too general, then the condition may still give a necessary and sufficient condition, but it may be impossible in principle to check the existence of $P(\rho)$, $S(\rho)$.

In this paper, we consider the parameter dependent matrices $P(\rho)$, $S(\rho)$ as polynomial matrices of ρ:

$$P_\rho = \sum_{i=0}^{d_p} \rho^i P_i, \quad S_\rho = \sum_{i=0}^{d_s} \rho^i S_i. \quad (14)$$

We show that in this section, even if we restrict the class of parameter-dependent matrices $P(\rho)$, $S(\rho)$ within the polynomial matrices, the condition gives still a necessary and sufficient condition of robust stability. We also derive an upper bound of the degree of the polynomial matrices with respect to ρ.

We need a few preliminary results to prove the theorem. The following is related to the eigenvalues of a Kronecker sum of the form $A \otimes I_n + I_n \otimes A$ [9].

Lemma 4: For a matrix $A \in \mathbb{R}^{n\times n}$, the eigenvalues of the matrix $A \otimes I_n + I_n \otimes A$ are given by $\lambda_i + \lambda_j, 1 \le i \le j \le n$, where λ_i, λ_j are the eigenvalues of A.

The following fact can be derived straightforwardly from the definitions of the determinant of a matrix and its adjoint matrix.

Fact 1: Let $H(\rho) \in \mathbb{R}^{n\times n}$ be a polynomial matrix of ρ with degree d. Then,

$$\deg[\det(H(\rho))] \le nd, \quad (15)$$

$$\deg[\mathrm{adj}\,(H(\rho))] \le (n-1)d \quad (16)$$

holds, where $\deg[\psi(\rho)]$ means degree of a polynomial (matrix) $\psi(\rho)$ with respect to ρ, and $\mathrm{adj}\,(H(\rho))$ means the adjoint matrix of $H(\rho)$.

Then, we state one of the main theorems of this paper. This is a polynomial (generalized) Lyapunov inequality for exact robust stability analysis, where an upper bound of the degree of the polynomial matrices $P(\rho)$, $S(\rho)$ is stated explicitly.

Theorem 1: A descriptor system (8) is robustly stable if and only if there exists matrices $P_i = P_i^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $i = 0, 1, \ldots, d_p$, $S_i \in \mathbb{R}^{(n-r)\times(n-r)}$, $i = 0, 1, \ldots, d_s$ such that

$$A_\rho(P_\rho E^{\mathrm{T}} + VS_\rho U^{\mathrm{T}}) + (EP_\rho + US_\rho^{\mathrm{T}}V^{\mathrm{T}})A^{\mathrm{T}} < 0, \quad (17)$$

$$P_\rho = \sum_{i=0}^{d_p} \rho^i P_i > 0, \qquad S_\rho = \sum_{i=0}^{d_s} \rho^i S_i \quad (18)$$

holds for all $\rho \in \Omega$, where $d_p = d_s = r^2(n - r + 1) - 1$.

Proof: (sufficiency) It is obvious.
(necessity) First, we write the system (8) as an SVD form:

$$T^{\mathrm{T}}ER = \tilde{E} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad (19)$$

$$T^{\mathrm{T}}A_\rho R = \tilde{A}_\rho = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad (20)$$

$$\tilde{A}_{11} \in \mathbb{R}^{r\times r}, \tilde{A}_{12} \in \mathbb{R}^{r\times(n-r)}, \quad (21)$$

$$\tilde{A}_{21} \in \mathbb{R}^{(n-r)\times r}, \tilde{A}_{22} \in \mathbb{R}^{(n-r)\times(n-r)} \quad (22)$$

where $R = [R_1 \quad R_2]$, $R_1 \in \mathbb{R}^{n\times r}$, $R_2 \in \mathbb{R}^{n\times(n-r)}$, R is an orthogonal matrix, and T is a nonsingular matrix. Since R, T does not contain the parameter ρ, the matrices $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{A}_{21}$, $\tilde{A}_{22}$ are still affine functions of ρ. Now, let us consider the equation

$$A_\rho(PE^{\mathrm{T}} + VSU^{\mathrm{T}}) + (EP + US^{\mathrm{T}}V^{\mathrm{T}})A_\rho^{\mathrm{T}} + Q_\rho = 0, \quad (23)$$

