Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005



# Realization Theory For Bilinear Switched Systems: Formal Power Series Approach

Mihály Petreczky
Centrum voor Wiskunde en Informatica (CWI)
P.O.Box 94079, Amsterdam
M.Petreczky@cwi.nl

*Abstract*— **The paper deals with the realization theory of bilinear switched systems. Necessary and sufficient conditions are formulated for a family of input-output maps to be realizable by a bilinear switched system. Characterization of minimal realizations is presented. The paper treats two types of bilinear switched systems. The first one is when all switching sequences are allowed. The second one is when only a subset of switching sequences is admissible, but within this restricted set the switching times are arbitrary. The paper uses the theory of formal power series to derive the results on realization theory. Partial realization theory is also discussed in the paper.**

## I. Introduction

Switched systems are one of the best studied subclasses of hybrid systems. A vast literature is available on various issues concerning switched systems, for a comprehensive survey see [7]. Yet, to the author's knowledge, the only works available on the realization theory of switched systems are [8], [9], which develop realization theory for linear switched systems. Most of the material of the current paper together with the proofs can be found in [11].

The current paper develops realization theory for *bilinear switched systems*. More specifically, the paper presents solutions to the following problems.

`(i)` If $\Phi$ is a subset of input-output maps generated by a bilinear switched system, then find a minimal bilinear switched system generating the input-output maps of $\Phi$,

`(ii)` Find necessary and sufficient condition for the existence of a bilinear switched system realizing a given set of input-output maps,

`(iii)` Find conditions, under which a realization of a set of input output maps can be constructed from finite data.

`(iv)` Find sufficient and necessary conditions for the existence of a bilinear switched system realizing $\Phi$ under the following conditions. Assume that a set of admissible switching sequences is defined. Assume that the switching times of the admissible switching sequences are arbitrary. The input-output maps from $\Phi$ are defined only for the admissible sequences.

The motivation of this problem is the following. Assume that the switching is controlled by a finite automaton, which is specified in advance, and the discrete modes are the states of this automaton. Assume that discrete-state transitions can be triggered only by discrete control input signals, which can be generated at any time. Then the traces of this automaton combined with the switching times ( which are arbitrary ) give us the admissible switching sequences. If we can solve the realization problem for the case of restricted switching, then we can solve the realization problem for the hybrid system described above.

The following results are proved in the paper.

- A bilinear switched system realization is minimal if and only if it is observable and *semi-reachable*. Minimal bilinear switched system which realize a given set of input-output maps are isomorphic. Each bilinear switched system realization can be transformed to a minimal one.
- A set of input/output maps is realizable by a bilinear switched system if and only if it has *generalized Fliess-series expansion* and the rank of its Hankel-matrix is finite. There is a procedure to construct the realization from the columns of the Hankel-matrix, and this procedure yields a minimal realization. Under certain conditions, similar to those for bilinear systems ([4]), a bilinear switched system realization can be constructed from finite data.
- Consider a set of input-output maps $\Phi$ defined on some subset of switching sequences. Assume that the switching sequences of this subset have arbitrary switching times and that their discrete mode parts form a regular language $L$. Then $\Phi$ has a realization by a bilinear switched system if and only if it has a generalized Fliess-series expansion and its Hankel-matrix is of finite rank. Again, there exists a procedure to construct a realization from the columns of the Hankel-matrix. The procedure yields an observable and semi-reachable realization of $\Phi$. But this realization need not to be the realization with the smallest state-space dimension possible.

The main tool used in the paper is the theory of rational formal power series. Rational formal power series were used in systems theory earlier, for application of rational formal power series, see [6], [5], [3], [1]. There are a number of definitions for representation of rational formal power series, see [2], [14], [13], [12]. All the cited works deal with representations of a *single formal power series*. In this paper, we will look at representations of *families of formal power* series instead. This requires a slight but straightforward extension of the existing theory, see [9], [11], [10] for details.

The outline of the paper is the following. Section II introduces the notation and describes some properties and concepts related to bilinear switched systems. Section III contains the necessary results on formal power series. Section IV presents the notion of generalized Fliess-series expansion and gives a characterization of input-output maps generated by bilinear switched systems. Section V presents realization theory of bilinear switched systems.

## II. Bilinear Switched Systems

For sets $A, B$, denote by $PC(A,B)$ the class of piecewise-continuous maps from $A$ to $B$. For a set $\Sigma$ denote by $\Sigma^*$ the set of finite strings of elements of $\Sigma$. For $w = a_1 a_2 \cdots a_k \in \Sigma^*$ the length of $w$ is denoted by $|w|$, i.e. $|w| = k$. The empty sequence is denoted by $\epsilon$. The length of $\epsilon$ is zero: $|\epsilon| = 0$. Let $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$. The concatenation of two strings $v = v_1 \cdots v_k, w = w_1 \cdots w_m \in \Sigma^*$ is the string $vw = v_1 \cdots v_k w_1 \cdots w_m$. If $w \in Q^+$ then $w^k$ denotes the word $\underbrace{ww\cdots w}_{k-times}$. The word $w^0$ is just the empty word $\epsilon$. Denote by $T$ the set $[0, +\infty) \subseteq \mathbb{R}$. Denote by $\mathbb{N}$ the set of natural numbers including 0. Denote by $F(A,B)$ the set of all functions from the set $A$ to the set $B$. By abuse of notation we will denote any constant function $f : T \to A$ by its value. That is, if $f(t) = a \in A$ for all $t \in T$, then $f$ will be denoted by $a$. For any function $f$ the range of $f$ will be denoted by $\mathrm{Im} f$. If $A, B$ are two sets, then the set $(A \times B)^*$ will be identified with the set $\{(u,w) \in A^* \times B^* \mid |u| = |w|\}$. For any two sets $J, X$ the surjective function $A : J \to X$ is called an *indexed subset* of $X$ or simply an *indexed set*. It will be denoted by $A = \{a_j \in X \mid j \in J\}$. Let $T = [0, +\infty)$ and let $f, g \in PC(T, A)$ for some suitable set $A$. Define for any $\tau \in T$ the concatenation $f \#_\tau g \in PC(T, A)$ of $f$ and $g$ by $f \#_\tau g(t) = \begin{cases} f(t) & \text{if } t \le \tau \\ g(t) & \text{if } t > \tau \end{cases}$. If $f : T \to A$, then for each $\tau \in T$ define $\mathrm{Shift}_\tau(f) : T \to A$ by $\mathrm{Shift}_\tau(f)(t) = f(t + \tau)$.

