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Tools for Stability of Switching Linear Systems: Gain Automata and Delay Compensation

Rom Langerak
Department of Computer Science
University of Twente
P.O.Box 217
7500 AE Enschede, The Netherlands
langerak@cs.utwente.nl

Jan Willem Polderman
Department of Applied Mathematics
University of Twente
P.O.Box 217
7500 AE Enschede, The Netherlands
J.W.Polderman@math.utwente.nl

Abstract— The topic of this paper is the analysis of stability for a class of switched linear systems, modeled by hybrid automata. In each location of the hybrid automaton the dynamics is assumed to be linear and asymptotically stable; the guards on the transitions are hyperplanes in the state space. For each location an estimate is made of the gain via a Lyapunov function for the dynamics in that location, given a pair of ingoing and outgoing transitions. It is shown how to obtain the best possible estimate by optimizing the Lyapunov function.

The estimated gains are used in defining a so-called gain automaton that forms the basis of an algorithmic criterion for the stability of the hybrid automaton. The associated gain automaton provides a systematic tool to detect potential sources of instability as well as an indication on to how to stabilize the hybrid systems by requiring appropriate delays for specific transitions.

I. INTRODUCTION

The hybrid automata model [2], [7] can be seen as an extension of the timed automata model [3]. A hybrid automaton is an automaton with locations and transitions (or switches) between the transitions, together with continuous dynamics in the locations, usually described by differential equations, and constraints on both locations and transitions. This model is a prominent model for the study of hybrid systems [19].

An important and interesting problem is that of the stability of a hybrid system. Even in the case of switched linear systems with asymptotically stable dynamics in each location, it is possible that the global behavior of the system is unstable (see e.g. [4]). For an overview of results on hybrid stability see [6], [12], [14]. Some results assume arbitrary switching between locations [14], [20]; it is then possible to look for a Lyapunov function common to all locations [1], [12]. The arbitrary switching assumption would be unsuitable in general for hybrid automata, since there the possible switchings are restricted by the guards of the transitions and the invariants of the locations.

Another stability criterion is that of multiple Lyapunov functions [4], [5], [12]. Each location is assumed to have a Lyapunov function. Now all behaviors of the system should satisfy the so-called non-increasing sequence property: if a location is visited again, the value of the Lyapunov function should be less than what it was at the previous time the location was visited. This is a sufficient condition for the stability of a hybrid automaton. In general checking the non-increasing sequence property may be difficult, as checking all possible behaviors of a hybrid system is clearly not an option.

Another approach is the construction of Lyapunov functions that are either piecewise linear [11] or piecewise quadratic [10], [15], [17]. In the latter case the piecewise quadratic function should be continuous on the switching boundaries, which can be checked efficiently by solving a linear matrix inequality. The approach has originally been formulated for piecewise affine systems, where the state space is divided into regions, and to each region corresponds a dynamics. It is not so easy to adapt the approach to the more general model of hybrid automata.

In this paper we present an alternative approach that still makes use of the existence of a Lyapunov function in each location, but in addition incorporates the underlying automaton structure of the system model. Moreover, we demonstrate how a potentially unstable system may be stabilized by introducing appropriate transition delays.

The remainder of the paper is organized as follows. Section II gives basic definitions about hybrid automata and stability, and defines the class of hybrid automata used in this paper. Section III shows how to calculate an estimate of the gain using Lyapunov functions. It is shown how an optimal Lyapunov function can be chosen. Section IV shows how to use gain estimates in constructing an automaton that is used in an algorithm that detects non-contractive cycles in hybrid automata. The absence of such non-contractive cycles is a sufficient condition for the stability of the hybrid automaton. The occurrence, however, of non-contractive cycles does not imply instability of the hybrid system, rather it indicates that no simple guarantees for stability can be given. Due to the complex interaction between discrete switching between locations and continuous dynamics within location, a hybrid system having non-contractive cycles may be considered unsafe if stability is crucial. Section V describes how a cycle that is detected to be non-contractive may be stabilized by sufficiently delaying transitions belonging to that cycle. Then, in Section VI we introduce the notion of gain automaton. This is an automaton that is associated to the

hybrid system under consideration and is used to detect, in a systematic way, non-contractive cycles. We present two algorithms: a passive detection algorithm, and an algorithm that not only detects non-contractive cycles, but also has the built-in option to determine how specific transitions in the cycle should be delayed such that the delayed cycle becomes contractive. Finally, in Section VII, we give conclusions and directions for future research. For basic notions of systems theory used in this paper we refer to [18].

