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Consistent Nonparametric Estimation of NARX Systems Using Convex Optimization

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Abstract— **In this paper, a nonparametric method based on quadratic programming (QP) for identification of nonlinear autoregressive systems with exogenous inputs (NARX systems) is presented. We consider a mixed parametric/nonparametric model structure. The output is assumed to be the sum of a parametric linear part and a nonparametric Lipschitz continuous part. The consistency of the estimator is shown assuming only that an upper bound on the true Lipschitz constant is given. In addition, different types of prior knowledge about the system can easily be incorporated. Examples show that the method can give accurate estimates also for small data sets and that the estimate of the linear part sometimes can be improved compared to the linear least squares estimate.**

I. Introduction

Nonlinear autoregressive systems with exogenous inputs (NARX systems) [14] are a straightforward generalization of linear ARX systems that has been used in many applications. For an NARX system, the optimal one step ahead predictor is a nonlinear function of a finite number of past output and input components. Using a version of the prediction-error method [8], we will here simultaneously estimate both a nonparametric NARX model and a parametric ARX model such that their sum give an as good prediction of the output as possible. Related model structures have been used in semiparametric or partially linear models (see, for example, [7], [3]).

It is interesting to consider nonparametric methods for nonlinear system identification since the assumptions about the true system are usually weaker for such methods than for parametric methods. For a nonlinear system, it can be hard to tell in advance whether a specific assumption about, for example, the shape of the nonlinearities is reasonable or not. In this paper, the only assumption about the true NARX system is that its nonlinearities are Lipschitz continuous.

This assumption makes it possible to use an approach where the identification problem is formulated as a quadratic programming (QP) problem. By solving this problem, both the parameters of the linear ARX model and the nonparametric NARX model can be estimated at the same time. A version of this idea, without the linear, parametric part, has previously been used for nonparametric regression and for maximum likelihood estimation of unknown parameters in probability density functions [1]. Other methods for nonparametric regression can be found in, for example, [5].

Lipschitz conditions are a common way to guarantee that a function, or some of its derivatives, will be smooth. For example, functions with a Lipschitz continuous gradient can be identified using local modeling such that the worst-case mean-square error is minimized [12].

A small nonlinear system component can have a large influence on an estimated linear approximation of the system if standard methods for linear identification are used [9], [4]. In some cases, this behavior can be understood if the nonlinear contribution to the system output is viewed as a nonlinear disturbance [11], [13]. The method presented in this paper will make the estimate of the linear model more robust against nonlinearities in the system since the nonparametric NARX model can compensate for some of the nonlinear effects. A related concept is the notion of unknown but bounded noise and set membership identification [6], since a bounded nonlinearity might affect the system output in a similar way as such a noise term.

II. NARX Identification

Consider an NARX system with input $u(t)$ and output $y(t)$ that can be written

$$y(t) = \theta_0^T \varphi(t) + r_0(\varphi(t)) + e(t), \tag{1}$$

where

$$\varphi(t) = \begin{pmatrix} -y(t-1) \\ \vdots \\ -y(t-n_a) \\ u(t-n_k) \\ \vdots \\ u(t-n_k-n_b) \end{pmatrix} \tag{2}$$

is a regression vector and where $e(t)$ is white noise. The constant vector θ_0 defines a linear ARX part of the system while the function r_0 can be nonlinear. Assume that $e(t)$ and $\varphi(t)$ are independent for all t and that r_0 is a Lipschitz continuous function with Lipschitz constant L_0, i.e., that

$$|r_0(\varphi_1) - r_0(\varphi_2)| \leq L_0 \|\varphi_1 - \varphi_2\|_2, \ \forall \varphi_1, \varphi_2 \in \mathbb{R}^n, \tag{3}$$

where $n = n_a + n_b + 1$. Furthermore, assume that a dataset $(\varphi(t), y(t))_{t=1}^N$ consisting of N measurements of the regression vector and the system output is available.