$$Q_\rho = T^{-\mathrm{T}} \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} T^{-1}, \quad (24)$$

$$Q_{11} = \alpha|\det(\hat{A}_\rho)||\det(\tilde{A}_{22})|^{r^2} I_r - \tilde{A}_{12}\tilde{A}_{12}^{\mathrm{T}}, \quad (25)$$

$$Q_{22} = \tilde{A}_{22}\tilde{A}_{22}^{\mathrm{T}}, \quad (26)$$

where

$$\hat{A}_\rho = (\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}) \otimes I_r + I_r \otimes (\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}) \quad (27)$$

and α is a number determined as follows.

When the system (8) is robustly stable, from Lemma 3, $\tilde{A}_{22}$ is a nonsingular matrix and thus $\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$ is a stable matrix for all ρ. Hence, from Lemma 4, $\det(\hat{A}_\rho) \neq 0$, $\det(\tilde{A}_{22}) \neq 0$ holds for all ρ. Consequently, we set α so that

$$\alpha > \frac{\max_{\rho\in\Omega} \lambda_{\max}(\tilde{A}_{12}\tilde{A}_{12}^{\mathrm{T}})}{\min_{\rho\in\Omega} |\det(\hat{A}_\rho)||\det(\tilde{A}_{22})|^{r^2}} \quad (28)$$

which implies that $Q_{11} > 0$, $Q_{22} > 0$ for all ρ, and $Q_\rho > 0$ for all ρ.

Then, there exists a parameter-dependent and nonsingular matrix C_ρ such that $Q_\rho = C_\rho^{\mathrm{T}}C_\rho$, which meets the rank conditions (4), (5). From Lemma 2, there exists a positive definite matrix $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and a matrix $S \in \mathbb{R}^{(n-r)\times(n-r)}$ satisfying the equation (3). Pre- and post-multiplying this equation by T^{T} and by T respectively, and using the identity $RR^{\mathrm{T}} = I_n$ appropriately, we obtain

$$T^{\mathrm{T}}A_\rho RR^{\mathrm{T}}(PRR^{\mathrm{T}}E^{\mathrm{T}} + VSU^{\mathrm{T}})T + T^{\mathrm{T}}(ERR^{\mathrm{T}}P + US^{\mathrm{T}}V^{\mathrm{T}})RR^{\mathrm{T}}A_\rho^{\mathrm{T}}T + T^{\mathrm{T}}Q_\rho T = 0.$$

From (20), this equation can be rewritten as

$$\tilde{A}_\rho(\tilde{P}\tilde{E}^{\mathrm{T}} + R^{\mathrm{T}}VSU^{\mathrm{T}}T) + (\tilde{E}\tilde{P} + T^{\mathrm{T}}US^{\mathrm{T}}V^{\mathrm{T}}R)\tilde{A}_\rho^{\mathrm{T}} = -\tilde{Q}_\rho, \quad (29)$$

where

$$\tilde{P} = R^{\mathrm{T}} P R, \tag{30}$$

$$\tilde{Q}_\rho = T^{\mathrm{T}} Q_\rho T = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}. \tag{31}$$

Now, since both R_2 and V are matrices composed of basis of $\mathrm{Ker}\, E$, there exists a nonsingular matrix W such that $V = R_2 W$. With $\tilde{S} = WSU^{\mathrm{T}}T = [\tilde{S}_1 \ \ \tilde{S}_2]$, $\tilde{S}_1 \in \mathbb{R}^{(n-r)\times r}$, $\tilde{S}_2 \in \mathbb{R}^{(n-r)\times(n-r)}$, we have

$$\begin{aligned} R^{\mathrm{T}} VSU^{\mathrm{T}} T = R^{\mathrm{T}} R_2 \tilde{S} &= \begin{bmatrix} R_1^{\mathrm{T}} R_2 \tilde{S}_1 & R_1^{\mathrm{T}} R_2 \tilde{S}_2 \\ R_2^{\mathrm{T}} R_2 \tilde{S}_1 & R_2^{\mathrm{T}} R_2 \tilde{S}_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ \tilde{S}_1 & \tilde{S}_2 \end{bmatrix}. \end{aligned} \tag{32}$$