II. HYBRID AUTOMATA AND STABILITY

The hybrid automaton model [2], [7] extends the classical notion of automaton by incorporating continuous dynamics in the locations, together with constraints at both locations and transitions.

Definition II.1. A *hybrid automaton* is a tuple $H = (X, L, Init, Inv, f, E, Guard, Assign, \Sigma)$ where:
- $X \subseteq \mathbb{R}^n$ is the *continuous state space* ranged over by the state vector x.
- L is a finite set of *locations*.
- $Init \subseteq L \times \mathbb{R}^n$ is a set of *initial location state pairs*.
- $Inv : L \to 2^X$ assigns to each location ℓ an *invariant* to be satisfied by state x while in location ℓ.
- $f : L \to (X \to \mathbb{R}^n)$ assigns to each location ℓ a *continuous vector field* f_ℓ such that the state x should satisfy $\frac{d}{dt}x = f_\ell(x)$.
- $E \subseteq L \times L$ is the set of *transitions*, also called *switches*.
- $Guard : E \to 2^X$ assigns to each transition a *guard* that has to be satisfied by state x if the transition is taken.
- $Assign : E \to (X \to X)$ assigns to each transition an *assignment* that may alter state x when the transition is taken.
- Σ a set of *transition labels*. We assume a labeling function $lab : E \to \Sigma$ and refer to transitions by their labels (assuming uniqueness).

In this paper we make a few additional assumptions:
- We assume that $Init = L' \times \mathbb{R}^n$ for a set $L' \subseteq L$ of initial locations, so that for a given initial location we can start with any state (which is technically convenient when studying stability).
- We assume that there are no invariants, i.e., Inv maps each location to the trivial condition $true$. This means that transitions are never forced, and it is possible to remain in a location forever.
- we assume that the dynamics in each location is linear, so $\frac{d}{dt}x = f_\ell(x) = A_l x$, $A_\ell \in \mathbb{R}^{n \times n}$.
- We assume that for each transition e the guard is a hyperplane defined by an equation of the form $v_e^{\mathrm{T}} x = 0$ for some $v_e \in \mathbb{R}^n$.
- We assume that the state is left unchanged by transitions (also called *continuous switching*), so for each transition e, $Assign_e(x) = x$.

We call a hybrid automaton that satisfies these assumptions a *Linear Continuous Hyperplane* (LCH) hybrid automaton.

Example II.2. Consider the hybrid automaton consisting of four locations, $\ell_1, \ldots, \ell_4$. The dynamics in location ℓ_i is given by $\frac{d}{dt}x = A_{\ell_i}x$, $A_{\ell_i} \in \mathbb{R}^{2 \times 2}$, $i = 1, \ldots 4$. The following events can occur: $E = \{(\ell_1, \ell_2), (\ell_1, \ell_4), (\ell_2, \ell_1), (\ell_2, \ell_3), (\ell_3, \ell_4), (\ell_4, \ell_2)\}$, to which correspond labels a to f respectively. To each event there corresponds a switching line L_{ij}. For instance if the automaton is in location ℓ_2 there are two possible transitions: to ℓ_1 and to ℓ_3. The transition to ℓ_1 is enabled if and only if $x \in L_{21}$, whereas the transition to ℓ_3 is possible when $x \in L_{23}$.

Definition II.3. A *hybrid trace* of an LCH hybrid automaton is a finite or infinite sequence of the form $\sigma = x_1 e_1 x_2 e_2 \ldots x_{m-1} e_{m-1} x_m$, with an associated monotonically increasing timing sequence $\tau_0 \tau_1 \ldots \tau_m$ (with $\tau_0 = 0$, $\tau_i \in \mathbb{R} \cup \{\infty\}$), such that
- each e_i is a transition from location ℓ_i to location ℓ_{i+1}
- each x_i is a mapping from $[\tau_{i-1}, \tau_i]$ to $\mathbb{R}^n$ satisfying $\frac{d}{dt}x_i = A_{\ell_i} x_i$
- initial and switching constraints and assignments are respected, so $(\ell_1, x_1(0)) \in Init$, and for all $1 \leq i \leq m-1$: $v_{e_i}^{\mathrm{T}} x_i(\tau_i) = 0$ and $x_i(\tau_i) = x_{i+1}(\tau_i)$.

