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# Nonlinear Output Regulation Without Immersion

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*Abstract*— **The main goal of this paper is to show that the so-called "immersion assumption" is not in principle needed in solving a problem of nonlinear output regulation by output feedback. Under the only assumption that the zero dynamics of the controlled system have bounded trajectories, it is shown that there exists a controller solving the problem. The design procedure illustrated in the paper is based on some recent results, developed in [1], on the theory of nonlinear state observers originally proposed in [16]. The internal model obtained in this way is a linear Hurwitz system with nonlinear output map.**

## I. Introduction

In the problem of nonlinear output regulation (see [14], [5]), it is well-known that a central role in the design of the regulator is played by the so-called *internal model unit*. The latter is a subsystem, embedded in the regulator, which is required to posses two fundamental properties. First, it is required to have the capability of reproducing all possible "feed-forward inputs" securing perfect tracking/rejection of trajectories/disturbances generated by the so-called *exosystem* (the latter being the dynamical system which, in the framework of output regulation, is supposed to generate exogenous reference/disturbance signals). Second, it is required to have the property that the *augmented system*, namely the controlled plant augmented with the internal model unit, be stabilizable by output feedback. The need of satisfying simultaneously the previous two properties has been the primary reason of being of a crucial "classical" assumption characterizing most of the existing related literature: the so-called "immersion assumption". The latter essentially consists in the requirement that the dynamical system defining all possible "feed-forward inputs" which force an identically zero regulation error be "immersed" into a system exhibiting certain structural properties. In this framework, the recent literature has shown steady development of more general and less restrictive conditions. At the beginning, the system in question was assumed to be immersed into a linear *known* observable system (see [13], [19], [5], [20]). This assumption has been then weakened, in the framework of adaptive nonlinear regulation (see [21]), by asking immersion into a linear *un-known* (but linearly parameterized) system. Subsequent extensions have been presented in [8] (where immersion into a linear system having a nonlinear output map is assumed) and in an important contribution in the development of in [9] (where immersion into a nonlinear system linearizable by output injection is assumed). Finally the recent works in [3] and [10] (see also [11]) have definitely focused the attention on the design of nonlinear internal models requiring immersion into *nonlinear* systems described, respectively, in a canonical observability form and in a nonlinear adaptive observability form.

The contribution of the present paper is to go a step further and to present the important conceptual result that no immersion assumption is needed at all for the regulator to exist. In particular we shall show that under the only assumption that the zero dynamics of the extended system (namely of the controlled plant and of the exosystem) have bounded trajectories, there exists an internal model-based controller solving the problem of output regulation. In order to prove this

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result, we will take advantage of a recently proposed theory of observers for nonlinear systems pioneered in [16] subsequently refined in [17], [18] and [1]. In linee with the results obtained in [1] for observer design, the internal model unit is constituted by a linear Hurwitz system of suitable dimension having a continuous nonlinear output map.

The solution provided in this paper will be derived in the general "non-equilibrium" framework proposed in [2] in which the zero dynamics of regulated plant and the dynamics of the exosystem are not required to posses an equilibrium point but rather a possible complex, though bounded, attractor. In this general framework the proof of our result passes through a number of technical details which, for obvious reasons, can not be reported in a conference paper and which will be presented in a journal version, in preparation, to which the interested reader is referred. Here we just limit ourself to provide the main result and the main theorems and propositions supporting the main result.

## II. Problem Statement

In [3], as an illustration of how the non-equilibrium approach presented in [2] can be applied to the design of regulators, we have shown how the problem of output regulation can be solved, under appropriate assumptions, for a system which can be put in the form

$$\begin{array}{rcl} \dot z &=& f(w,z,\zeta) \\ \dot\zeta &=& q(w,z,\zeta)+u \\ e &=& \zeta \\ y &=& \zeta\,, \end{array} \tag{1}$$

with state $(z,\zeta) \in \mathbb{R}^n \times \mathbb{R}$, control input $u \in \mathbb{R}$, regulated output $e \in \mathbb{R}$, measured output $y \in \mathbb{R}$ and exogenous (disturbance) input $w \in \mathbb{R}^r$ generated by an exosystem

$$\dot w = s(w)\,. \tag{2}$$

The functions $f(w,z,\zeta), q(w,z,\zeta)$ and $s(w)$ are $C^k$ functions (for some large $k$) of their arguments. The initial conditions of (1) range on a set $Z \times E$, in which $Z$ and $E$ are fixed *compact* subsets of $\mathbb{R}^n$ and, respectively, $\mathbb{R}$. The initial conditions of the exosystem (2) range on a *compact* subset $W$ of $\mathbb{R}^r$.