We therefore rewrite (29) as

$$\begin{aligned} &\begin{bmatrix} \Lambda & * \\ \tilde{S}_2^{\mathrm{T}} \tilde{A}_{12}^{\mathrm{T}} + \tilde{A}_{21}\tilde{P}_{11} + \tilde{A}_{22}(\tilde{P}_{12}^{\mathrm{T}} + \tilde{S}_1) & \tilde{A}_{22}\tilde{S}_2 + \tilde{S}_2^{\mathrm{T}} \tilde{A}_{22}^{\mathrm{T}} \end{bmatrix} \\ &= \begin{bmatrix} -Q_{11} & 0 \\ 0 & -Q_{22} \end{bmatrix} \end{aligned} \tag{33}$$

where

$$\begin{aligned} \Lambda = {} & \tilde{A}_{11}\tilde{P}_{11} + \tilde{A}_{12}(\tilde{P}_{12}^{\mathrm{T}} + \tilde{S}_1) \\ & + \tilde{P}_{11}\tilde{A}_{11}^{\mathrm{T}} + (\tilde{P}_{12}^{\mathrm{T}} + \tilde{S}_1)^{\mathrm{T}} \tilde{A}_{12}^{\mathrm{T}}, \end{aligned} \tag{34}$$

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{12}^{\mathrm{T}} & \tilde{P}_{22} \end{bmatrix}, \tag{35}$$

and $*$ represent the transpose of its $(2,1)$ block.

The equations corresponding to each block are given by

$$\begin{aligned} \tilde{A}_{11}\tilde{P}_{11} + \tilde{A}_{12}(\tilde{P}_{12}^{\mathrm{T}} + \tilde{S}_1) & \\ + \tilde{P}_{11}\tilde{A}_{11}^{\mathrm{T}} + (\tilde{P}_{12}^{\mathrm{T}} + \tilde{S}_1)^{\mathrm{T}} \tilde{A}_{12}^{\mathrm{T}} &= -Q_{11}, \end{aligned} \tag{36}$$

$$\tilde{S}_2^{\mathrm{T}} \tilde{A}_{12}^{\mathrm{T}} + \tilde{A}_{21}\tilde{P}_{11} + \tilde{A}_{22}(\tilde{P}_{12}^{\mathrm{T}} + \tilde{S}_1) = 0, \tag{37}$$

$$\tilde{A}_{22}\tilde{S}_2 + \tilde{S}_2^{\mathrm{T}} \tilde{A}_{22}^{\mathrm{T}} = -Q_{22}. \tag{38}$$

From (37) and (38), we see that the equality (36) can be further rewritten as

$$\begin{aligned} &(\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21})\tilde{P}_{11} + \tilde{P}_{11}(\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21})^{\mathrm{T}} \\ &= -(\tilde{A}_{12}\tilde{A}_{22}^{-1} Q_{22} \tilde{A}_{22}^{-\mathrm{T}} \tilde{A}_{12}^{\mathrm{T}} + Q_{11}). \end{aligned} \tag{39}$$

It turns out that $\tilde{P}_{11}$ is the solution of (39), $\tilde{P}_{22}$, $\tilde{P}_{12}$ are arbitrary matrices, and $\tilde{S}_2$ is a solution of (38). Since the matrix $\tilde{P}_{12}^{\mathrm{T}}$ has no constraint except for (37), we set $\tilde{P}_{12}^{\mathrm{T}} = 0$ without loss of generality. Then, we obtain

$$\tilde{S}_1 = -\tilde{A}_{22}^{-1}(\tilde{S}_2^{\mathrm{T}} \tilde{A}_{12}^{\mathrm{T}} + \tilde{A}_{21}\tilde{P}_{11}). \tag{40}$$

We first consider the solution $\tilde{P}_{11}$. By column spreading both side of (39), we get

$$\hat{A}_\rho \mathrm{vec}\,(\tilde{P}_{11}) = -\mathrm{vec}\,(\tilde{A}_{12}\tilde{A}_{22}^{-1} Q_{22} \tilde{A}_{22}^{-\mathrm{T}} \tilde{A}_{12}^{\mathrm{T}} + Q_{11}) \tag{41}$$

$$\begin{aligned} \mathrm{vec}\,(\tilde{P}_{11}) = {} & -\mathrm{sign}\,(\det(\hat{A}_\rho))\alpha |\det(\tilde{A}_{22})|^{r^2} \\ & \times \mathrm{adj}\,(\hat{A}_\rho) \mathrm{vec}\,(I_r). \end{aligned} \tag{42}$$