Remark II.4. To avoid Zeno behavior we impose that all transitions are delayed by a fixed minimal dwell time. This implies in particular that for infinite traces there holds $\lim_{i \to \infty} \tau_i = \infty$.

Definition II.5. An LCH hybrid automaton is *stable* iff $\forall \epsilon > 0 \; \exists \delta > 0 : \|x^0\| < \delta \Rightarrow$ for all hybrid traces $x_1 e_1 x_2 e_2 \ldots$ with $x_1(0) = x^0$ and $\forall i \forall t \in [\tau_{i-1}, \tau_i] : \|x_i(t)\| < \epsilon$. An automaton that is not stable is called *unstable*.

It is well known that even if for each location ℓ the dynamics is asymptotically stable, so the matrix A_ℓ is Hurwitz (i.e., all eigenvalues have negative real part, see [18]), still the hybrid automaton can be unstable (see e.g. [4] for a simple example). We say that a hybrid automaton has *stable locations* (with some abuse of terminology) if for each location ℓ the matrix A_ℓ is Hurwitz.

Now, our problem setting consists of two parts. Firstly, find sufficient conditions for the stability of an LCH hybrid automaton with stable locations. Secondly, if the sufficient conditions are not met, then provide systematic and quantified guidelines as to which transitions should be delayed to ensure stability.

III. ESTIMATE OF GAINS VIA LYAPUNOV FUNCTIONS

Suppose a location ℓ is entered via a transition a with a state vector x_a and is left via a transition b with a state vector x_b. An indication as to how the location contributes to the stability or instability is the ratio of the norm of the outbound state and the inbound state. A ratio below one is in favor of stability whereas a ratio above one points at instability. Of course, since the ratio depends on the actual trace and state trajectory it does not provide a feasible stability indicator. Therefore we propose to use an
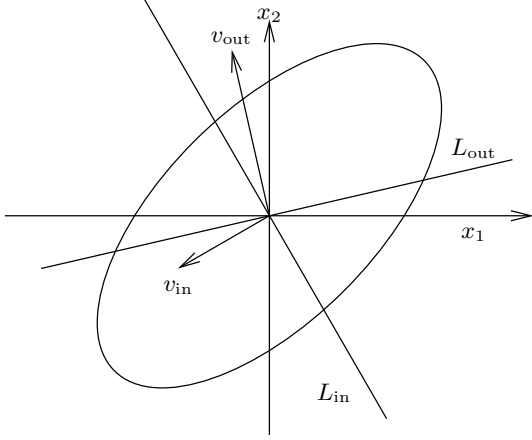
Fig. 1. Switching lines in a location

upperbound that only depends on the particular location and its corresponding inbound and outbound transitions.

Definition III.1. Consider an LCH hybrid automaton H with Hurwitz locations. With each location ℓ we associate a symmetric positive definite matrix P_ℓ such that $A_\ell^{\mathrm{T}} P_\ell + P_\ell A_\ell \leq 0$. Let e_{in} represent a transition to ℓ and e_{out} a transition from ℓ and let L_{in}, given by $v_{\mathrm{in}}^{\mathrm{T}} x = 0$, and L_{out}, given by $v_{\mathrm{out}}^{\mathrm{T}} x = 0$, denote the corresponding switching hyperplanes. Define ellipsoids E_{in} and E_{out} as $E_{\mathrm{in}} = \{x \in L_{\mathrm{in}} \mid x^{\mathrm{T}} P_\ell x = 1\}$ and $E_{\mathrm{out}} = \{x \in L_{\mathrm{out}} \mid x^{\mathrm{T}} P_\ell x = 1\}$. The corresponding gain $\alpha_{\mathrm{in/out}}$ is defined as

$$\alpha_{\mathrm{in/out}} = \max_{x_{\mathrm{i}} \in E_{\mathrm{in}}, x_{\mathrm{o}} \in E_{\mathrm{out}}} \frac{x_{\mathrm{o}}^{\mathrm{T}} x_{\mathrm{o}}}{x_{\mathrm{i}}^{\mathrm{T}} x_{\mathrm{i}}} \qquad (1)$$