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Using this dataset, estimates $\hat{\theta}_N$ and $\hat{r}_N$ of θ_0 and r_0, respectively, can be obtained by solving the QP problem

$$\begin{array}{ll} \underset{\theta_N,\rho_N}{\text{minimize}} & \frac{1}{N}\sum_{t=1}^{N}(y(t)-\theta_N^T\varphi(t)-\rho_N(t))^2 \\ \text{subject to} & \rho_N(t)-\rho_N(s)\leq L\|\varphi(t)-\varphi(s)\|_2 \\ & \forall s,t\in\{1,2,\ldots,N\}. \end{array} \quad (4)$$

In this problem, ρ_N is a vector with N elements $\rho_N(t)$ which can be viewed as estimates of $r_0(\varphi(t))$. The constraints on the variables $\rho_N(t)$ imply that these variables will satisfy

$$|\rho_N(t)-\rho_N(s)|\leq L\|\varphi(t)-\varphi(s)\|_2$$

for all $s,t\in\{1,2,\ldots,N\}$. If the variables $\rho_N(t)$ are viewed as samples from some function, this implies that a Lipschitz condition holds for the sample points $(\varphi(t))_{t=1}^N$. Note that N of the constraints in (4) are trivial ($0\leq 0$) and present in (4) only for notational convenience. These constraints can be removed without changing the solution of the problem.

An optimal solution $(\hat{\theta}_N,\hat{\rho}_N)$ to the problem (4) can be used to construct one step ahead predictions

$$\hat{y}_N(\varphi(t))=\hat{\theta}_N^T\varphi(t)+\hat{\rho}_N(t) \quad (5)$$

of the system output for the observed regression vectors $(\varphi(t))_{t=1}^N$. In order to obtain a predictor which can be used for an arbitrary regression vector, the nonparametric function estimate $\hat{\rho}_N$ has to be interpolated.

When $\varphi(t)$ is a scalar, linear interpolation is probably the most natural type of interpolation. However, for $\varphi(t)\in\mathbb{R}^n$ with $n>1$, linear interpolation of the variables $\hat{\rho}_N(t)$ will in general not result in a function that satisfies the Lipschitz condition for the choice of L used in (4). Instead, for $n>1$, an estimate $\hat{r}_N$ of r_0 can be defined as

$$\begin{aligned}\hat{r}_N(\varphi)=\frac{1}{2}\max_{1\leq t\leq N}(\hat{\rho}_N(t)-L\|\varphi-\varphi(t)\|_2) \\ +\frac{1}{2}\min_{1\leq t\leq N}(\hat{\rho}_N(t)+L\|\varphi-\varphi(t)\|_2)\end{aligned} \quad (6)$$

using a similar construction as in [1]. The function $\hat{r}_N$ is Lipschitz continuous since it is the mean of two Lipschitz continuous functions. Using $\hat{\theta}_N$ and $\hat{r}_N$, a general one step ahead predictor

$$\hat{y}_N(\varphi)=\hat{\theta}_N^T\varphi+\hat{r}_N(\varphi) \quad (7)$$

can be constructed. At first sight, it might seem that the $N+n$ variables used in the problem (4) and for the construction of the model (7) are too many since there are only N measurements. However, thanks to the randomness of the disturbance $e(t)$ in (1), the constraints in (4) will impose an averaging effect on the nonparametric function estimate.

Without these constraints, one optimal solution to (4) is $\theta_N=0$, $\rho_N(t)=y(t)$ for $t=1,2,\ldots,N$. Of course, since the measurements of the output are noisy, such a solution does not give a good model of the true system. By adding constraints like in (4), two variables $\rho_N(t)$ and $\rho_N(s)$ are allowed to differ only marginally from each other if the distance $\|\varphi(t)-\varphi(s)\|_2$ between the corresponding regression vectors is small. In this way, the ρ variables are imposed to have similar properties as samples from the true Lipschitz continuous function r_0. If the set of regression vectors gets more dense when N increases, $\hat{\theta}_N^T\varphi(t)+\hat{\rho}_N(t)$ will approach $\theta_0^T\varphi(t)+r_0(\varphi(t))$. For an intuitive understanding of this convergence, consider a small region in $\mathbb{R}^n$ which contains many regression vectors. The corresponding ρ variables will with a high probability be close to the mean of $y(t)-\hat{\theta}_N^T\varphi(t)$ since the constraints in (4) implies that the ρ variables should have values close to each other. The consistency of the predictor function estimator (7) will be discussed in Section III.