Since

$$\begin{aligned} & |\det(\tilde{A}_{22})|^{r^2} \mathrm{adj}\,(\hat{A}_\rho) \\ &= \mathrm{sign}\,(\det(\tilde{A}_{22}))|\det(\tilde{A}_{22})| \\ &\quad \times \mathrm{adj}\left[\{\det(\tilde{A}_{22})\tilde{A}_{11} - \tilde{A}_{12}\mathrm{adj}\,(\tilde{A}_{22})\tilde{A}_{21}\} \otimes I_r \right. \\ &\quad \left. + I_r \otimes \{\det(\tilde{A}_{22})\tilde{A}_{11} - \tilde{A}_{12}\mathrm{adj}\,(\tilde{A}_{22})\tilde{A}_{21}\}\right] \end{aligned} \tag{43}$$

we have

$$\begin{aligned} & \deg[|\det(\tilde{A}_{22})|^{r^2} \mathrm{adj}\,(\hat{A}_\rho)] \\ & \leq \deg[|\det(\tilde{A}_{22})|] \\ & \quad + (r^2 - 1) \max \{ \deg[|\det(\tilde{A}_{22})|\tilde{A}_{11}], \\ & \qquad \deg[\tilde{A}_{12}\mathrm{adj}\,(\tilde{A}_{22})\tilde{A}_{21}]\} \\ & \leq n - r + (r^2 - 1)(n - r + 1) \\ & = r^2(n - r + 1) - 1. \end{aligned} \tag{44}$$

That is, the degree of the entries of $\tilde{P}_{11}$ is at most $r^2(n - r + 1) - 1$, which means that $\tilde{P}_{11}$ is a polynomial matrix of ρ with the degree less than or equal to $r^2(n - r + 1) - 1$. Here we rewrite $\tilde{P}_{11}$ as

$$\begin{aligned} \tilde{P}_{11} = {} & -\mathrm{sign}\left(\det(\hat{A}_\rho)\det(\tilde{A}_{22})\right) \\ & \times \alpha |\det(\tilde{A}_{22})| \sum_{i=0}^{(r^2-1)(n-r+1)} \rho^i L_i. \end{aligned} \tag{45}$$

for further discussion.

Let

$$\tilde{S}_2 = -\frac{1}{2} A_{22}^{\mathrm{T}},. \tag{46}$$

Then, from

$$\begin{aligned} \tilde{A}_{22}\tilde{S}_2 + \tilde{S}_2^{\mathrm{T}} \tilde{A}_{22}^{\mathrm{T}} &= -\frac{1}{2}(\tilde{A}_{22}\tilde{A}_{22}^{\mathrm{T}} + \tilde{A}_{22}\tilde{A}_{22}^{\mathrm{T}}) \\ &= -Q_{22}, \end{aligned} \tag{47}$$

the equation (38) holds. Hence, one of the solution $\tilde{S}_2$ of (33) is a polynomial matrix of ρ with degree 1.

On the other hand, $\tilde{S}_1$ is given from (40) as

$$\begin{aligned} \tilde{S}_1 &= -\tilde{A}_{22}^{-1}(\tilde{S}_2^{\mathrm{T}} \tilde{A}_{12}^{\mathrm{T}} + \tilde{A}_{21}\tilde{P}_{11}) \\ &= \frac{1}{2}\tilde{A}_{12}^{\mathrm{T}} + \mathrm{sign}\left(\det(\hat{A}_\rho)\det(\tilde{A}_{22})\right) \\ &\quad \times \alpha \mathrm{adj}\,(\tilde{A}_{22})\tilde{A}_{21} \sum_{i=0}^{(r^2-1)(n-r+1)} \rho^i L_i. \end{aligned} \tag{48}$$