Obviously, since $V(x) = x^{\mathrm{T}} P_\ell x$ is a Lyapunov function for $\frac{d}{dt} x = A_\ell x$ we have that any trajectory that enters the location through L_{in} and leaves it through L_{out} has the property that the ratio of the norms of outbound and inbound states is upperbounded by $\sqrt{\alpha_{\mathrm{in/out}}}$.

Two questions arise. Firstly, given P_ℓ how can we calculate $\alpha_{\mathrm{in/out}}$? Secondly, it appears that the choice of P_ℓ affects $\alpha_{\mathrm{in/out}}$. How to choose P_ℓ such that $\alpha_{\mathrm{in/out}}$ is minimal? The latter question is treated in Section III-B.

A. Calculation of the gains

Let us explain how the calculation for given P_ℓ works. For ease of exposition we treat the two dimensional case first, see Figure 1.

The switching lines are given by $v_{\mathrm{in}}^{\mathrm{T}} x = 0$ and $v_{\mathrm{out}}^{\mathrm{T}} x = 0$ respectively. Let $\tilde{v}_{\mathrm{in}}$ and $\tilde{v}_{\mathrm{out}}$ be orthogonal to v_{in} and v_{out} respectively. Then it is not difficult to verify that

$$\alpha_{\mathrm{in/out}} = \frac{\tilde{v}_{\mathrm{out}}^{\mathrm{T}} \tilde{v}_{\mathrm{out}}}{\tilde{v}_{\mathrm{in}}^{\mathrm{T}} \tilde{v}_{\mathrm{in}}} \frac{\tilde{v}_{\mathrm{in}}^{\mathrm{T}} P_\ell \tilde{v}_{\mathrm{in}}}{\tilde{v}_{\mathrm{out}}^{\mathrm{T}} P_\ell \tilde{v}_{\mathrm{out}}}. \qquad (2)$$

If we choose $\tilde{v}_{\mathrm{in}}$ and $\tilde{v}_{\mathrm{out}}$ both on the same level curve, then (2) reduces to $\frac{\tilde{v}_{\mathrm{out}}^{\mathrm{T}} \tilde{v}_{\mathrm{out}}}{\tilde{v}_{\mathrm{in}}^{\mathrm{T}} \tilde{v}_{\mathrm{in}}}$.

For dimensions $n > 2$ the situation is a bit more complicated since the maximization in (2) comes into play.

First notice that

$$\alpha_{\mathrm{in/out}} = \frac{\max_{x_{\mathrm{o}} \in E_{\mathrm{out}}} x_{\mathrm{o}}^{\mathrm{T}} x_{\mathrm{o}}}{\min_{x_{\mathrm{i}} \in E_{\mathrm{in}}} x_{\mathrm{i}}^{\mathrm{T}} x_{\mathrm{i}}}. \qquad (3)$$

Next, e.g., the numerator of (3) can easily be calculated as follows. First assume that v_{out} has norm one (otherwise normalize), then determine an orthogonal matrix V_{out} such that the first column of V_{out} is v_{out}. Define $P_{\mathrm{out}} = V_{\mathrm{out}}^{\mathrm{T}} P_\ell V_{\mathrm{out}}$ and $\tilde{E}_{\mathrm{out}} = \{z \in \mathbb{R}^n \mid z_1 = 0, z^{\mathrm{T}} P_{\mathrm{out}} z = 1\}$. Then

$$\max_{x \in E_{\mathrm{out}}} x^{\mathrm{T}} x = \max_{z \in \tilde{E}_{\mathrm{out}}} z^{\mathrm{T}} z = \frac{1}{\lambda_{\min}(\tilde{P}_{\mathrm{out}})}, \qquad (4)$$

where $\tilde{P}_{\mathrm{out}}$ is obtained from P_{out} by deleting the first row and and the first column. Furthermore $\lambda_{\min}(P)$ denotes the smallest eigenvalue of matrix P. In a similar way the denominator of (3) is obtained, resulting in

$$\alpha_{\mathrm{in/out}} = \frac{\lambda_{\max}(\tilde{P}_{\mathrm{in}})}{\lambda_{\min}(\tilde{P}_{\mathrm{out}})}. \qquad (5)$$

Remark III.2. For $n \geq 3$ two hyperplanes always intersect and hence also the sets E_{out} and E_{in} intersect. As a consequence the gain $\alpha_{\mathrm{in/out}} \geq 1$.