*Remark* System (1) may look very particular, as it has relative degree 1 between control input $u$ and regulated output $e$. However, the design methodology described in [3], and pursued in what follows under much weaker hypotheses, lends itself to a straightforward extension to systems with higher relative degree. Details are not included here (see, for instance, [6]). $\triangleleft$

The analysis in [3] was based on three standing hypotheses. The first of these hypotheses was that the exosystem is "in steady-state":

*Assumption* 0. The set $W$ is a differential submanifold (with boundary) invariant for (2). $\triangleleft$

The second hypothesis was that the trajectories of the zero dynamics of (1), augmented with (2), are *bounded*, namely that:

*Assumption* 1. There exists a bounded subset $B$ of $W \times \mathbb{R}^n$ which contains the positive orbit of the set $W \times Z$ under the flow of

$$\begin{array}{rcl} \dot w &=& s(w) \\ \dot z &=& f(w,z,0) \end{array} \tag{3}$$

and the set $\omega(W \times Z)$ is a differential submanifold (with boundary) of $W \times \mathbb{R}^n$. Moreover, there exists a number $d_1 > 0$ such that

$$\begin{array}{l} (w,z) \in W \times \mathbb{R}^n \\ |(w,z)|_{\omega(W\times Z)} \le d_1 \end{array} \quad \Rightarrow \quad (w,z) \in W \times Z\,. \quad \triangleleft$$

It follows from this assumption (see [12]) that the set

$$\mathcal{A}_0 := \omega(W \times Z)\,,$$

i.e the $\omega$-limit set – under the flow of (3) – of the set $W \times Z$, is a nonempty, compact, invariant set which is stable in the sense of Lyapunov and uniformly attracts $W \times Z$.

The third assumption was the so-called *immersion assumption* described as follows.

*Assumption* 2-nl. There exists an integer $d$ and a locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ such that, for any $(w_0,z_0) \in \mathcal{A}_0$, the solution $(w(t),z(t))$ of (3) passing through $(w_0,z_0)$ at $t=0$ is such that the function $\varphi(t) := -q(w(t),z(t),0)$ satisfies

$$\varphi^{(d)} + f(\varphi,\varphi^{(1)},\ldots,\varphi^{(d-1)}) = 0\,.\ \triangleleft \tag{4}$$

The purpose of this paper it to show how this last assumption can be removed, showing in this way the important conceptual result that no immersion assumption is needed in the the design of controllers which solve the problem of output regulation for nonlinear systems.

## III. Output Regulation

We consider in what follows a control law of the form

$$\begin{aligned} \dot{\xi} &= F\xi + Gu \\ u &= \gamma(\xi) + v \\ v &= -ky\,, \end{aligned} \tag{5}$$

in which $(F, G) \in I\!R^{m\times m} \times I\!R^{m\times 1}$ is a controllable pair with $F$ a Hurwitz matrix and $\gamma : I\!R^m \to I\!R$ is a continuous map. The initial conditions of (5) range on a *compact* subset $\Xi$ of $I\!R^m$.