Several types of extensions can be made to the identification method presented here. For example, if any prior knowledge about the true system can be written as linear constraints on θ_N and ρ_N, this knowledge can easily be incorporated in the QP problem (4). Examples of such prior knowledge are:

- Bounds on the function r_0 are known in a subset of its domain.
- Different Lipschitz constants can be used in different parts of the domain of r_0.
- The function r_0 is known to be odd or even.
- An expression for the function r_0 is known in a subset of its domain.

Sometimes it could also be interesting to consider the case when only a Lipschitz continuous function should be estimated (setting $\theta_N=\theta_0=0$ in (1) and (4)). Analogously to (4), we can handle this case by solving a QP

$$\begin{array}{ll} \underset{\rho_N}{\text{minimize}} & \frac{1}{N}\sum_{t=1}^{N}(y(t)-\rho_N(t))^2 \\ \text{subject to} & \rho_N(t)-\rho_N(s)\leq L\|\varphi(t)-\varphi(s)\|_2 \\ & \forall s,t\in\{1,2,\ldots,N\}. \end{array} \quad (8)$$

The construction of $\hat{r}_N$ using the interpolation method (6) can be used also in this case. In the next section, the consistency of both presented nonparametric identification methods will be shown.

III. Consistency

Before we consider the consistency of the approaches, let us study the behavior of the mean of the predicted outputs at $(\varphi(t))_{t=1}^N$. As the following lemmas show, it is quite simple to show consistency for these.

Lemma 1: The optimum of (4) satisfies

$$\frac{1}{N}\sum_{t=1}^{N}\left(\hat{\theta}_N^T\varphi(t)+\hat{\rho}_N(t)\right)=\frac{1}{N}\sum_{t=1}^{N}y(t), \quad (9)$$

and, for NFIR systems,

$$\begin{aligned}\mathrm{E}\left[\frac{1}{N}\sum_{t=1}^{N}\left(\hat{\theta}_N^T\varphi(t)+\hat{\rho}_N(t)\right)\middle|(\varphi(t))_{t=1}^N\right] \\ =\frac{1}{N}\sum_{t=1}^{N}\theta_0^T\varphi(t)+r_0(\varphi(t)).\end{aligned} \quad (10)$$

Proof: The Lagrangian of (4) (see [2]) can be written

$$\mathcal{L}(\theta_N,\rho_N;\lambda) = \frac{1}{N}\sum_{t=1}^{N}(y(t) - \theta_N^T\varphi(t) - \rho_N(t))^2 \qquad (11)$$
$$- \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}\left(L\|\varphi(i) - \varphi(j)\|_2 - \rho_N(i) + \rho_N(j)\right).$$

The optimum should satisfy $\frac{\partial \mathcal{L}}{\partial \rho_N(k)} = 0$ for $k = 1, \ldots, N$:

$$-\frac{2}{N}(y(k) - \hat{\theta}_N^T\varphi(k) - \hat{\rho}_N(k)) + \sum_{i=1}^{N}(\hat{\lambda}_{ki} - \hat{\lambda}_{ik}) = 0. \quad (12)$$

Summing (12) over k gives (9). Taking expectations over both sides of (9) then gives (10). ∎

Lemma 2: The optimum of (8) satisfies

$$\frac{1}{N}\sum_{t=1}^{N}\hat{\rho}_N(t) = \frac{1}{N}\sum_{t=1}^{N}y(t), \qquad (13)$$

and, for NFIR systems,

$$\mathrm{E}\left[\frac{1}{N}\sum_{t=1}^{N}\hat{\rho}_N(t)\middle|(\varphi(t))_{t=1}^{N}\right] = \frac{1}{N}\sum_{t=1}^{N}r_0(\varphi(t)). \qquad (14)$$

Proof: As for Lemma 1. ∎

As it now turns out, the identification methods given by (4) and (8), respectively, have fairly attractive properties. Let us start by considering the estimates we get by using (8) together with (6). (A related result was shown in [1]. Here we give an alternative proof.)