B. Optimizing the choice of Lyapunov function

The gains that provide a stability indicator obviously depend on the Lyapunov functions in each location. Loosely, the fit of the level curves with respect to the trajectories determines the amount of conservatism. The better the fit, the less conservative the gain. Since Lyapunov functions are not unique, this suggests that we might be able to choose in each location a Lyapunov function that is optimal with respect to the switching planes. In this section we explain how this can indeed be done. We confine ourselves to quadratic Lyapunov functions. We show that for a given stable matrix A the set of quadratic Lyapunov functions is convex and compact. Furthermore the stability gain corresponding to A and given switching hyperplanes depends continuously on the Lyapunov function so that at least the optimum exists. We illustrate the effectiveness of these results by means of a two dimensional example.

Let $A \in \mathbb{R}^{n \times n}$ be a stable matrix. We are interested in the set of quadratic Lyapunov functions, or, more precisely, the set of level curves corresponding to quadratic Lyapunov functions. To enable the comparison of different Lyapunov functions we choose a nonzero $x_0 \in \mathbb{R}^n$ and define the set

$$\Omega_{x_0} = \{P \in \mathbb{R}^{n \times n} \mid A^{\mathrm{T}} P + PA \leq 0, x_0^{\mathrm{T}} P x_0 = 1\}.$$

In fact, Ω_{x_0} is a parametrization of the level curves corresponding to quadratic Lyapunov functions and level unity.

Lemma III.3. *Let $A \in \mathbb{R}^{n \times n}$ and let $x_0 \in \mathbb{R}$ be a nonzero vector that does not belong to an A-invariant subspace of dimension at most $n - 1$. Let U be an open neighborhood of $0 \in \mathbb{R}$. Then $\operatorname{span}_{t \in U}(\exp(At)) x_0 = \mathbb{R}^n$.*

Theorem III.4. *Let $A \in \mathbb{R}^{n \times n}$ be a stable matrix and let $x_0 \in \mathbb{R}$ be a nonzero vector. Define $\Omega = \{P \in \mathbb{R}^{n \times n} \mid A^T P + PA \leq 0$ and $x_0^T P x_0 = 1\}$.*

1. If x_0 does not belong to a proper A-invariant subspace then Ω is compact.

2. Every $P \in \Omega$ is positive semi-definite.

3. Ω is convex.

Example III.5. Let the dynamics in a given location be given by $\frac{d}{dt}x = Ax$ with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

The switching lines are given by

$$L_{\text{in}} = \lambda \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{a_{\text{in}}} \quad L_{\text{out}} = \lambda \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{a_{\text{out}}} \tag{6}$$

As explained in Section III-A, for a given Lyapunov function $V(x) = x^{\text{T}} P x$ the gain is defined as

$$\alpha_P = \frac{a_{\text{out}}^{\text{T}} a_{\text{out}}}{a_{\text{in}}^{\text{T}} a_{\text{in}}} \frac{a_{\text{in}}^{\text{T}} P a_{\text{in}}}{a_{\text{out}}^{\text{T}} P a_{\text{out}}} = \frac{p_{22}}{p_{11}} \tag{7}$$

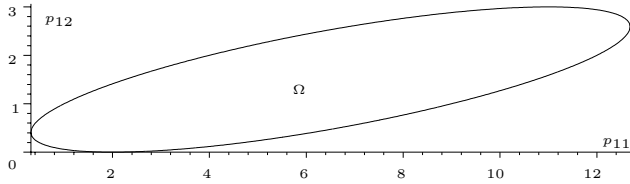The set $\Omega_{a_{\text{in}}}$ is depicted in Figure 2. To find the optimal

Fig. 2. The set $\Omega_{a_{\text{in}}}$ of all Lyapunov functions.