By considering the degree of $\tilde{P}_{11}$, we obtain

$$\deg[\tilde{S}_1] \leq r^2(n - r + 1) - 1. \tag{49}$$

As a result of the above observation, the solutions $\tilde{P}$, $\tilde{S}$

of (33) are given by

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & 0 \\ 0 & \tilde{P}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}_{22} \end{bmatrix} + \sum_{i=0}^{r^2(n-r+1)-1} \rho^i \begin{bmatrix} \tilde{L}_i & 0 \\ 0 & 0 \end{bmatrix} \quad (50)$$

$$\tilde{S} = \begin{bmatrix} \tilde{S}_1 & \tilde{S}_2 \end{bmatrix} = \sum_{i=0}^{r^2(n-r+1)-1} \rho^i \begin{bmatrix} M_i & N_i \end{bmatrix} \quad (51)$$

where $\tilde{P}_{22}$, $\tilde{L}_i$, M_i, N_i are appropriate matrices independent of ρ. Furthermore, the relation between the solutions P, S of (23) and $\tilde{P}$, $\tilde{S}$ is represented as

$$P = R\tilde{P}R^{\mathrm{T}}, \quad S\,U^{\mathrm{T}} = W^{-1}\tilde{S}\,T^{-1}. \quad (52)$$

Since $U^{\mathrm{T}} \in \mathbb{R}^{(n-r)\times n}$ is of full row rank, there exists nonsingular matrices $\Gamma_1 \in \mathbb{R}^{(n-r)\times(n-r)}$, $\Gamma_2 \in \mathbb{R}^{n\times n}$ such that

$$\Gamma_1 U^{\mathrm{T}}\Gamma_2 = \begin{bmatrix} I_{n-r} & 0_{(n-r)\times r} \end{bmatrix}. \quad (53)$$

We therefore see that

$$S\Gamma_1^{-1}\Gamma_1 U^{\mathrm{T}}\Gamma_2 = W^{-1}\tilde{S}\,T^{-1}\Gamma_2 \quad (54)$$

$$\begin{bmatrix} S\Gamma_1^{-1} & 0_{(n-r)\times r} \end{bmatrix} = W^{-1}\tilde{S}\,T^{-1}\Gamma_2 \quad (55)$$

holds. Rewriting $W^{-1}\tilde{S}\,T^{-1}\Gamma_2$ as $W^{-1}\tilde{S}\,T^{-1}\Gamma_2 = [\hat{S}_1 \quad \hat{S}_2]$, $\hat{S}_1 \in \mathbb{R}^{(n-r)\times(n-r)}$, $\hat{S}_2 \in \mathbb{R}^{(n-r)\times r}$, we see that $S = \hat{S}_1\Gamma_1$. Since the degree of $\hat{S}_1$ is less than or equal to $\tilde{S}$, the degree of S is also the same. Furthermore, since R does not depend on ρ, the degree of P is less than or equal to $\tilde{P}$. This completes the proof. ■

B. Parameter-Independent LMI Feasibility Condition

Since the condition of Theorem 1 is a parameter dependent LMI, it is hard to check directly the existence of variable matrices. In this subsection, we recast the LMI as a parameter-independent and finite-dimensional LMI, which enables us to check the condition of Theorem 1 exactly by using a standard LMI solver. To this end, we first define a vector

$$\rho^{[k]} = \begin{bmatrix} 1 & \rho & \cdots & \rho^{k-1} \end{bmatrix}^{\mathrm{T}}. \quad (56)$$

Using this vector, a polynomial matrix P_ρ of ρ with degree d_p can be represented as

$$P_\rho = (\rho^{[k]} \otimes I_n)^{\mathrm{T}} P_\Sigma (\rho^{[k]} \otimes I_n), \quad (57)$$

where $P_\Sigma = P_\Sigma^{\mathrm{T}} \in \mathbb{R}^{nk\times nk}$ is a constant symmetric matrix and $k = \lceil (d_p/2) \rceil + 1$. Here $\lceil (d_p/2) \rceil$ denotes the minimum integer larger than or equal to $d_p/2$. Since P_Σ is not unique, we have several choices. An example is given by

$$P_\Sigma = \frac{1}{2}\begin{bmatrix} 2P_0 & P_1 & & & 0 \\ P_1 & 2P_2 & P_3 & & \\ & P_3 & 2P_4 & \ddots & \\ & & \ddots & \ddots & P_{d_p-1} \\ 0 & & & P_{d_p-1} & 2P_{d_p} \end{bmatrix} \quad (58)$$

if d_p is even, and

$$P_\Sigma = \frac{1}{2}\begin{bmatrix} 2P_0 & P_1 & & & & 0 \\ P_1 & 2P_2 & P_3 & & & \\ & P_3 & 2P_4 & \ddots & & \\ & & \ddots & \ddots & P_{d_p-2} & \\ & & & P_{d_p-2} & 2P_{d_p-1} & P_{d_p} \\ 0 & & & & P_{d_p} & 0 \end{bmatrix} \quad (59)$$

if d_p is odd. We also represent S_ρ in a similar way.