Theorem 1: Let

$$y(t) = r_0(\varphi(t)) + e(t), \qquad (15)$$

where $e(t)$ is a stationary white noise process with zero mean and bounded variance σ^2. Let $\hat{\rho}_N(t)$ be the optimal solution to (8). Suppose that

1) $\varphi(t) \in \Phi$, where Φ is a compact set such that for any $\varepsilon > 0$, Φ can be partitioned
$$\Phi = \bigcup_{i=1}^{d}\Phi_i, \qquad (16)$$
where $\varphi_1, \varphi_2 \in \Phi_i \Rightarrow \|\varphi_1 - \varphi_2\|_2 \le \varepsilon$ and $P(\varphi(t) \in \Phi_i) > 0$ for all $i = 1, 2, \ldots, d$,
2) the stochastic process $\varphi(t)$ is such that $N_i \to \infty$ when $N \to \infty$ w.p.1 for all i in any ε-partitioning (16) where
$$N_i = \mathrm{card}(T_i) \text{ and } T_i = \{t|\varphi(t) \in \Phi_i,\ t \le N\}, \qquad (17)$$
3) $e(t)$ and $\varphi(t)$ are independent, but $\varphi(t)$ may depend on past $e(s)$,
4) $|r_0(\varphi_1) - r_0(\varphi_2)| \le L_0\|\varphi_1 - \varphi_2\|_2\ \forall \varphi_1, \varphi_2 \in \Phi$,
5) $L_0 \le L$.

Then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}(\hat{\rho}_N(t) - r_0(\varphi(t))^2 = 0 \text{ w.p.1}. \qquad (18)$$

Proof: Take an arbitrary $\varepsilon > 0$ and consider an ε-partitioning such that assumption 1 is satisfied. Consider arbitrary realizations of the processes $\varphi(t)$ and $e(t)$. With probability one, these realizations are such that $N_i \to \infty$ as $N \to \infty$ and that

$$\lim_{N_i\to\infty}\frac{1}{N_i}\sum_{t\in T_i}e(t) = 0, \qquad (19a)$$
$$\lim_{N_i\to\infty}\frac{1}{N_i}\sum_{t\in T_i}|e(t)| \le C, \qquad (19b)$$
$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}e(t)^2 = \sigma^2 \qquad (19c)$$

for some constant C and for all i. (The limits (19) follow since $\{e(t), t \in T_i\}$ is a sequence of independent random variables such that the law of large numbers can be applied.) For two fixed such realizations of $\varphi(t)$ and $e(t)$, we can thus find an $N'(\varepsilon)$ such that for $N > N'(\varepsilon)$

$$\left|\frac{1}{N_i}\sum_{t\in T_i}e(t)\right| \le \varepsilon \quad \forall i, \qquad (20a)$$
$$\frac{1}{N_i}\sum_{t\in T_i}|e(t)| \le 2C \quad \forall i, \qquad (20b)$$
$$\frac{1}{N}\sum_{t=1}^{N}e(t)^2 \le 2\sigma^2. \qquad (20c)$$

This follows since the partitioning is finite for any ε.

Since $r_N(t) = r_0(\varphi(t))$ is a feasible choice in the minimization problem, we have

$$\frac{1}{N}\sum_{t=1}^{N}(y(t) - \hat{\rho}_N(t))^2 \le \frac{1}{N}\sum_{t=1}^{N}(y(t) - r_0(\varphi(t))^2$$
$$= \frac{1}{N}\sum_{t=1}^{N}e(t)^2, \qquad (21)$$

which means that

$$\frac{1}{N}\sum_{t=1}^{N}(r_0(\varphi(t)) - \hat{\rho}_N(t))^2$$
$$\le \left|\frac{2}{N}\sum_{t=1}^{N}e(t)(r_0(\varphi(t)) - \hat{\rho}_N(t))\right|.$$

Note first that, by applying Cauchy-Schwarz' inequality to the right hand side, we find that

$$\frac{1}{N}\sum_{t=1}^{N}(r_0(\varphi(t)) - \hat{\rho}_N(t))^2 \le \frac{4}{N}\sum_{t=1}^{N}e(t)^2.$$