Lyapunov function we want to minimize α_P over $\Omega_{a_{\text{in}}}$. For this example this amounts to the minimization of $\frac{1}{p_{11}}$ or, equivalently, the maximization of p_{11}. Theorem III.4 guarantees the optimum exists. Using Maple we found two extreme Lyapunov functions. One that minimizes α_P and one that maximizes α_P:

$$P_{\text{min}} = \begin{bmatrix} 12.70 & 2.59 \\ 2.59 & 1 \end{bmatrix} \quad P_{\text{max}} = \begin{bmatrix} .32 & .41 \\ .41 & 1 \end{bmatrix} \tag{8}$$

The corresponding minimum and maximum values of the gains are

$$\alpha_{\text{min}} \approx 0.38 \quad \alpha_{\text{max}} \approx 2.61 \tag{9}$$

In Figure 3 the level curves of the two Lyapunov functions are drawn together with the phase portrait of $\frac{d}{dt}x = Ax$. This clearly shows the benefit of using the non-uniqueness of Lyapunov function. The one that minimizes α is obviously dramatically less conservative than the one that maximizes α.

Remark III.6. In higher dimensions the minimization of the gain may be cumbersome to perform. Indeed, the expression in (5) is a non-convex function of P. Notice, however, that $\lambda_{\text{max}}(P)$ is convex and $\lambda_{\text{min}}(P)$ is concave. A suboptimal solution is therefore obtained by disregarding either the numerator or the denominator in (5).
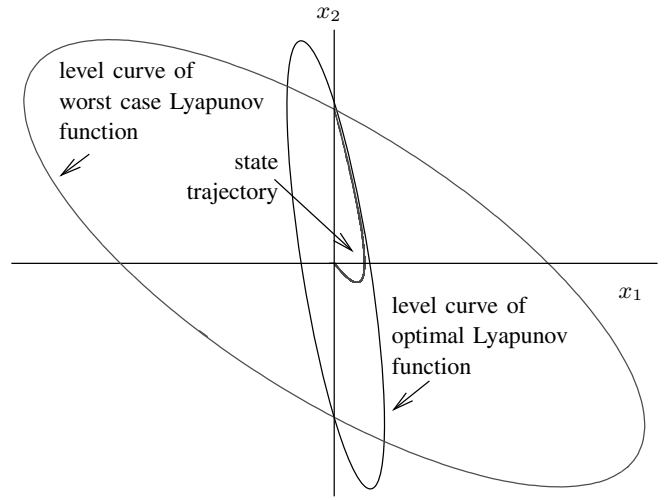
Fig. 3. Level curves of the extreme Lyapunov functions

IV. Non-contractive cycles

Suppose we have a hybrid automaton with Hurwitz locations. If for each location that can be visited more than once the estimated gain is ≤ 1, then it can be seen that the hybrid automaton is stable: since the number of locations in a trace that are visited only once (seen as a function of a trace) is bounded, there is a bound to the gains corresponding to the traces. However, such a condition is unnecessarily restrictive as it does not take into account situations where a higher gain in one location is compensated by a lower gain in another location. So we need a more sophisticated condition.

Definition IV.1. Let H be a hybrid automaton, then a *contractive cycle* of H is a sequence of transitions $C = e_1 e_2 \ldots e_m$ such that each e_i is a transition from ℓ_i to ℓ_{i+1}, with $\ell_1 = \ell_{m+1}$, and $\gamma_C = \alpha_{e_1 e_2} \cdot \alpha_{e_2 e_3} \cdot \ldots \cdot \alpha_{e_m e_1} \leq 1$. The scalar γ_C is called the cycle gain.

Theorem IV.2. *Let H be an LCH hybrid automaton with Hurwitz locations. Then: H is unstable $\Rightarrow$ H has a non-contractive cycle.*

V. Stabilizing non-contractive cycles by delaying transitions

If a cycle is non-contractive, that is, the product of the gains exceeds one, then the cycle may still be stabilized by appropriately delaying transitions. Indeed, since all locations have asymptotically stable dynamics, sufficiently delaying transitions reduce all separate gains to a value not exceeding unity. This, however, is not necessary. It suffices to delay one or more transitions in the cycle such that the cycle becomes contractive, that is, such that the product of all gains does not exceed one. How to exactly achieve this may be quantified through the following lemma.