The control of (1) by means of (5) results in a system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w, z, \zeta) \\ \dot{\zeta} &= q(w, z, \zeta) + \gamma(\xi) + v \\ \dot{\xi} &= F\xi + G(\gamma(\xi) + v)\,, \end{aligned} \tag{6}$$

which, via the change of variables $\xi \mapsto x = \xi - G\zeta$, can be put in normal form as

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w, z, \zeta) \\ \dot{x} &= Fx - Gq(w, z, \zeta) + FG\zeta \\ \dot{\zeta} &= q(w, z, \zeta) + \gamma(x + G\zeta) + v\,. \end{aligned} \tag{7}$$

This system, viewing $v$ as input and $\zeta$ as output, has relative degree 1 and a zero-dynamics

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w, z, 0) \\ \dot{x} &= Fx - Gq(w, z, 0)\,. \end{aligned} \tag{8}$$

The asymptotic properties of the latter are summarized in the following results. Set $\mathbf{z} := \mathrm{col}(w, z)$, set $\mathbf{Z} := W \times Z$ and rewrite (8) as

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) \\ \dot{x} &= Fx - G\mathbf{q}_0(\mathbf{z})\,. \end{aligned} \tag{9}$$

Consider the map

$$\begin{aligned} \tau \;:\; \mathcal{A}_0 &\to I\!R^m \\ \mathbf{z} &\mapsto -\int_{-\infty}^{0} e^{-Fs} G\mathbf{q}_0(\mathbf{z}(s, \mathbf{z})))ds \end{aligned}$$

and set

$$\mathrm{gr}(\tau) := \{(\mathbf{z}, x) : \mathbf{z} \in \mathcal{A}_0, x = \tau(\mathbf{z})\}\,.$$

Without loss of generality, let $\Xi$ be such that $\tau(\mathcal{A}_0) \subset \mathrm{int}(\Xi)$.

*Lemma 1:* The positive orbit of $\mathbf{Z} \times \Xi$ under the flow of (9) is bounded and

$$\omega(\mathbf{Z} \times \Xi) = \mathrm{gr}(\tau)\,.$$

If $\mathcal{A}_0$ is also locally exponentially stable for (3), so is $\mathrm{gr}(\tau)$ for (9).

*Proof:* Let $\mathbf{z}(t, \mathbf{z}_0)$ denote the solution of $\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})$ passing through $\mathbf{z}_0$ at time $t = 0$ and note that, if $\mathbf{z}_0 \in \mathcal{A}_0$, then $\mathbf{z}(t, \mathbf{z}_0) \in \mathcal{A}_0$ for all $t$ (thus, in particular, since $\mathcal{A}_0$ is compact, $|\mathbf{z}(t, \mathbf{z}_0)|$ is bounded by a number which depends only on $\mathcal{A}_0$). Then, since $F$ is a Hurwitz matrix, the map $\tau(\cdot)$ is well defined. A simple calculation shows that

$$\tau(\mathbf{z}(t, \mathbf{z}_0)) = e^{Ft}\tau(\mathbf{z}_0) + \int_0^t e^{F(t-s)} G\mathbf{q}_0(\mathbf{z}(s, \mathbf{z}_0))ds\,.$$

This shows that $\mathrm{gr}(\tau)$ is invariant for (9). From this the claim of the lemma can be easily obtained as in Lemma 5 in [10] (see also Proposition 1 in [4]) considering the change of variable $\xi \mapsto \tilde{\xi} := \xi - \tau(\mathbf{z})$. ■

Motivated by this result, we look at the closed loop system (7) as a system whose zero dynamics have an asymptotically stable compact attractor and we proceed with the design of the parameter $k$ and map $\gamma(\cdot)$. The idea is prove that, if $k$ is large enough, $\zeta$ and $(\mathbf{z}, x)$ asymptotically approach respectively 0 and $\mathrm{gr}(\tau)$. To this end, with an eye to the last of (7), it is crucial to make sure that the map $\gamma(\cdot)$ is such that $q(w, z, \zeta) + \gamma(x + G\zeta) \equiv 0$ on the set $\{\zeta = 0, (\mathbf{z}, x) \in \mathrm{gr}(\tau)\}$. Bearing in mind the expression of $\mathrm{gr}(\tau)$ and the notation introduced above, this is to say that

$$\gamma \circ \tau(\mathbf{z}) = -\mathbf{q}_0(\mathbf{z}) \qquad \text{for all } \mathbf{z} \in \mathcal{A}_0\,. \tag{10}$$

It is easy to realize that the possibility of choosing $\gamma(\cdot)$ in this way is intimately related to the fact that the map $\tau$ satisfies the *partial (with respect to* $\mathbf{q}_0(\cdot)$*) injectivity condition*

$$\tau(\mathbf{z}_1) = \tau(\mathbf{z}_2) \qquad \Rightarrow \qquad \mathbf{q}_0(\mathbf{z}_1) = \mathbf{q}_0(\mathbf{z}_2) \tag{11}$$

for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{A}_0$.