Since r_0 is bounded (let $C_{r_0} = \sup_{\varphi\in\Phi}|r_0(\varphi)|$), so must $\hat{r}_N(\varphi)$ as defined by (6) be. (Recall that $\hat{r}_N$ is Lipschitz-continuous over a compact set, so if one value tends to infinity, all values will tend to infinity.) Hence, we can choose a constant $C_{\hat{r}}$, such that for $N > N'(\varepsilon)$ we have $C_{\hat{r}} > \sup_{\varphi\in\Phi}|\hat{r}_N(\varphi)|$. Now, choose $t_i^* \in T_i$ and let

$$r_i = r_0(\varphi(t_i^*)), \text{ and } \hat{r}_i = \hat{\rho}_N(t_i^*).$$

This means that for $t \in T_i$

$$|r_0(\varphi(t)) - r_i| \le L_0\varepsilon \text{ and } |\hat{\rho}_N(t) - \hat{r}_i| \le L\varepsilon.$$

Inserting this into the expression above gives

$$\begin{aligned}
&\frac{1}{N}\sum_{t=1}^{N}(r_0(\varphi(t)) - \hat{\rho}_N(t))^2 \\
&\le \left|\frac{2}{N}\sum_{t=1}^{N} e(t)(r_0(\varphi(t)) - \hat{\rho}_N(t))\right| \\
&= \left|\frac{2}{N}\sum_{i=1}^{d}\sum_{t\in T_i} e(t)(r_0(\varphi(t)) - r_i + r_i - \hat{\rho}_N(t) + \hat{r}_i - \hat{r}_i)\right| \\
&= \left|\frac{2}{N}\sum_{i=1}^{d} N_i\left(\left[\frac{1}{N_i}\sum_{t\in T_i} e(t)\right][r_i - \hat{r}_i]\right.\right. \\
&\quad \left.\left. + \left[\frac{1}{N_i}\sum_{t\in T_i} e(t)(r_0(\varphi(t)) - r_i - \hat{\rho}_N(t) + \hat{r}_i)\right]\right)\right| \\
&\le \frac{1}{N}\sum_{i=1}^{d} N_i\left[\varepsilon\max_i |r_i - \hat{r}_i| + \frac{1}{N_i}\sum_{t\in T_i}|e(t)|2L\varepsilon\right] \\
&\le C'\varepsilon \qquad \text{for } N > N'(\varepsilon),
\end{aligned}$$

where C' is $C_{r_0} + C_{\hat{r}} + 4LC$. Since ε and the realizations are arbitrary, (18) has been proven. ■

Remark 1: The theorem is still true if $\{e(t)\}$ is a mixing process, independent of the process $\{\varphi(t)\}$, since the only thing that matters is that (19) holds.

Now, using the Lipschitz continuity of $\hat{r}$ and r_0, it is easy to prove the consistency of the method, summarized in the following corollary.

Corollary 1: Let $\varphi(t)$, $y(t)$, and $\hat{\rho}_N(t)$ satisfy the conditions of Theorem 1, and let $\hat{r}_N(\varphi)$ be defined by (6). Then, with probability one, $\hat{r}_N$ converges uniformly to r_0 on Φ as $N \to \infty$.

For identifying systems in the form (1) by using (4), consistency is a bit harder. If using a too large Lipschitz constant, the separation of the system into a linear and a Lipschitz continuous part is obviously not uniquely determined. However, we can show that the one step ahead predictor converges uniformly to the true one.

Theorem 2: Let $(\varphi(t), y(t))_{t=1}^{N}$ be generated from

$$y(t) = \theta_0^T\varphi(t) + r_0(\varphi(t)) + e(t), \tag{22}$$

where r_0, $e(t)$ and $\varphi(t)$ satisfy the conditions of Theorem 1, and where $\|\theta_0\|_2 \le M_\theta$ for some known constant M_θ. Let $\hat{\theta}_N$ and $\hat{r}_N$ be estimates of θ_0 and r_0 on Φ, obtained by (4) with the extra requirement $\|\theta_N\|_2 \le M_\theta$, and by (6) with Lipschitz constant $L \ge L_0$. Then, with probability one, $\hat{y}_N(\varphi)$ as defined in (7) converges uniformly to $\theta_0^T\varphi + r_0(\varphi)$ as $N \to \infty$.

Proof: Analogous to Theorem 1 and Corollary 1. ■

IV. Examples

The previously presented method for combined parametric and nonparametric estimation of NARX systems has been used in a couple of numerical examples. The first example concerns identification of a static nonlinearity.