A *switched ( control ) system* is a tuple

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\})$$

where $\mathcal{X} = \mathbb{R}^n$ is the state-space, $\mathcal{Y} = \mathbb{R}^p$ is the output-space, $\mathcal{U} = \mathbb{R}^m$ is the input-space, $Q$ is the finite set of discrete modes, $f_q(x,u)$, is a smooth function and globally Lipschitz in $x$ for each $q \in Q$, $h_\sigma : \mathcal{X} \to \mathcal{Y}$ is smooth map for each $\sigma \in Q$.

Elements of the set $(Q \times T)^+$ are called *switching sequences*. The inputs of the switched system $\Sigma$ are functions from $PC(T, \mathcal{U})$ and sequences from $(Q \times T)^+$. That is, the switching sequences are part of the input, they are specified externally and we allow any switching sequence to occur. The state space evolution of a switched system takes place as follows. Between two switches the state trajectory is a solution to the differential equation corresponding to the current discrete mode. The solution of the differential equation is taken with an initial condition which coincides with the value of the state trajectory at the moment when the switch took place.

Let $u \in PC(T, \mathcal{U})$ and $w = (q_1, t_2)(q_2, t_2) \cdots (q_k, t_k) \in (Q \times T)^+$. The inputs $u$ and $w$ steer the system $\Sigma$ from state $x_0$ to the state $x_\Sigma(x_0, u, w)$ given by

$$x_\Sigma(x_0,u,w) = F(q_k, \mathrm{Shift}_{\sum_1^{k-1} t_i}(u), t_k) \circ \\ \circ F(q_{k-1}, \mathrm{Shift}_{\sum_1^{k-2} t_i}(u), t_{k-1}) \circ \cdots \circ F(q_1, u, t_1)(x_0)$$

where $F(q,u,t) : \mathcal{X} \to \mathcal{X}$ and for each $x \in \mathcal{X}$ the function $F(q,u,t,x) : t \mapsto F(q,u,t)(x)$ is the solution of the differential equation $\frac{d}{dt} F(q,u,t,x) = f_q(F(q,u,t,x), u(t))$, $F(q,u,0,x) = x$. The empty sequence $\epsilon \in (Q \times T)^*$ leaves the state intact: $x_\Sigma(x_0, u, \epsilon) = x_0$. The *reachable set* of a system $\Sigma$ from a set of initial states $\mathcal{X}_0$ is defined by $Reach(\Sigma, \mathcal{X}_0) = \{x_\Sigma(x_0, u, w) \in \mathcal{X} \mid u \in PC(T, \mathcal{U}), w \in (Q \times T)^*, x_0 \in \mathcal{X}_0\}$. $\Sigma$ is said to be *reachable* from $\mathcal{X}_0$ if $Reach(\Sigma, \mathcal{X}_0) = \mathcal{X}$ holds. $\Sigma$ is *semi-reachable* from $\mathcal{X}_0$ if $\mathcal{X}$ is the smallest vector space containing $Reach(\Sigma, \mathcal{X}_0)$, that is, $\mathcal{X} = \mathrm{Span}\{z \in \mathcal{X} \mid z \in Reach(\Sigma, \mathcal{X}_0)\}$. Define the function $y_\Sigma : \mathcal{X} \times PC(T, \mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}$ by $y_\Sigma(x,u,w) = h_{q_k}(x_\Sigma(x,u,w))$, $\forall x \in \mathcal{X}, u \in PC(T,U), w = (q_1,t_1)(q_2,t_2)\cdots(q_k,t_k) \in (Q \times T)^+$. For each $x \in \mathcal{X}$ define the *input-output map of* $\Sigma$ *induced by* $x$ as $y_\Sigma(x,.,.) : PC(T, \mathcal{U}) \times (Q \times T)^+ \ni (u,w) \mapsto y_\Sigma(x,u,w) \in \mathcal{Y}$. Two states $x_1 \neq x_2 \in \mathcal{X}$ of the switched system $\Sigma$ are *indistinguishable* if $y_\Sigma(x_1,.,.) = y_\Sigma(x_2,.,.)$. $\Sigma$ is called *observable* if it has no pair of indistinguishable states. A set $\Phi \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$ is said to be *realized* by a switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\})$ if there exists $\mu : \Phi \to \mathcal{X}$ such that $y_\Sigma(\mu(f),.,.) = f$. By abuse of terminology, both $\Sigma$ and $(\Sigma, \mu)$ will be called a *realization* of $\Phi$. That is, $\Sigma$ realizes $\Phi$ if and only if for each $f \in \Phi$ there exists a state $x \in \mathcal{X}$ such that $y_\Sigma(x,.,.) = f$. Denote by $\dim \Sigma := \dim \mathcal{X}$ the dimension of the state space of the switched system $\Sigma$. Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. A switched system $\Sigma$ is a *minimal realization of* $\Phi$ if $\Sigma$ is a realization of $\Phi$ and for each switched system $\Sigma_1$ such that $\Sigma_1$ is a realization of $\Phi$ it holds that $\dim \Sigma \le \dim \Sigma_1$. For any $L \subseteq Q^+$ define the subset of admissible switching sequences $TL \subseteq (Q \times T)^+$ by $TL := \{(w, \tau) \in (Q \times T)^+ \mid w \in L, \tau \in T^{|w|}\}$. That is, $TL$ is the set of all those switching sequences, for which the sequence of discrete modes belongs to $L$ and the sequence of times is arbitrary. Notice that if $L = Q^+$ then $TL = (Q \times T)^+$. Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$. The system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\})$ *realizes* $\Phi$ *with constraint* $L$ if there exists $\mu : \Phi \to \mathcal{X}$ such that $y_\Sigma(\mu(f), u, w) = f(u,w)$ for each $u \in PC(T, \mathcal{U})$ and $w \in TL$. We will call both $(\Sigma, \mu)$ and $\Sigma$ a *realization* of $\Phi$. Notice that if $L = Q^+$ then $\Sigma$ realizes $\Phi$ with constraint $L$ if and only if $\Sigma$ realizes $\Phi$. If $\Sigma$ is a switched system, then we say that the realization $(\Sigma, \mu)$ is *semi-reachable* , if $\Sigma$ is semi-reachable from $\mathrm{Im}\mu$.

A switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\})$ is called *bilinear* if for each $q \in Q$ there exist linear mappings $A_q : \mathcal{X} \to \mathcal{X}$, $B_{q,j} : \mathcal{X} \to \mathcal{X}$, $j = 1, 2, \ldots, m$ , $C_q : \mathcal{X} \to \mathcal{Y}$ such that $f_q(x,u) = A_q x + \sum_{j=1}^m u_j B_{q,j} x$ and $h_q = C_q x$,

$\forall x \in \mathcal{X}, u = (u_1, \ldots, u_m)^T \in \mathcal{U} = \mathbb{R}^m, q \in Q$. We will use the notation $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$ to denote bilinear switched systems. Similarly to bilinear systems, the state- and output trajectories of switched bilinear systems can be expressed by series of iterated integrals. For each $u = (u_1, \ldots, u_k) \in \mathcal{U}$ denote $d\zeta_j[u] = u_j, j = 1, 2, \ldots, m, \quad d\zeta_0[u] = 1$. Denote the set $\{0, 1, \ldots, m\}$ by $\mathrm{Z}_m$. For each $j_1 \cdots j_k \in \mathrm{Z}_m^*, j_1, \ldots, j_k \in \mathrm{Z}_m$, $k \geq 0, t \in T$, $u \in PC(T,\mathcal{U})$ define $V_{j_1 \cdots j_k}[u](t) = \begin{cases} 1 & k = 0 \\ \int_0^t d\zeta_{j_k}[u(\tau)] V_{j_1,\ldots,j_{k-1}}[u](\tau) d\tau & k > 1 \end{cases}$. For each $w_1, \ldots, w_k \in \mathrm{Z}_m^*$, $(t_1, \cdots, t_k) \in T^k$, $u \in PC(T,\mathcal{U})$ define $V_{w_1,\ldots,w_k}[u](t_1, \ldots, t_k) = V_{w_1}(t_1)[u] V_{w_2}(t_2)[\mathrm{Shift}_1(u)] \cdots V_{w_k}[\mathrm{Shift}_{k-1}(u)](t_k)$. where $\mathrm{Shift}_i(u) = \mathrm{Shift}_{\sum_1^i t_i}(u)$, $i = 1, 2, \ldots, k-1$. For each $q \in Q$ and $w = j_1 \cdots j_k, k \geq 0, j_1, \ldots, j_k \in \mathrm{Z}_m$ let us introduce the following notation $B_{q,0} := A_q$, $B_{q,\epsilon} := Id_{\mathcal{X}}$, , $B_{q,w} := B_{q,j_k} B_{q,j_{k-1}} \cdots B_{q,j_1}$, where $Id_X$ is the identity map on $\mathcal{X}$. From the well-known result on iterated integral series expansion of state trajectories of bilinear systems it follows by induction that

$$x_\Sigma(x_0, u, s) = \sum_{w_1,\ldots,w_k \in \mathrm{Z}_m^*} B_{q_k,w_k} \cdots B_{q_1,w_1} x_0 \times \times V_{w_1,\ldots,w_k}[u](t_1, \ldots, t_k)$$

$$y_\Sigma(x_0, u, s) = \sum_{w_1,\ldots,w_k \in \mathrm{Z}_m^*} C_{q_k} B_{q_k,w_k} \cdots B_{q_1,w_1} x_0 \times \times V_{w_1,\ldots,w_k}[u](t_1, \ldots, t_k)$$

$x_0 \in \mathcal{X}$, $u \in PC(T,\mathcal{U})$ and $s = (q_1, t_1) \cdots (q_k, t_k) \in (Q \times T)^*$. Reachability and observability properties of bilinear switched systems can be easily derived from the formulas above. Let $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$ be a bilinear switched system. The following holds.

*Proposition 1:* (i) Let $W(\mathcal{X}_0) = \mathrm{Span}\{z \in \mathcal{X} \mid z \in Reach(\mathcal{X}_0, \Sigma)\}$. Then

$$W(\mathcal{X}_0) = \mathrm{Span}\{B_{q_k,w_k} \cdots B_{q_1,w_1} x_0 \mid q_k, \ldots q_1 \in Q, k \geq 0, w_k, \ldots, w_1 \in \mathrm{Z}_m^*, x_0 \in \mathcal{X}_0\}$$

(ii) Let

$$O_\Sigma = \bigcap_{q_1,\ldots,q_k \in Q, k \geq 0, w_1,\ldots,w_k \in \mathrm{Z}_m^*} \ker C_{q_k} B_{q_k,w_k} \cdots B_{q_1,w_1}$$

Then $x_1, x_2 \in \mathcal{X}$ are indistinguishable if and only if $x_1 - x_2 \in O_\Sigma$. $\Sigma$ is observable if and only if $O_\Sigma = \{0\}$.

Let $\Sigma_1 = (\mathcal{X}_1,\mathcal{U},\mathcal{Y}, Q, \{(A^1_q, \{B^1_{q,j}\}_{j=1,2,\ldots,m}, C^1_q) \mid q \in Q\})$ and $\Sigma_2 = (\mathcal{X}_2,\mathcal{U},\mathcal{Y}, Q, \{(A^2_q, \{B^2_{q,j}\}_{j=1,2,\ldots,m}, C^2_q) \mid q \in Q\})$. A linear map $T : \mathcal{X}_1 \to \mathcal{X}_2$ is called a *homomorphism* from $\Sigma_1$ to $\Sigma_2$, denoted by $T : \Sigma_1 \to \Sigma_2$, if for each $q \in Q$, $j = 1, \ldots, m$ the following holds:

$$TA^1_q = A^2_q T \quad C^1_q = C^2_q T \quad TB^1_{q,j} = B^2_{q,j} \ ,$$

If $T$ is a linear isomorphism then $\Sigma_1$ and $\Sigma_2$ are said to be *isomorphic* or *algebraically similar*. By abuse of terminology $T$ is said to be a bilinear switched system morphism from $(\Sigma, \mu)$ to $(\Sigma', \mu')$, denoted by $T : (\Sigma, \mu) \to (\Sigma', \mu')$, if $T : \Sigma \to \Sigma'$ is a bilinear switched system morphism and $T \circ \mu = \mu'$.

Note that switched systems defined above can be viewed as general non-linear systems with discrete inputs. In particular, bilinear switched systems can be viewed as ordinary bilinear systems with particular inputs. Thus, the realization problem for bilinear switched systems might be reduced to the realization problem for the bilinear systems above. One could attempt to develop realization theory of bilinear switched systems relying on the realization theory for bilinear systems. In this paper we will not pursue this approach. The reason for that is the following First, dealing with restricted switching would require dealing with the realization problem of bilinear systems with input constraints. The author is not aware of any work on this topic. Second, the author thinks that using bilinear realization theory would not substantially simplify the solution to realization problem for bilinear switched systems. Notice however, that the equivalence of realization problems mentioned above does explain the role of rational formal power series in realization theory of bilinear switched systems.