As $\tau$ is dependent on the pair $(F, G)$, the next natural point to be addressed is if there exists a choice of $(F, G)$ yielding the desired property for $\tau(\cdot)$. This is possible as claimed in the next lemma whose proof can be obtained by adapting the arguments of Theorem 3 in [1].

*Lemma 2:* There exist an integer $m > 0$ and a controllable pair $(F, G) \in I\!R^{m\times m} \times I\!R^{m\times 1}$ with $F$ a Hurwitz matrix, such that (11) holds for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{A}_0$.

It turns out that the injectivity property (11) is a sufficient condition for the map $\gamma(\cdot)$, with the desired property, to exist as formalized in the next lemma.

*Lemma 3:* Suppose (11) holds for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{A}_0$. Then there exists a continuous map $\gamma : I\!R^m \to I\!R$ such that (10) holds.

From these Lemma it is possible to conclude the following final result which provides the solution to the problem of output regulation.

*Proposition 1:* Consider system (1) controlled by (5). Let $W, Z, E, \Xi$ be fixed compact sets of initial conditions and suppose Assumptions 0, 1 hold. Let $(F, G)$ be such that the condition indicated in Lemma 2 holds and $\gamma(\cdot)$ such that the condition indicated in Lemma 3 holds. For every $\varepsilon > 0$, there exists a number $k^*$ such that, if $k \geq k^*$, the positive orbit of $W \times Z \times E \times \Xi$ is bounded and there exists $\bar{t}$ such that $|e(t)| \leq \varepsilon$ for all $t \geq \bar{t}$. If, in addition, $\mathcal{A}_0$ is locally exponentially attractive for (3) and $\gamma(\xi)$ is locally Lypschitz at $\xi = 0$, there is a number $k^*$ such that, if $k \geq k^*$, the positive orbit of $W \times Z \times E \times \Xi$ is bounded and $e(t) \to 0$ as $t \to \infty$.

The proof of this Proposition follows a standard paradigm (see, for instance, [22]). System (7) can be viewed as interconnection of two subsystems, one with state $(w, z, x)$ and input $\zeta$, the other with state $\zeta$ and input $(w, z, x)$. As shown in [7], this system is semiglobally practically stabilizable, in the parameter $k$, to the set $\mathrm{gr}(\tau) \times \{0\}$, and this proves the first part of the Proposition. If $\mathcal{A}_0$ is locally exponentially attractive for (3) and $\gamma(\xi)$ is locally Lypschitz at $\xi = 0$, the two subsystems in question are both input-to-state stable (relative to the set $\mathrm{gr}(\tau)$ and to the set $\{0\}$, respectively), with restrictions, with a linear gain function (see again [7] for details). Thus an easy extension (to the case of systems which are input-to-state stable relative to compact attractors) of the small-gain theorem of [15] can be invoked to show that, if $k$ is large enough, the results of the second part of the Proposition holds.

*Remark* The reader familiar with recent developments on nonlinear state observers will find interesting to compare the previous results with those on the design of nonlinear observers design by Kazantzis and Kravaris in [16] and recently developed in [1] (see also [17]). In the framework of [16], system (9) can be identified with the cascade of an "observed" system $\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})$ with output $y_{\mathbf{z}} = \mathbf{q}_0(\mathbf{z})$ driving an "observer" $\dot{x} = Fx - Gy_{\mathbf{z}}$. If the map $\tau(\cdot)$ has a left inverse $\tau_\ell^{-1}(\cdot)$, the observer in question provides a state estimate $\hat{\mathbf{z}} = \tau_\ell^{-1}(x)$. Such a left-inverse, as shown in [1], always exists provided that the dimension of $x$ is sufficiently large and certain *observability conditions* for the system $(\mathbf{f}_0, \mathbf{q}_0)$ hold. In the present context of output regulation, though, left invertibility of $\tau(\cdot)$ (and thus the observability conditions) is not needed. In fact, it is not necessary to recover the full state $\mathbf{z}$ but rather only the output $\mathbf{q}_0(\mathbf{z})$ of the observed system. This motivates the absence of observability conditions for the system $(\mathbf{f}_0, \mathbf{q}_0)$ and, in turn, the absence of *immersion conditions* in the above framework. $\triangleleft$