Let us define the matrices

$$\hat{J}_k = [I_k \quad 0_{k\times 1}], \quad \check{J}_k = [0_{k\times 1} \quad I_k]. \quad (60)$$

Then, we remark the following useful identities [1]:

$$\hat{J}_k \rho^{[k+1]} = \rho^{[k]}, \quad \check{J}_k \rho^{[k+1]} = \rho\rho^{[k]}. \quad (61)$$

It is also shown that

$$(\rho^{[k]} \otimes I_p)M = (I_k \otimes M)(\rho^{[k]} \otimes I_q) \quad (62)$$

holds for all $M \in \mathbb{R}^{p\times q}$.

With the definitions above, we can show the following lemma, which says that the inequality (17) can be represented as a quadratic form similar to (57).

Lemma 5: Let us define

$$R_\rho = A_\rho(P_\rho E^{\mathrm{T}} + VS_\rho U^{\mathrm{T}}) + (EP_\rho + US_\rho^{\mathrm{T}}V^{\mathrm{T}})A_\rho^{\mathrm{T}} \quad (63)$$

where $A_\rho = A_0 + \rho A_1$, $\rho \in \mathbb{R}$ is a parameter, $E, A_0, A_1 \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{n\times(n-r)}$, $U \in \mathbb{R}^{n\times(n-r)}$ are constant matrices, $P_\rho \in \mathbb{R}^{n\times n}$ is a parameter-dependent symmetric matrix and $S_\rho \in \mathbb{R}^{(n-r)\times(n-r)}$ is a parameter-dependent matrix of the forms

$$P_\rho = (\rho^{[k]} \otimes I_n)^{\mathrm{T}} P_\Sigma (\rho^{[k]} \otimes I_n), \quad (64)$$

$$S_\rho = (\rho^{[k]} \otimes I_{n-r})^{\mathrm{T}} S_\Sigma (\rho^{[k]} \otimes I_{n-r}). \quad (65)$$

Then, R_ρ can be represented as

$$R_\rho = (\rho^{[k+1]} \otimes I_n)^{\mathrm{T}} R_\Sigma (\rho^{[k+1]} \otimes I_n) \quad (66)$$

where

$$R_\Sigma = H_1^{\mathrm{T}} P_\Sigma F_1 + F_1^{\mathrm{T}} P_\Sigma H_1 + H_2^{\mathrm{T}} S_\Sigma F_2 + F_2^{\mathrm{T}} S_\Sigma^{\mathrm{T}} H_2, \quad (67)$$

$$H_1 = \hat{J}_k \otimes A_0^{\mathrm{T}} + \check{J}_k \otimes A_1^{\mathrm{T}}, \quad (68)$$

$$F_1 = \hat{J}_k \otimes E^{\mathrm{T}}, \quad (69)$$

$$H_2 = \hat{J}_k \otimes (A_0 V)^{\mathrm{T}} + \check{J}_k \otimes (A_1 V)^{\mathrm{T}}, \quad (70)$$

$$F_2 = \hat{J}_k \otimes U^{\mathrm{T}}. \quad (71)$$

This lemma can be confirmed by a direct computation.

For a polynomial matrix represented as a quadratic form of $(\rho^{[k]} \otimes I_n)$, it is known that positive definiteness of the polynomial matrix for all ρ can be checked by solving a finite-dimensional LMI feasibility problem which does not contain ρ. The following lemma is taken from [2], which is a modified version of the result in [10].