## III. Formal Power Series

The material of this section is based on the classical theory of formal power series, see [14], [2]. A more detailed discussion on the topic can be found in [11], [9]. Let $X$ be a finite alphabet. A *formal power series* $S$ with coefficients in $\mathbb{R}^p$ is a map $S : X^* \to \mathbb{R}^p$. We denote by $\mathbb{R}^p \ll X^* \gg$ the set of all formal power series with coefficients in $\mathbb{R}^p$. An indexed set of formal power series $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ is called *rational* if there exists a vector space $\mathcal{X}$ over $\mathbb{R}$, $\dim \mathcal{X} < +\infty$, linear maps $C : \mathcal{X} \to \mathbb{R}^p$, $A_\sigma : \mathcal{X} \to \mathcal{X}$, $\sigma \in X$ and an indexed set $B = \{B_j \in \mathcal{X} \mid j \in J\}$ of elements of $\mathcal{X}$ such that for all $\sigma_1, \ldots, \sigma_k \in X, k \geq 0$, $S_j(\sigma_1\sigma_2 \cdots \sigma_k) = CA_{\sigma_k} A_{\sigma_{k-1}} \cdots A_{\sigma_1} B_j$. The 4-tuple $R = (\mathcal{X}, \{A_x\}_{x \in X}, B, C)$ is called a *representation* of $S$. The number $\dim \mathcal{X}$ is called the *dimension* of $R$ and it is denoted by $\dim R$. In the sequel the following short-hand notation will be used $A_w := A_{w_k} A_{w_{k-1}} \cdots A_{w_1}$ for $w = w_1 \cdots w_k$. $A_\epsilon$ is the identity map. A representation $R_{min}$ of $\Psi$ is called *minimal* if for each representation $R$ of $\Psi$ it holds that $\dim R_{min} \leq \dim R$. Let $R_i = (\mathcal{X}_i, \{A_{i,\sigma}\}_{\sigma \in X}, B_i, C_i)$, $i = 1, 2$ be two representations. A *representation morphism* $T : R_1 \to R_2$ is a linear map $T : \mathcal{X}_1 \to \mathcal{X}_2$ such that the following holds $TA_{1,x} = A_{2,x} T, \forall x \in X$, $TB_{1j} = B_{2j}, \forall j \in J$, $C_1 = C_2 T$ . The homomorphism $T$ is called surjective, injective, isomorphism if $T$ is a surjective, injective or isomorphism respectively. Let $L \subseteq X^*$. If $L$ is a regular language then the power series $\bar{L} \in \mathbb{R} \ll X^* \gg$, $\bar{L}(w) = \begin{cases} 1 & \text{if } w \in L \\ 0 & \text{otherwise} \end{cases}$ is a rational power series. Consider two power series $S, T \in \mathbb{R}^p \ll X^* \gg$. Define the *Hadamard product* $S \odot T \in \mathbb{R}^p \ll X^* \gg$ by $(S \odot T)_i(w) = S_i(w) T_i(w), i = 1, \ldots, p$. Let $w \in X^*$ and define $w \circ S \in \mathbb{R}^p \ll X^* \gg$ – the *left shift of $S$ by $w$* by $\forall v \in X^* : w \circ S(v) = S(wv)$. The following statements are generalizations of the results on rational power series from

[2]. Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. Define $W_\Psi = \mathrm{Span}\{w \circ S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J, w \in X^*\}$. Define the Hankel-matrix $H_\Psi$ of $\Psi$ as $H_\Psi \in \mathbb{R}^{(X^* \times I) \times (X^* \times J)}$, $I = \{1, 2, \dots, p\}$ and $(H_\Psi)_{(u,i)(v,j)} = (S_j)_i(vu)$. Notice that $\mathrm{Im} H_\Psi$ is isomorphic to $W_\Psi$ and thus $W_\Psi = \mathrm{rank}\ H_\Psi$.

*Theorem 1:* Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. Then $\Psi$ is rational if and only if $\dim W_\Psi = \mathrm{rank}\ H_\Psi < +\infty$.

*Lemma 1:* Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ and $\Theta = \{T_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ be rational indexed sets. Then $\Psi \odot \Theta := \{S_j \odot T_j \mid j \in J\}$ is a rational set. Moreover, $\mathrm{rank}\ H_{\Psi\odot\Theta} \leq \mathrm{rank}\ H_\Psi \cdot \mathrm{rank}\ H_\Theta$.

Let $R = (\mathcal{X}, \{A_\sigma\}_{\sigma \in X}, B, C)$ be a representation of $\Psi \subseteq \mathbb{R}^p \ll X^* \gg$. Define the following subspaces of $\mathcal{X}$: $W_R = \mathrm{Span}\{A_w B_j \mid w \in X^*, j \in J\}$ and $O_R = \bigcap_{w \in X^*} \ker C A_w$. The representation $R$ is called *reachable* if $\dim W_R = \dim R$ and $R$ is called *observable* if $O_R = \{0\}$. It can be shown, that if $J$ is a finite set, then observability and reachability of representations can be checked by a numerical algorithm. Moreover, in this case $R$ can be transformed to a reachable and observable representation by a numerical algorithm. See [10] on this issue.

*Theorem 2 (Minimal representation):* Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. The following are equivalent. (i) $R_{min} = (\mathcal{X}, \{A^{min}_\sigma\}_{\sigma \in X}, B^{min}, C^{min})$ is a minimal representation of $\Psi$, (ii) $R_{min}$ is reachable and observable. (iii) $\mathrm{rank}\ H_\Psi = \dim W_\Psi = \dim R_{min}$, (iv) If $R$ is a reachable representation of $\Psi$, then there exists a surjective representation morphism $T : R \to R_{min}$. In particular, all minimal representations of $R$ are isomorphic.

Note that if $\Psi$ is rational, one can construct a minimal representation of $\Psi$ over the space of column vectors of $H_\Psi$. Without loss of generality we can always assume that $\mathcal{X} = \mathbb{R}^n$ holds for any representation considered.