## IV. A WEAKENED ASSUMPTION

Assumption 1 is rather restrictive, as it requires the controlled system (1) augmented with (2) to be "weakly minimum-phase". Indeed, such an assumption is not strictly speaking necessary as it is well known that problems of output regulation can be solved also for systems which are not "weakly minimum-phase". In this section we show how Assumption 1 can be weakened and replaced by another assumption, which is closer to being "necessary", at least for linear systems.

Consider a system having the same dynamics as system (1), driven by an exogenous input $w$ generated by an exosystem of the form (2), but in which the regulated output $e \in I\!R$ and the measurable output $y \in I\!R^p$ are generic $C^k$ functions of $(w, z, \zeta)$. In other words, consider a system modelled by equations of the form

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z} &=& f(w, z, \zeta) \\ \dot{\zeta} &=& q(w, z, \zeta) + u \\ e &=& h(w, z, \zeta) \\ y &=& k(w, z, \zeta)\,. \end{array} \tag{12}$$

Let, as above, initial conditions vary in compact sets $W, Z, E$.

We retain Assumption 0 above but we replace Assumption 1 by the following one:

*Assumption* 1-*wk*. There exists a bounded subset $B$ of $W \times I\!R^n$, a $C^k$ function $\alpha : I\!R^r \times I\!R^n \to I\!R$ and a $C^k$ map $\Phi : I\!R^p \to I\!R$ such that:

($\mathbf{a}_1$) the set $B$ contains the positive orbit of the set $W \times Z$ under the flow of

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z} &=& f(w, z, \alpha(w, z)) \end{array} \tag{13}$$

and the set $\omega(W \times Z)$ is a differential submanifold (with boundary) of $W \times I\!R^n$. Moreover, there exists a number $d_1 > 0$ such that

$$\begin{array}{l} (w,z) \in W \times I\!R^n \\ |(w,z)|_{\omega(W\times Z)} \leq d_1 \end{array} \quad \Rightarrow \quad (w,z) \in W \times Z\,.$$

**($\mathbf{a}_2$)** $h(w, z, \alpha(w, z)) = 0$ for all $(w, z) \in \omega(W \times Z)$.

**($\mathbf{a}_3$)** $\Phi(k(w, z, \zeta)) = \zeta - \alpha(w, z)$ for all $(w, z, \zeta) \in I\!R^r \times I\!R^n \times I\!R$.

*Remark* Of course, in the case of system (1), where $h(w, z, \zeta) = \zeta$ and $k(w, z, \zeta) = \zeta$, Assumption 1-*wk* is a trivial consequence of Assumption 1 (the function $\alpha(w, z)$ being in this case the trivial function $\alpha(w, z) = 0$). $\triangleleft$

Few words to explain the previous assumption are in order. First of all note that assumption ($\mathbf{a}_1$) can be interpreted as a stabilizability (by state feedback) property of

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z} &=& f(w, z, \zeta) \end{array}$$

in which the variable $\zeta$ is considered as "virtual" control input. Indeed (see also [2], [12]) the conditions presented in ($\mathbf{a}_1$) imply that the set $\omega(Z \times W)$ is asymptotically stable for (13) with a domain of attraction containing $Z \times W$. To take advantage of this property, one may seek a steady state behavior of the overall closed-loop system in which $\zeta$ converges to $\alpha(w, z)$ and $(w, z)$ converges to $\omega(Z \times W)$. Bearing this in mind, also the two additional assumptions ($\mathbf{a}_2$) and ($\mathbf{a}_3$) can be easily justified. In particular, in order to have the regulated error asymptotically converging to zero, assumption ($\mathbf{a}_2$) requires that the map $h(\cdot)$ vanishes at the desired steady state. Finally, assumption ($\mathbf{a}_3$) makes sure that the "mismatch" between $\zeta$ and $\alpha(w, z)$ is available to the controller via the measurable output $y$.