Lemma V.1. *Let A be a Hurwitz matrix, $Q = Q^T$ is a positive definite matrix and let P satisfy the associated*

Lyapunov equation $A^T P + PA = -Q$. Furthermore let $x(t)$ be the solution of $\frac{d}{dt}x(t) = Ax(t)$ with $x(0) = x_0$. Then

$$x(t)^T P x(t) \leq x_0^T P x_0 e^{-\lambda t}, \tag{10}$$

where $\lambda = \sigma_{\min}(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}})$.

Next we describe how to use Lemma V.1 to stabilize a non-contractive cycle. Let C be a non-contractive cycle. This means that the product of the estimated gains is larger than one. Let α be an estimated gain corresponding to location ℓ belong to the cycle C. Denote the dynamics in ℓ by $\frac{d}{dt}x = A_\ell x$ and let $A^T P + PA = -Q$ be the corresponding Lyapunov equation on the basis of which α has been calculated. This may be the optimized P obtained as described in Section III-B provided it is positive definite. However if that P is positive semi-definite, then the gain estimate has to be recalculated on the basis of positive definite P followed by the calculation of the delay.

As noted, the interpretation of the transition gain α is that it reflects an upper bound on the state gain corresponding to the event 'enter ℓ followed by 'leave ℓ. The gain α is based on level sets of the Lyapunov function $x^T P x$ only, time evolution is not taken into account. In that sense α may be somewhat conservative. On the other hand, it would be very difficult to even estimate how long we stay in ℓ. However, by imposing a minimum delay it becomes feasible to estimate the reduction in gain estimate.

More precisely: Let H be a hybrid automaton and let C be cycle of H, i.e., a sequence of transitions $e_1 e_2 \ldots e_m$ such that each e_i is a transition from ℓ_i to ℓ_i, with $\ell_1 = \ell_{m+1}$, and let $\alpha_{e_1 e_2} \cdot \alpha_{e_2 e_3} \cdot \ldots \cdot \alpha_{e_m e_1}$ be the corresponding gain estimates. Denote by $\delta_{e_{i-1}, e_i} \geq 0$ the delay corresponding to the transition pair (e_{i-1}, e_i). The interpretation of $\delta_{e_{i-1} e_i}$ is that after transition e_{i-1} a delay of at least $\delta_{e_{i-1} e_i}$ has to be respected before transition e_i may be taken. In this way $\delta_{e_{i-1} e_i}$ acts as a kind of guard. During the delay $\delta_{e_{i-1} e_i}$ location ℓ_i is active. The delayed cycle gain is now defined as:

$$\gamma_C(\delta) = \prod_{i=1}^{m+1} \alpha_i e^{-\lambda_i \delta_{e_{i-1} e_i}}. \tag{11}$$

It now suffices to choose the delays $\delta_{e_{i-1} e_i}$s such that the cycle gain becomes smaller than or equal to one.

The above is, of course, closely related to the concept of dwell-time as introduced studied in [8], [16]. However, there dwell times are defined in terms of the dynamics in locations as isolated entities. In our context delays are studied in relation to all transitions in a given cycle.

VI. GAIN AUTOMATA AND DETECTION OF NON-CONTRACTIVE CYCLES

Theorem IV.2 together with the transition delays provides us with a sufficient condition for stability, namely the absence of non-contractive cycles. In order to check for non-contractive cycles we first transform a hybrid automaton into what we call *gain automaton*.
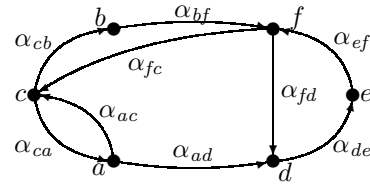

Fig. 4.   Example of a gain automaton

Definition VI.1. A *gain automaton* is a tuple $GA = (S, S^0, G)$ where

- S is the set of nodes,
- S^0 is the set of initial nodes,
- $G \subseteq S \times \mathbb{R}^+ \times S$ is the set of edges labeled with gains.