Below we will present conditions, under which a representation of $\Psi$ can be constructed from finite data. The approach is similar to [4]. A more detailed discussion can be found in [10]. For each $S \in \mathbb{R}^p \ll X^*$ define $S_N = S_{\{w \in X^*, |w| \leq N\}}$. Let $H_{\Psi,N,M} \in \mathbb{R}^{I_M \times J_N}$, $I_M = \{(v, i) \mid v \in X^*, |v| \leq M, i = 1, \dots, p\}$, $J_N = \{(u, j) \mid j \in J, u \in X^*, |u| \leq N\}$ and $(H_{\Psi,N,M})_{(v,i),(u,j)} = (S_j(uv))_i$. Notice that $H_{\Psi,N,M}$ is a finite matrix, if $J$ is finite. Define $W_{\Psi,N,M} = \{(w \circ S_j)_M \mid w \in X^*, |w| \leq N, j \in J\}$. Notice that $\mathrm{rank}\ H_{\Psi,N,M} = \dim W_{\Psi,N,M}$.

*Theorem 3 (Partial representation):* (i) If $R$ is a representation of $\Psi$, $\dim R \leq N$, then $\mathrm{rank}\ H_\Psi = \mathrm{rank}\ H_{\Psi,N,N}$, (ii) Assume that $\mathrm{rank}\ H_{\Psi,N,N} = \mathrm{rank}\ H_{\Psi,N,N+1} = \mathrm{rank}\ H_{\Psi,N+1,N}$. Then there exists a representation $R_N = (W_{\Psi,N,N}, \{A_x\}_{x \in X}, C, B)$, such that $A_x((w \circ S_j)_N) = (wx \circ S_j)_N$, $C(T) = T(\epsilon)$, $B_j = (S_j)|_N, j \in J$ and for which the following holds. If $\Psi$ has a representation $R$ such that $N \geq \dim R$, then $R_N$ is a minimal representation of $\Psi$.

## IV. Input/output maps of bilinear switched systems

Let $L \subseteq Q^+$. Let $\widetilde{\Gamma} = Q \times \mathbb{Z}_m^*$. Let $JL = \{(q_1, w_1) \cdots (q_k, w_k) \in \widetilde{\Gamma}^* \mid (q_1, w_1) \dots, (q_k, w_k) \in \widetilde{\Gamma}, k \geq 0, q_1 \cdots q_k \in L\}$. Define the relation $R \subseteq \widetilde{\Gamma}^* \times \widetilde{\Gamma}^*$ by requiring that $(q, w_1)(q, w_2) R (q, w_1 w_2)$, and $(q, \epsilon)(q', w) R (q', w)$ hold for any $q \in Q, (q', w) \in \widetilde{\Gamma}$, $(q, w_1), (q, w_2) \in \widetilde{\Gamma}$ Let $R^*$ be smallest congruence relation containing $R$. That is, $R^*$ is the smallest relation such that $R \subseteq R^*$, $R^*$ is symmetric, reflexive, transitive and $(v, v') \in R^*$ implies $(wvu, wv'u) \in R^*$, for each $w, u \in \widetilde{\Gamma}^*$. A $c : JL \to \mathcal{Y}$ is called a *generating convergent series on* $JL$ if (1) $(w, v) \in R^*, w, v \in JL \implies c(w) = c(v)$, (2) There exists $K, M > 0$ such that for each $(q_1, w_1) \cdots (q_k, w_k) \in JL$, $(q_1, w_1) \dots (q_k, w_k) \in \widetilde{\Gamma}$: $c((q_1, w_1) \cdots (q_k, w_k)) < K M^{|w_1|} \cdots M^{|w_k|}$. The notion of generating convergent series is an extension of the notion of convergent power series from [6].

Let $c : JL \to \mathcal{Y}$ be a generating convergent series. For each $u \in PC(T, \mathcal{U})$ and $s = (q_1, t_1) \cdots (q_k, t_k) \in TL$ define the convergent series $F_c(u, s) = \sum_{w_1, \dots, w_k \in \mathbb{Z}_m^*} c((q_1, w_1) \cdots (q_k, w_k)) \times \times V_{w_1, \dots, w_k}[u](t_1, \dots, t_k)$. By induction, using the well-known result for classical Fliess-series expansion, one can show that the series above are absolutely convergent. In fact we can define a function $F_c \in F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$ by $F_c : (u, w) \mapsto F_c(u, w)$. It can be shown that $F_c$ is uniquely determined by $c$. That is, if $d, c : JL \to \mathcal{Y}$ are two convergent generating series, then $F_c = F_d \iff c = d$. Now we are ready to define the concept of generalized Fliess-series representation of a set of input/output maps. The set of input-output maps $\Phi \subseteq F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$ is said to admit *a generalized Fliess-series expansion* if for each $f \in \Phi$ there exists a generating convergent series $c_f : JL \to \mathcal{Y}$ such that $F_{c_f} = f$. The following proposition gives a description of the Fliess-series expansion of $\Phi$ in the case when $\Phi$ is realized by a bilinear switched system.

*Proposition 2:* $(\Sigma, \mu)$ is a bilinear switched system realization of $\Phi$ with constraint $L$ if and only if $\Phi$ has a generalized Fliess-series expansion such that for each $f \in \Phi, (q_1, w_1) \cdots (q_k, w_k) \in JL$

$$c_f((q_1, w_1) \cdots (q_k, w_k)) = C_{q_k} B_{q_k, w_k} \cdots B_{q_1, w_1} \mu(f)$$

## V. Realization theory for bilinear switched systems

In this section realization theory for bilinear switched systems will be developed. We start with the case when the input/output maps are defined over all the switching sequences. Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$ and assume that $\Phi$ has a generalized Fliess-series expansion. As in the case of linear switched systems [11], [9], we will associate with $\Phi$ an indexed set of formal power series $\Psi_\Phi$. It turns out that every representation of $\Psi_\Phi$ determines a realization of $\Phi$ and vice versa. We will use the theory of formal power series to derive the results on realization theory. The proofs of the theorems of this section can be found in [11].