Change now variables as

$$\chi = \zeta - \alpha(w, z)$$

in the dynamics of (12) to obtain

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z} &=& f(w, z, \chi + \alpha(w, z)) \\ \dot{\chi} &=& q(w, z, \chi + \alpha(w, z)) - \dfrac{\partial \alpha}{\partial w} s(w) - \\ && \qquad \dfrac{\partial \alpha}{\partial z} f(w, z, \chi + \alpha(w, z)) + u\,. \end{array} \tag{14}$$

The latter, viewed as a system with control $u$ and regulated output $\chi$ is a system which can be handled by means of the design method presented in the first part of the paper. In particular, as a consequence of Assumption 1-*wk* , system (14) satisfies the two basic Assumptions on which the results of Proposition 1 were based. Appealing to this Proposition it is therefore possible to conclude that there exist an integer $m$, a controllable pair $(F, G)$ with $F$ a Hurwitz matrix and a continuous map $\gamma : I\!R^m \to I\!R$ such that, if (14) is controlled by

$$\begin{array}{rcl} \dot{\xi} &=& F\xi + Gu \\ u &=& \gamma(\xi) + v \\ v &=& -k\chi\,, \end{array} \tag{15}$$

(with initial conditions in a compact set $\Xi$), the following properties hold:

- for every $\varepsilon > 0$, there exists a number $k^*$ such that, if $k \geq k^*$, the positive orbit of $W \times Z \times E \times \Xi$ is bounded and there exists $\bar{t}$ such that $|\chi(t)| \leq \varepsilon$ for all $t \geq \bar{t}$.
- if, in addition, $\omega(W \times Z)$ is locally exponentially attractive for (13) and $\gamma(\xi)$ is locally Lypschitz at $\xi = 0$, there is a number $k^*$ such that, if $k \geq k^*$, the positive orbit of $W \times Z \times E \times \Xi$ is bounded and $\chi(t) \to 0$ as $t \to \infty$.

This result clearly shows that a regulator of the form

$$\begin{array}{rcl} \dot{\xi} &=& F\xi + Gu \\ u &=& \gamma(\xi) + v \\ v &=& -k(\zeta - \alpha(w, z)) = -k\Phi(y)\,, \end{array} \tag{16}$$

is able to solve the problem of output regulation for the plant (12). In fact, looking at the property ($\mathbf{a}_2$), it is immediately concluded that the following holds.

*Proposition 2:* Consider system (12) controlled by (16). Let $W, Z, E, \Xi$ be fixed compact sets of initial conditions and suppose Assumptions 0 and 1-wk hold. There exist an integer $m$, a controllable pair $(F, G)$ with $F$ a Hurwitz matrix and a continuous map $\gamma : I\!R^m \to I\!R$ such that the following properties hold. For every $\varepsilon > 0$, there exists a number $k^*$ such that, if $k \geq k^*$, the positive orbit of $W \times Z \times E \times \Xi$ is bounded and there exists $\bar{t}$ such that $|e(t)| \leq \varepsilon$ for all $t \geq \bar{t}$. If, in addition, $\omega(W \times Z)$ is locally exponentially attractive for (13) and $\gamma(\xi)$ is locally Lypschitz at $\xi = 0$, there is a number $k^*$ such that, if $k \geq k^*$, the positive orbit of $W \times Z \times E \times \Xi$ is bounded and $e(t) \to 0$ as $t \to \infty$.

## V. CONCLUSIONS

In this paper the problem of nonlinear output regulation without immersion assumption has been addressed. It has been shown that, under only a suitable "weakened" minimum-phase assumption, it is possible to design a regulator solving the problem at issue. The regulator design strongly relies upon the ideas proposed in [16] and [1] in the context of nonlinear state observers. It is also stressed that the weakened assumptions make it possible to deal with output regulation of nonminimum-phase systems. Future developments will deal with constructive procedures to design the map $\gamma$ whose existence is guaranteed by Lemma 3.

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