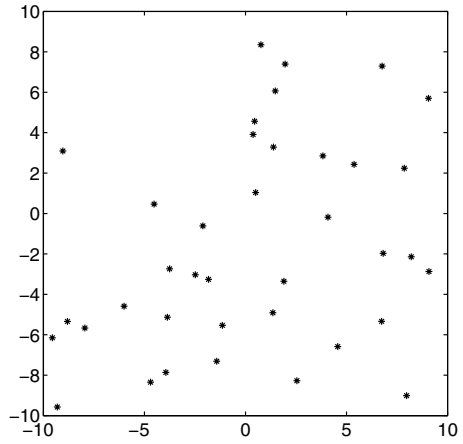


Fig. 1. The values of $y(t)$ plotted against $u(t)$ for the dataset with 40 measurements used in Example 1.

Example 1: Consider the system

$$y(t) = 0.4u(t) + r_0(u(t)) + e(t), \tag{23}$$

where both $u(t)$ and $e(t)$ are white noise processes and independent of each other. The input $u(t)$ has uniform distribution on the interval $[-10, 10]$ while the noise $e(t)$ is normally distributed such that its mean is zero and its variance is 25. The nonlinearity in this system is

$$\begin{aligned}
r_0(u(t)) = \frac{40}{5 + |u(t)|}\left(\frac{u(t)}{1 + |u(t)|} - \frac{u(t) - 3}{1 + |u(t) - 3|}\right. \\
\left. - \frac{u(t) + 6}{1 + |u(t) + 6|} + \frac{3}{28}\right).
\end{aligned} \tag{24}$$

This function is Lipschitz continuous with $L_0 = 7.4$ and bounded since $|r_0'(x)| < 7.4$ and $|r_0(x)| < 3.1$, for all $x \in \mathbb{R}$.

A small dataset consisting of 40 realizations of the input and output in (23) has been generated and is shown in Figure 1. Note that the shape of the nonlinear function is not obvious in this figure. The method (4) with $L = 7.4$ has been used with this dataset and linear interpolation has been used to construct $\hat{r}_N$. The resulting predictor function $\hat{y}_N(\varphi)$ is shown in Figure 2. From this figure, it seems that the function estimate has managed to pick up some key features of the true function, despite the small number of measurements.

In this case, the L value used in the method is equal to L_0. In a more realistic example, the true Lipschitz constant would typically be unknown. An alternative would then be to divide the dataset into estimation data and validation data and try different values of L. By evaluating the predictor (7) on the validation data for different choices of Lipschitz constant, it would be possible to find a good choice of L.

A larger dataset consisting of 500 realizations of the input and output in (23) has also been generated and a couple of models have been estimated using an extended version of (4) where bounds $\pm\rho_N(t) \le 4$ have been added. One model was estimated using $L = 15$ and the resulting predictor function is shown in Figure 3(a). The choices $L = 7.4$ and $L = 4$ gave the results shown in Figure 3(b) and 3(c), respectively. From these figures, it seems that the function estimates contain

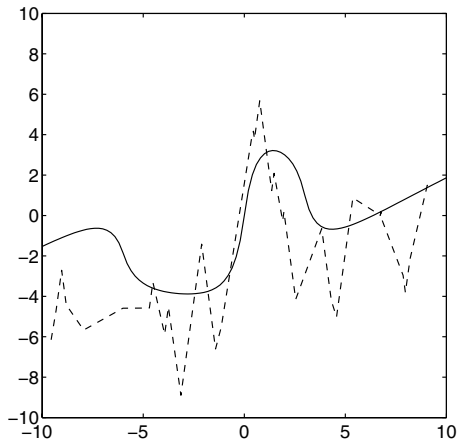


Fig. 2. The predictor function estimated from 40 measurements (dashed) and the true predictor function (solid) from Example 1.

no significant systematic errors and that a larger value of L gives more variations. Note that for $L = 4$, the true function r_0 is not a feasible solution to the identification problem. However, the obtained function estimate gives a rather good approximation of r_0 anyway.