Let $\Gamma = \{(q, j) \mid q \in Q, j \in \mathbb{Z}_m\}$. Define $\phi : \widetilde{\Gamma} \to \Gamma^*$ by $\phi((q, j_1 \cdots j_k)) = (q, j_1) \cdots (q, j_k)$, $\phi((q, \epsilon)) = \epsilon$ where $j_1, \dots, j_k \in \mathbb{Z}_m, k \geq 0$. The map $\phi$ determines a semigroup morphism $\phi : \widetilde{\Gamma}^* \to \Gamma^*$ given by $\phi((q_1, w_1) \cdots (q_k, w_k)) = \phi((q_1, w_1)) \cdots \phi((q_k, w_k))$ for each $(q_1, w_1), \dots, (q_k, w_k) \in$

$\widetilde{\Gamma}, k \geq 0$ and $\phi(\epsilon) = \epsilon$. It is also clear that any element of $\Gamma$ can be thought of as an element of $\widetilde{\Gamma}$, i.e. we can define the monoid morphism $i : \Gamma^* \rightarrow \widetilde{\Gamma}^*$ by $i(\epsilon) = \epsilon$ and $i((q_1, j_1) \cdots (q_k, j_k)) = (q_1, j_1) \cdots (q_k, j_k)$, $(q_1, j_1), \ldots, (q_k, j_k) \in \Gamma \subseteq \widetilde{\Gamma}$. It is also easy to see that $\phi(i(w)) = w, \forall w \in \Gamma^*$ and $w(q, \epsilon) R^* i(\phi(w))(q, \epsilon)$.

For each $f \in \Phi, q \in Q$ define formal power series $S_{f,q} \in \mathbb{R}^p \ll \Gamma^* \gg$ as follows:

$$S_{f,q}(s) = c_f(i(s)(q, \epsilon)), \forall s \in \Gamma^*$$

It is easy to see that in fact $c_f(v(q, \epsilon)) = S_{f,q}(\phi(v)) = c_f(i(\phi(v))(q, \epsilon))$, since $(v(q, \epsilon), i(\phi(v))(q, \epsilon)) \in R^*$. Assume that $Q = \{q_1, \ldots, q_N\}$. Define the formal power series $S_f \in \mathbb{R}^{Np} \ll \Gamma^* \gg$ by

$$S_f = [S^T_{f,q_1}, \ldots, S^T_{f,q_N}]^T$$

Define the *set of formal power series* $\Psi_\Phi$ *associated with* $\Phi$ by

$$\Psi_\Phi = \{S_f \in \mathbb{R}^{Np} \ll \Gamma^* \gg \mid f \in \Phi\}$$

Define the *Hankel-matrix* $H_\Phi$ of $\Phi$ as the Hankel-matrix of $\Psi_\Phi$. i.e. $H_\Phi = H_{\Psi_\Phi}$. Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$. Define the representation $R_{\Sigma,\mu}$ *associated with the realization* $(\Sigma, \mu)$ of $\Phi$ by

$$R_\Sigma = (\mathcal{X}, \{B_{(q,j)}\}_{(q,j)\in\Gamma}, \widetilde{C}, I)$$

where $B_{(q,j)} = B_{q,j}, B_{q,0} = A_q, q \in Q, j = 1, \ldots, m$, $\widetilde{C} = \begin{bmatrix} C^T_{q_1} & \ldots & C^T_{q_N} \end{bmatrix}^T$ and $I_f = \mu(f)$. Let $R = (\mathcal{X}, \{M_{(q,j)}\}_{(q,j)\in\Gamma}, \widetilde{C}, I)$ be a representation such that $I = \{I_f \in \mathcal{X} \mid f \in \Phi\}$. Define the realization $(\Sigma_R, \mu_R)$ *associated with* $R$ by

$$\Sigma_R = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$$

where $\mu_R(f) = I_f, f \in \Phi$, $B_{q,j} = M_{(q,j)}, A_q = M_{(q,0)}, q \in Q, j = 1, \ldots, m$, and $\widetilde{C} = \begin{bmatrix} C^T_{q_1} & \ldots & C^T_{q_N} \end{bmatrix}^T$. It is easy to see that $R_{\Sigma_R,\mu_R} = R$. Assume that $\Phi$ admits a generalized Fliess-series expansion. Then, (a) $(\Sigma, \mu)$ realization of $\Phi$ if and only if $R_{\Sigma,\mu}$ is a representation of $\Psi_\Phi$, (b) Conversely, $R$ is a representation of $\Psi_\Phi$ if and only if $(\Sigma_R, \mu_R)$ is a realization of $\Phi$. From the discussion above using Theorem 1 one gets the following characterization of realizability. Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$.

*Theorem 4 (Existence of a realization):* The following are equivalent (i) $\Phi$ has a realization by a bilinear switched system, (ii) $\Phi$ has a generalized Fliess-series expansion and $\Psi_\Phi$ is rational, (iii) $\Phi$ has a generalized Fliess-series expansion and $\operatorname{rank} H_\Phi < +\infty$

Assume that $(\Sigma, \mu)$ is a realization of $\Phi$. Let $R = R_{\Sigma,\mu}$. Then it is easy to see that $(\Sigma, \mu)$ is observable if and only if $R$ is observable, and $(\Sigma, \mu)$ is semi-reachable from $\operatorname{Im} \mu$ if and only if $R$ is reachable. It is also easy to see that $\dim \Sigma = \dim R_{\Sigma,\mu}$ and $\dim R = \dim \Sigma_R$. In fact, if $R$ is a minimal representation of $\Psi_\Phi$ then $(\Sigma_R, \mu_R)$ is a minimal realization of $\Phi$. Conversely, if $(\Sigma, \mu)$ is a minimal realization of $\Phi$, then $R_{\Sigma,\mu}$ is a minimal representation of $\Psi_\Phi$. Moreover, $T : (\Sigma, \mu) \rightarrow (\Sigma', \mu')$ is a bilinear switched system morphism if and only if $T : R_{\Sigma,\mu} \rightarrow R_{\Sigma',\mu'}$ is a representation morphism. Using the theory of ration formal power series presented in Section III we get the following. Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$.

*Theorem 5 (Minimal realization):* The following are equivalent (i) $(\Sigma_{min}, \mu_{min})$ is a minimal realization of $\Phi$ by a bilinear switched system, (ii) $(\Sigma_{min}, \mu_{min})$ is semi-reachable and it is observable , (iii) $\dim \Sigma_{\min} = \operatorname{rank} H_\Phi$, (iv) For any bilinear switched system realization $(\Sigma, \mu)$ of $\Phi$, such that $(\Sigma, \mu)$ is semi-reachable , there exist a surjective homomorphism $T : (\Sigma, \mu) \rightarrow (\Sigma_{min}, \mu_{min})$. In particular, all minimal bilinear switched system realizations of $\Phi$ are isomorphic.