Definition VI.2. Let H be a hybrid automaton, then the gain automaton for H is defined by $GA(H) = (S_H, S_H^0, G_H)$ where

- The nodes of the gain automaton are the transitions of H, so $S_H = E$.
- The initial nodes S_H^0 are the transitions from an initial location of H.
- For each pair of transitions e, e' in H such that $\xrightarrow{e} l \xrightarrow{e'}$ there is an edge $e \xrightarrow{\alpha_{ee'}} e'$ in G_H.

Example VI.3. Let H be the hybrid automata of Example II.2, then the gain automaton $GA(H)$ is given in Figure 4.

We define an algorithm on the gain automaton of a hybrid automaton for the detection of non-contractive cycles. This algorithm is inspired by the well-known algorithm for transforming an automaton into an equivalent regular expression (see e.g. [9], [13]). It works by successively deleting nodes of the gain automaton, while transforming the edges. The basic steps of the algorithm are:

- **Node elimination:** a node is eliminated. Each possible pair of an incoming and outgoing edge of this node leads to a new edge, labeled with the product of the gains, as illustrated in Figure 5(a).
- **Double edge elimination:** if two edges have the same initial and final node they are transformed into a single edge, labeled with the maximum of the gains, as illustrated in Figure 5(c).
- **Loop edge analysis:** it is possible that deleting a node creates a loop edge, as illustrated in Figure 5(b). If the gain of such a loop edge is > 1 (we call this a *non-contractive* loop edge) the algorithm is terminated. Otherwise, the loop edge is removed.

Algorithm VI.4. *Input: a gain automaton GA.*
   check all loop edges;
if *a non-contractive loop edge is found*
      then *terminate;*
remove all loop edges;
repeat
        eliminate a state;
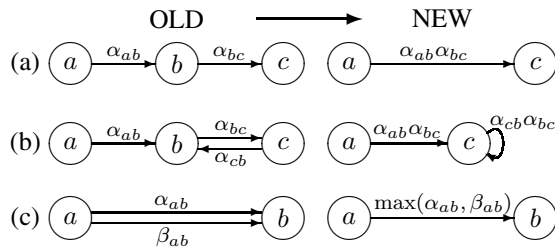   *eliminate all resulting double edges;*

Fig. 5. Basic steps of the algorithm

analyze all resulting loop edges;
if *a non-contractive loop edge is found*
    then *terminate;*
remove all loop edges
until *there is only one state*

Theorem VI.5. *Let H be an LCH hybrid automaton with Hurwitz locations. Then: Algorithm VI.4 detects a non-contractive loop edge in $GA(H) \Leftrightarrow H$ contains a non-contractive cycle.*

Of course, detecting a non-contractive cycle may not be the end of the story. In case transition delay is feasible non-contractive cycles may be stabilized by delay as detailed in Section V. The algorithm VI.4 then becomes:

Algorithm VI.6. *Input: a gain automaton GA.*
  check all loop edges;
if *a non-contractive loop edge is found*
    then *terminate;*
remove all loop edges;
repeat
    eliminate a state;
    eliminate all resulting double edges;
    analyze all resulting loop edges;
    if *a non-contractive loop edge is found*
        then *stabilize by delay;*
    remove all loop edges
until *there is only one state*

Algorithm VI.6 provides a systematic means to detect potential unstable cycles and subsequently stabilizing these.

The number of nodes in $GA(H)$ is quadratic in the number of nodes of H, and the complexity of Algorithm VI.4 and VI.6 is linear in the number of nodes of GA, so the complexity of Algorithm VI.4 and VI.6 is quadratic in the number of nodes of H. This means we have a computationally efficient way of checking the sufficiency condition for stability, viz. the absence of non-contractive cycles.

VII. Conclusions

We have derived a sufficient condition for the stability of a hybrid automaton, viz. the absence of non-contractive cycles, together with an algorithm for efficiently checking this condition. In addition, we have proposed a systematic method through which transition delays may be introduced

to stabilize potentially unstable cycles. In principle the incorporation of the delays could be part of the criterion according to which the optimal Lyapunov function is constructed. At this moment, however, this appears to yield an unfeasible optimization problem.

We have made use of both systems theoretic concepts (in calculating the estimated gains) and computer science concepts (in checking the cycles in the gain automaton), thereby doing justice to both the continuous and the discrete aspects of hybrid systems.

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