In the case with $L = 15$, the obtained estimate of the linear regression parameter $\theta_0 = 0.4$ was $\hat{\theta}_N = 0.31$ while $L = 7.4$ gave $\hat{\theta}_N = 0.33$ and $L = 4$ gave $\hat{\theta}_N = 0.39$. Using the same dataset but with a completely linear model, the least-squares method gave an estimate $\hat{\theta}_{\mathrm{LS}} = 0.23$. Hence, it seems that including a bounded nonlinear Lipschitz continuous term in the model sometimes can improve the estimate of the linear part. □

The method (4) combined with the interpolation (6) has also been used on a NARX system where the regression vector consists of two past output components and one input component. The results of this numerical experiment are described in the following example.

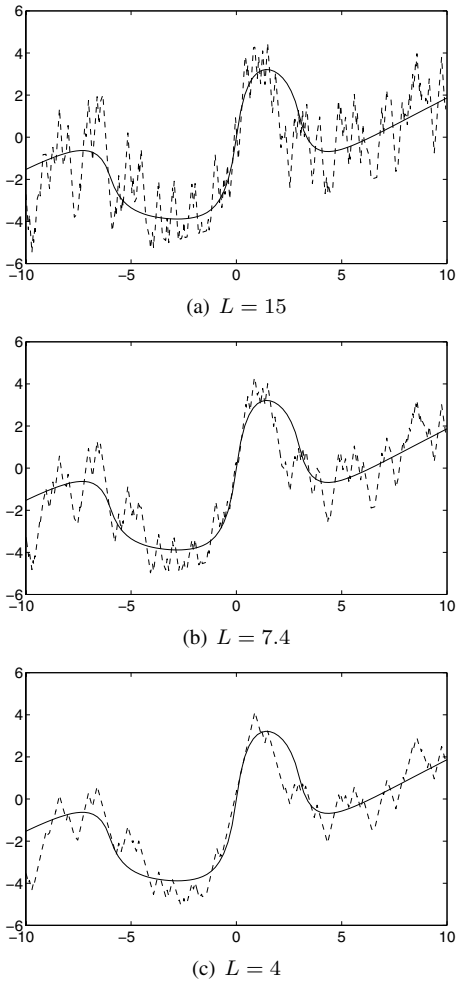
Example 2: Consider the following NARX system:

$$y(t) = -y(t-1) - 0.2y(t-2) + u(t-1) + \arctan(u(t-1) + y(t-1)) + \sin(y(t-2)) + e(t). \quad (25)$$

This system can be viewed as being composed by a linear part $\theta_0^T\varphi(t)$ (with $\theta_0 = (1\ \ 0.2\ \ 1)^T$) and a nonlinear part $r_0(t)$ with Lipschitz constant $L_0 = \sqrt{3}$. Furthermore, $|r_0(t)| \leq \pi/2+1$. The noise terms are independent, normally distributed variables with unit variance.

The system has been estimated using an estimation dataset of 500 samples generated from $u(t) \in N(0, 4)$. Three Lipschitz constants have been tried: $L = 4$, $L = \sqrt{3}$ and $L = 1.4$, together with the upper bound on $r_0(t)$. The obtained models have been evaluated on a validation dataset of 500 samples generated under the same conditions as the estimation data. As quality measure, the fit has been calculated according to

$$\left(1 - \sqrt{\frac{\sum_t (y(t) - \hat{y}_N(\varphi(t)))^2}{\sum_t (y(t) - \bar{y})^2}}\right) \cdot 100\%, \quad (26)$$



(a) $L = 15$

(b) $L = 7.4$

(c) $L = 4$

Fig. 3. The predictor function estimated from 500 measurements for three choices of L (dashed) and the true predictor function (solid) from Example 1.

where $\hat{y}_N(\varphi(t))$ is the output value predicted by the model and $\bar{y}$ is the arithmetic mean of $(y(t))_{t=1}^N$.

The results are given in Table I. As comparison, a linear ARX model has also been identified. Furthermore, the fit has been calculated for a one step ahead predictor using the true parameter values and nonlinearities. Clearly, the NARX models outperform the linear ARX model. They also get rather close in performance to the true model. Note that the NARX model with a "too small" Lipschitz constant performs best. The reason for this is that in the region where data is available, we can decrease the Lipschitz constant of the nonlinear part by "tilting it" and properly adjust the linear part of the model. □

TABLE I

FITS FOR THE ESTIMATED MODELS IN EXAMPLE 2.