In fact, it is easy to see that if $R$ is a minimal representation $\Psi_\Phi$, then $(\Sigma_R, \mu_R)$ is a minimal realization of $\Phi$. By the remark in Section III, it means that we can construct a realization of $\Phi$ on the column space of $H_\Phi$. From any $(\Sigma, \mu)$ bilinear realization of $\Phi$ we can construct a minimal realization of $\Phi$, by constructing from $R_{\Sigma,\mu}$ a minimal representation $R$ of $\Psi$ and then constructing $(\Sigma_R, \mu_R)$. The discussion in Section III yields that $R$ is computable from $R_{\Sigma,\mu}$ if $\Phi$ is finite, and thus $(\Sigma_R, \mu_R)$ is computable from $(\Sigma, \mu)$ if $\Phi$ is finite. The theory of rational formal power series also enables us to formulate partial realization theory for bilinear switched systems. With the notation of Theorem 3 the following holds.

*Theorem 6 (Partial realization):* Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. Assume that $\operatorname{rank} H_{\Psi_\Phi,N,N} = \operatorname{rank} H_{\Psi_\Phi,N+1,N} = \operatorname{rank} H_{\Psi_\Phi,N,N+1}$. Let $R_N$ be the representation from Theorem 3. Let $(\Sigma_N, \mu_N) = (\Sigma_{R_N}, \mu_{R_N})$. If $\Phi$ has a realization $(\Sigma, \mu)$ such that $N \geq \dim \Sigma$, then $(\Sigma_N, \mu_N)$ is a minimal realization of $\Phi$.

The theorem above implies that if it is known that $\Phi$ has a realization by a bilinear switched system of dimension at most $N$, then a minimal realization of $\Phi$ can be computed from finitely many data.

The case of restricted switching is slightly more involved. As in the case of arbitrary switching, we will associate a set $\Psi_\Phi$ of formal power series with the set of input-output maps $\Phi \subseteq F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$. If $L$ is regular then there is a correspondence between realizations of $\Phi$ and representations of $\Psi_\Phi$. However, minimal representations of $\Psi_\Phi$ need not yield realizations of $\Phi$ of the smallest possible dimension.

Recall the definition of the relation $R^* \subseteq \widetilde{\Gamma}^* \times \widetilde{\Gamma}^*$ from Subsection IV. Define the set $\widetilde{JL} \subseteq \widetilde{\Gamma}^*$ by $\widetilde{JL} = \{s \in \widetilde{\Gamma}^* \mid \exists w \in JL : (w, s) \in R^*\}$. In fact, $\widetilde{JL}$ contains all those sequences in $\widetilde{\Gamma}^*$ for which we can derive some information based on the values of a convergent generating series for the sequences from $JL$. More precisely, if $c : JL \rightarrow \mathcal{Y}$ is a generating convergent series, then $c$ can be extended to a generating convergent series $\widetilde{c} : \widetilde{JL} \rightarrow \mathcal{Y}$ by defining $\widetilde{c}(s) = c(w)$ for each $s \in \widetilde{JL}, w \in JL, (s, w) \in R^*$. By abuse of notation we will denote $\widetilde{c}$ simply by $c$. For each $q \in Q$ define $JL_q = \{v(q, w) \in \widetilde{JL} \mid v \in \widetilde{\Gamma}^*, (q, w) \in \widetilde{\Gamma}\}$.

Let $L_q = \{w \in \Gamma^* \mid \exists v \in JL_q : \phi(v) = w\}$. Notice that $w \in L_q \iff i(w)(q, \epsilon) \in JL_q$. Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$. For each $q \in Q$, $f \in \Phi$ define $T_{f,q} \in \mathbb{R}^p \ll \Gamma^* \gg$ by

$$T_{f,q}(s) = \begin{cases} c_f(i(s)(q,\epsilon)) & \text{if } s \in L_q \\ 0 & \text{otherwise} \end{cases}$$

Notice that for each $s \in L_q$ there exists a $w = u(q, v) \in JL$ such that $(w, i(s)(q, \epsilon)) \in R^*$, which implies that $T_{f,q}(s) = c_f(w)$ for some $w \in JL$. Assume that $Q = \{q_1, \ldots, q_N\}$. Define the formal power series $T_f \in \mathbb{R}^{Np} \ll \Gamma^* \gg$ by

$$T_f = [T_{f,q_1}^T, \ldots, T_{f,q_N}^T]^T$$

Define the *set of formal power series* $\Psi_\Phi$ *associated with* $\Phi$ as

$$\Psi_\Phi = \{T_f \in \mathbb{R}^{Np} \ll \Gamma^* \gg \mid f \in \Phi\}$$

Define the *Hankel-matrix* $H_\Phi$ *of* $\Phi$ as the Hankel-matrix of $\Psi_\Phi$, that is, $H_\Phi = H_{\Psi_\Phi}$. Define $Z_q \in \mathbb{R}^p \ll \Gamma^* \gg$ by $Z_q(w) = \begin{cases} (1,1,\ldots,1)^T & \text{if } w \in L_q \\ 0 & \text{otherwise} \end{cases}$. Define $Z \in \mathbb{R}^{Np} \ll \Gamma \gg$ by $Z = \begin{bmatrix} Z_{q_1}^T & \cdots & Z_{q_N}^T \end{bmatrix}^T$, and let $\Omega$ be the indexed set $\{Z \mid f \in \Phi\}$, i.e $\Omega : \Phi \to \mathbb{R}^{Np} \ll \Gamma^* \gg$ and $\Omega(f) = Z, f \in \Phi$. Define the set $comp(L) = \{w_1 \cdots w_k \in Q^* \mid \forall v \in Q^* : vw_k \notin L, w_1, \ldots, w_k \in Q\}$.

*Lemma 2:* Assume $(\Sigma, \mu)$ is a bilinear switched system realization of $\Phi$ with constraint $L$. Let $\Phi^{'} = \{y_\Sigma(\mu(f), ., .) \in F(PC(T,\mathcal{U}) \times (Q \times T)^+, \mathcal{Y}) \mid f \in \Phi\}$ and let $\Psi_\Phi^{'}$ be the set of formal power series associated with $\Phi^{'}$ as defined for the case of arbitrary switching. That is, $\Psi_{\Phi'} = \{S_g \in \mathbb{R}^{Np} \ll \Gamma^* \gg \mid g \in \Phi^{'}\}$. Let $S_f = S_{y_\Sigma(\mu(f),.,.)}$ and let $\Theta = \{S_f \mid f \in \Phi\}$. Then $\Psi_\Phi = \Theta \odot \Omega$.