Lemma 6: Let $\Theta = \Theta^{\mathrm{T}} \in \mathbb{R}^{nk\times nk}$. Then,

$$(\rho^{[k]} \otimes I_n)^{\mathrm{T}} \Theta (\rho^{[k]} \otimes I_n) < 0 \quad (72)$$

holds for all $\rho \in \Omega$ if and only if there exist matrices $D \in \mathbb{R}^{n(k-1)\times n(k-1)}$, $G \in \mathbb{R}^{n(k-1)\times n(k-1)}$ satisfying

$$D = D^{\mathrm{T}} > 0, \quad G + G^{\mathrm{T}} = 0, \tag{73}$$

$$\Theta < \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -D & G \\ G^{\mathrm{T}} & D \end{bmatrix} \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}. \tag{74}$$

Then, we state the second result of this paper, which shows that robust stability analysis of the descriptor system (8) is reduced to a finite-dimensional and parameter-independent LMI feasibility problem.

Theorem 2: A descriptor system (8) is robustly stable if and only if there exist matrices $P_\Sigma = P_\Sigma^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $S_\Sigma \in \mathbb{R}^{(n-r)\times(n-r)}$, $D_1 \in \mathbb{R}^{n(k-1)\times n(k-1)}$, $D_2 \in \mathbb{R}^{nk\times nk}$, $G_1 \in \mathbb{R}^{n(k-1)\times n(k-1)}$, $G_2 \in \mathbb{R}^{nk\times nk}$ satisfying

$$D_1 = D_1^{\mathrm{T}} > 0, \quad G_1 + G_1^{\mathrm{T}} = 0$$

$$-P_\Sigma < \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -D_1 & G_1 \\ G_1^{\mathrm{T}} & D_1 \end{bmatrix} \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}$$

$$D_2 = D_2^{\mathrm{T}} > 0, \quad G_2 + G_2^{\mathrm{T}} = 0$$

$$R_\Sigma(P_\Sigma, S_\Sigma) < \begin{bmatrix} \hat{J}_{k} \otimes I_n \\ \check{J}_{k} \otimes I_n \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -D_2 & G_2 \\ G_2^{\mathrm{T}} & D_2 \end{bmatrix} \begin{bmatrix} \hat{J}_{k} \otimes I_n \\ \check{J}_{k} \otimes I_n \end{bmatrix}$$

where $k = \lceil \frac{d}{2} \rceil + 1$ and $d = r^2(n - r + 1) - 1$.

This theorem is a direct consequence of the above lemmas, thus the proof is omitted.

IV. Numerical Example

In this section, we report a numerical example. We considered a state space model of the form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \epsilon_1/(2+\rho) & 2 \\ 0 & \epsilon_2/(2+\rho) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{75}$$

where $\rho \in [-1, 1]$. Since the coefficient matrix has a triangular form, we immediately see that this system is stable for all $\rho \in [-1, 1]$ if and only if $\epsilon_1 < 0$ and $\epsilon_2 < 0$.

We represented the system as a descriptor system of the form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 2 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ -1 & 0 & 2+\rho & 0 \\ 0 & -1 & 0 & 2+\rho \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{76}$$

Then, we applied Theorem 2 to the following cases:

(i) $\epsilon_1 = -0.001$, $\epsilon_2 = -0.001$;
(ii) $\epsilon_1 = -0.001$, $\epsilon_2 = 0.001$;
(iii) $\epsilon_1 = 0.001$, $\epsilon_2 = -0.001$;
(iv) $\epsilon_1 = 0.001$, $\epsilon_2 = 0.001$.

In these computations, we took $d_p = d_s = 11$ from Theorem 1, and therefore $k = 7$ in Theorem 2.

Using the Matlab LMI Toolbox [11], we obtained a feasible solution only for the case (i), while we did not obtain a feasible solution for the other cases. That is, the numerical results based on Theorem 2 meet the theoretical consequence stated for the original system (75).

V. Concluding Remarks

In this paper, we have proposed a method of robust stability analysis for uncertain descriptor systems whose coefficient matrices are affine functions of a parameter. The robust stability condition we have derived is parameter-independent and finite dimensional LMI feasibility condition. Using the condition, we can check robust stability of this class of uncertain systems exactly with a finite-dimensional convex feasibility problem.

We have dealt with the descriptor systems with single parameter so far. For multi-parameter case, we can also obtain an exact robust stability condition as a parameterized LMI. However, it seems difficult to recast it as a parameter-independent LMI. Recent progress of probabilistic method [12] could be useful for solving this type of problems, which is currently under investigation.

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