Model	Fit (validation data)
NARX, $L = 1.4$	69.625
NARX, $L = \sqrt{3}$	69.050
NARX, $L = 4$	66.231
ARX	63.743
True model	72.039

V. Discussion

As mentioned previously, it is easy to incorporate various kinds of prior knowledge into the identification problem. In fact, we can regard the presented approach as a special instance of the more general identification problem

$$\begin{aligned} \underset{\theta_N,\rho_N}{\text{minimize}} \quad & \tfrac{1}{N}\textstyle\sum_{t=1}^{N}(y(t)-\theta_N^T\varphi(t)-\rho_N(t))^2 \\ \text{subject to} \quad & A\begin{pmatrix}\rho_N\\ \theta\end{pmatrix} \preccurlyeq b, \end{aligned} \tag{27}$$

where $\preccurlyeq$ denotes component-wise inequality. This is still a convex QP problem. An interesting special case of (27) is

$$\begin{aligned} \underset{\theta_N,\rho_N}{\text{minimize}} \quad & \tfrac{1}{N}\textstyle\sum_{t=1}^{N}(y(t)-\theta_N^T\varphi(t)-\rho_N(t))^2 \\ \text{subject to} \quad & |\rho_N(t)| \le M \\ & \forall t \in \{1,2,\ldots,N\}. \end{aligned} \tag{28}$$

It turns out that minimizing (28) gives exactly the same linear part as using an *ε-insensitive norm* for identification of ARX models, i.e.,

$$\underset{\theta_N}{\text{minimize}} \ \frac{1}{N}\sum_{t=1}^{N}\left|y(t)-\theta_N^T\varphi(t)\right|_\varepsilon^k \tag{29}$$

with

$$|x|_\varepsilon = \begin{cases} 0 & |x| \le \varepsilon \\ |x|-\varepsilon & |x| > \varepsilon \end{cases}$$

and with $k=2$ and $\varepsilon = M$. This norm (or the corresponding norm with $k=1$) is often used in support vector machines [15], and similar approaches are also used in robust adaptive control [10]. To see the equivalence between (28) and (29), define

$$\bar{r}(t,\theta) = \begin{cases} M & y(t)-\theta^T\varphi(t) > M, \\ y(t)-\theta^T\varphi(t) & -M \le y(t)-\theta^T\varphi(t) \le M, \\ -M & y(t)-\theta^T\varphi(t) < -M. \end{cases}$$

Then we can write (29) as

$$\underset{\theta_N}{\text{minimize}} \ \frac{1}{N}\sum_{t=1}^{N}\left|y(t)-(\theta_N^T\varphi(t)+\bar{r}(t,\theta_N))\right|^k.$$

On the other hand it is easy to see that, for a given θ, the minimum of (28) is obtained precisely when $\rho_N(t)=\bar{r}(t,\theta)$. Since $\bar{r}(t,\theta)$ automatically has a magnitude not greater than M, the desired equivalences follow.

The advantage with using the formulation (28) instead of (29) is that the explicit representation of ρ_N again makes it possible to combine different types of requirements on the nonlinearity, just as was done in Example 2.

Instead of assuming a nonlinearity in the system, we can also interpret the terms $\hat{\rho}_N$ as estimates of deterministic noise terms ρ_0. These could for instance be bounded (unknown but bounded noise) like in (28) or satisfy a Lipschitz condition as in (4). Another option would be that their variation over time could be bounded, i.e.,

$$|\rho_N(t+1)-\rho_N(t)| \le L_t.$$

VI. Conclusions

In this paper, NARX systems that can be written as the sum of a linear ARX part and a nonlinear, Lipschitz continuous, NARX part have been studied. It has been shown that a model with a linear, parametric ARX part and a nonparametric NARX part of such an NARX system can be estimated by solving a quadratic programming problem. A novel proof of the consistency of this method has been presented. It should be noted that the consistency does not rely on knowledge of the true Lipschitz constant L_0. In fact, the only knowledge necessary is an upper bound of L_0. The tighter the upper bound, however, the faster convergence to the true function we can expect. The examples indicate that the method is fairly robust to incorrect values of L.

The examples also show that the introduction of a nonlinear term in the model sometimes can improve the estimate of the linear ARX term. Furthermore, the described method can produce NARX models that can predict the output in a validation dataset much better than an ARX model. The method can be useful also when the dataset is relatively small.

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