*Theorem 7:* If $\Phi$ has a generalized Fliess-series expansion and $R$ is a representation of $\Psi_\Phi$, then $(\Sigma_R, \mu_R)$ is a realization of $\Phi$ with constraint $L$. Moreover, for each $f \in \Phi$, $w \in T(comp(L))$, $\forall u \in PC(T,\mathcal{U}) : y_\Sigma(\mu(f), u, w) = 0$.

We see that rationality of $\Psi_\Phi$, i.e. the condition $\mathrm{rank}\ H_\Phi < +\infty$, is a sufficient condition for realizability of $\Phi$. It turns out that if $L$ is regular, this is also a necessary condition, since then $\Omega$ is a rational indexed set.

*Theorem 8:* Assume that $L$ is regular. Then the following are equivalent. (i) $\Phi$ has a realization with constraint $L$ by a bilinear switched system , (ii) $\Phi$ has a generalized Fliess-series expansion and $\mathrm{rank}\ H_\Phi < +\infty$, (iii) There exists a realization with constraint $L$ of $\Phi$ by a bilinear switched system $(\Sigma, \mu)$ such that $\Sigma$ is observable and semi-reachable and $\forall f \in \Phi : y_\Sigma(\mu(f), ., .)|_{PC(T,\mathcal{U}) \times T(compl(L))} = 0$ and for any $(\Sigma^{'}, \mu^{'})$ bilinear switched system realization of $\Phi$ it holds that $\dim \Sigma \leq \mathrm{rank}\ H_\Omega \dim \Sigma^{'}$.

The following example demonstrates existence of a semi-reachable and observable realization of $\Phi$, which is non-minimal.

**Example** Let $Q = \{1, 2\}$, $L = \{q_1^k q_2 \mid k > 0\}$, $\mathcal{Y} = \mathcal{U} = \mathbb{R}$. Define the generating series $c : JL \to \mathbb{R}$ by $c((q_1, w_1)(q_2, w_2)) = 2^k$, where $w_2 = 0^{j_0} z_1 \cdots z_l 0^{j_l}$, $k = \sum_{i=0}^{l} j_l$, $z_i \in \{1\}^*, i = 1, \ldots, l$. Let $\Phi = \{F_c\}$. Define the system $\Sigma_1 = (\mathbb{R}, \mathbb{R}, \mathbb{R}, Q, \{(A_q, B_{q,1} C_q) \mid q \in \{q_1, q_2\}\})$ by $A_{q_1} = 1, B_{q_1,1} = 1, C_{q_1} = 1$ and $A_{q_2} = 2, B_{q_2,1} = 1, C_{q_2} = 1$ . Define the system $\Sigma_2 = (\mathbb{R}^2, \mathbb{R}, \mathbb{R}, Q, \{(\widetilde{A}_q, \widetilde{B}_{q,1}, \widetilde{C}_q) \mid q \in Q\})$ by

$$\widetilde{A}_{q_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \widetilde{B}_{q_1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \widetilde{C}_{q_1} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$
$$\widetilde{A}_{q_2} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \quad \widetilde{B}_{q_2,1} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \widetilde{C}_{q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Let $\mu_1 : F_c \mapsto 1$ and $\mu_2 : F_c \mapsto (1, 0)^T \in \mathbb{R}^2$. Both $(\Sigma_1, \mu_1)$ and $(\Sigma_2, \mu_2)$ are semi-reachable from $\mathrm{Im}\mu_1$ and $\mathrm{Im}\mu_2$ respectively and they are observable, therefore they are the minimal realizations of $y_{\Sigma_1}(1, ., .)$ and $y_{\Sigma_2}((1, 0)^T, ., .)$. Moreover, it is easy to see that $(\Sigma_i, \mu_i), i = 1, 2$ are both realizations of $\Phi$ with constraint $L$. Yet, $\dim \Sigma_1 = 1$ and $\dim \Sigma_2 = 2$. In fact, $\Sigma_2$ can be obtained by constructing the minimal representation of $\Psi_\Phi$, i.e., $\Sigma_2$ is a realization of $F_c$ satisfying part (iii) of Theorem 8.

## VI. Conclusions

Solution to the realization problem for bilinear switched systems was presented. The realization problem considered is to find a realization of a family of input-output maps. Moreover, it is allowed to restrict the input-output maps to some subsets of switching sequences. Topics of further research include realization theory for piecewise-affine systems, switched systems with switching controlled by an automaton or a timed automaton and non-linear switched systems.

**Acknowledgment** The author thanks Jan H. van Schuppen for the help with the preparation of the manuscript. The author thanks Luc Habets for the useful discussions and suggestions.

## References


[1] Joseph A. Ball, Gilbert Groenewald, and Tanit Malakorn. Structured noncommutative multidimensional linear systems. In *MTNS 2004*, 2004.

[2] J. Berstel and C. Reutenauer. In *Rational series and Their Languages*, EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1984.

[3] M Fliess. Sur la realisation des systemes dynamiques bilineares. *C. R. Acad. Sc. Paris, Series A*, 277:243–247, 1973.

[4] Dieter Gollmann. Partial realization by discrete-time internally bilinear systems: An algorithm. In *MTNS 1983*, 1983.

[5] Alberto Isidori. Direct construction of minimal bilinear realizations from nonlinear input/output maps. *IEEE Trans* Automatic Control*, AC-18:626–631, 1973.

[6] Alberto Isidori. *Nonlinear Control Systems*. Springer Verlag, 1989.

[7] Daniel Liberzon. *Switching in Systems and Control*. Birkhäuser, Boston, 2003.

[8] M. Petreczky. Realization theory for linear switched systems. In *MTNS, Proc.*, 2004.

[9] M. Petreczky. Realization theory for linear switched systems: Formal power series approach. Technical Report MAS-R0403, CWI, 2004.

[10] M. Petreczky. Realization theory for linear and bilinear hybrid systems. 2005. To appear as a CWI research report.

[11] M. Petreczky. Realization theory of linear and bilinear switched systems: A formal power series approach. 2005. To appear as a CWI research report.

[12] M. P. Schützenberger. On the definition of a family of automata. *Information and Control*, 4:245–270, 1961.

[13] Eduardo D. Sontag. *Polynomial Response Maps*, volume 13 of *Lecture Notes in Control and Information Sciences*. Springer Verlag, 1979.

[14] Eduardo D. Sontag. Realization theory of discrete-time nonlinear systems: Part I – the bounded case. *IEEE Transaction on Circuits and Systems*, CAS-26(4), April 1979.