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Structural Stability of Boundary Equilibria in a class of hybrid systems: Analysis and Use for Control System Design

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Abstract— **This paper is concerned with the structural stability of boundary equilibria in a class of hybrid systems, that of piecewise linear continuous systems. Specifically, we study the structural stability under parameter variations of equilibria lying on discontinuity boundaries in phase space dividing regions where the system under investigation is smooth. We show that it is possible to give a set of conditions to account for the possible dynamical scenarios that can be observed. We present a novel scenario where under perturbations, the equilibrium of interest does not persist and a family of stable limit cycles is generated with amplitudes increasing linearly under parameter perturbations. We propose a novel switching control strategy based on the use of the results presented in the paper.**

I. Introduction

In recent years, much research effort in applied science and engineering has focussed on piecewise smooth dynamical systems, a class of hybrid systems that can be described by sets of ordinary differential equations (ODEs) of the form:

$$\dot{x} = f(x, \mu); \qquad \text{(I.1)}$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a piecewise smooth function, $\mu \in \mathbb{R}$ is a parameter and $x \in \mathbb{R}^n$.

The phase space of a general piecewise-smooth (PWS) system, such as (I.1), can be split into countably many regions G_i, $i = 1, 2, \dots, N$ where f has a different smooth functional form, i.e. $f = f_i$ if $x \in G_i$ for $i = 1, ..., N$. Such regions are closed, connected subsets of phase space. Smoothness is lost as trajectories cross the boundaries, Σ_{ij}, between adjacent regions. We will refer to such boundaries as *nonsmoothness sets* or *switching manifolds*. Across these manifolds, a PWS system can be characterised by a different relative degree, r, or discontinuity type. Namely, we can have systems with discontinuos states x; discontinuous vector fields (i.e. $f_i \neq f_2$); discontinuos Jacobian, i.e. $f_i = f_j$ but $\frac{\partial f_i}{\partial x} \neq \frac{\partial f_j}{\partial x}$; higher order discontinuities, i.e. $f_i = f_j$, $\frac{\partial f_i}{\partial x} = \frac{\partial f_j}{\partial x}$, but $\frac{\partial^n f_i}{\partial x^n} \neq \frac{\partial^n f_j}{\partial x^n}$. Examples include, to mention just a few, vibro-impacting machines in mechanical engineering and systems with friction [1], switching circuits in power electronics [2], [3] physiological models [4], internal combustion engines [5], walking machines [6]; more generally, all those systems which are intrinsically non-smooth on macroscopic time-scales.

In control engineering, the benefits of switching control actions have often been exploited in applications. Variable structure control [7], relay feedback [8], pulse width modulation [9] and hybrid control [10] are all examples of theories which exploit control actions, giving rise to closed-loop systems which are nonsmooth. Despite their widespread use, there does not yet seem to be an effective systematic theory of such systems. While the last decades have witnessed an explosive development in the theory of smooth dynamical systems, many fundamental problems remain open for hybrid and switched ones. These include, for example, well-posedness, stability and numerical analysis (see for example [1]).

An important open problem from a control viewpoint, often neglected in the current literature, is the lack of a consistent theory of structural stability and robustness of hybrid and switched dynamical systems. In fact, as the system parameters are varied, novel transitions have been observed in a number of real-world nonsmooth systems leading to the loss of their structural stability [3], [11]–[17]. These phenomena cannot be explained in terms of well-known standard bifurcations for smooth systems. For instance, pulse-width-modulated feedback systems were shown to exhibit sudden transitions from periodic oscillations to irregular chaotic motion which were left unexplained for a long time (see [11] or [2] for a chronological literature review).

The aim of this paper is to present a set of conditions to analyse the structural stability of boundary equilibria, i.e. equilibria of piecewise linear systems which are located on one of the phase space discontinuity boundary. In this paper, we will consider the case of piecewise linear continuous systems which are described by sets of smooth equations in each of the phase space regions and are such that the state trajectories are continuous across the switching manifolds. These systems are used in many domains of applications. An example is the origin of a switching control system designed to guarantee its stability by switching whenever a manifold passing through zero is crossed. We will discuss the most relevant novel scenarios leading to the loss of structural stability of such equilibria in piecewise smooth control systems. After introducing appropriate notation and terminology, conditions will be derived to predict the asymptotic behaviour of the system under parameter variations. The topological implications of these events will be discussed and related with the observed dynamical scenarios in the

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system of interest. We will show that the conditions derived to analyse the structural stability of the equilibria of interest and characterise the ensuing dynamics, can be used to design a simple but effective switching control strategy to generate limit cycles of desired amplitude.

II. Structural Stability of Boundary-equilibria

A. Piecewise-smooth Continuous Systems

We focus on piecewise linear continuous systems. We restrict our attention to a region of phase space, say D, where the system under investigation can be described as follows in terms of a local set of coordinates. Namely, we have

$$\dot{x} = \begin{cases} F_1(x,\mu), & \text{if } H(x,\mu) < 0 \\ F_2(x,\mu), & \text{if } H(x,\mu) > 0 \end{cases} \tag{II.1}$$

where $x \in \mathbb{R}^n$, $F_1, F_2 : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$ are supposed to be sufficiently smooth and $H : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ is a sufficiently smooth scalar function of the system states. Because of the continuity assumption we must have

$$F_2(x,\mu) = F_1(x,\mu) + G(x,\mu)H(x,\mu), \tag{II.2}$$

so that when $H(x,\mu) = 0$ then $F_1 = F_2$ as required.

According to (II.1), H defines the switching manifold

$$\Sigma := \{x \in \mathbb{R}^n : H(x,\mu) = 0\}$$

Locally, Σ divides D in the two regions G_1 and G_2 where the system is smooth and defined by the vector fields F_1 and F_2 respectively; namely:

$$G_1 = \{x \in D : H(x,\mu) < 0\},$$
$$G_2 = \{x \in D : H(x,\mu) > 0\}.$$

We assume that both the vector fields F_1 and F_2 are defined over the entire local region of phase space under consideration, i.e., on both sides of Σ.

We can identify different types of equilibria of system (II.1). Namely, it is possible to give the following definitions.

Definition II.1 *We term a point $x \in D$ as an* admissible equilibrium *of (II.1) if, for a given μ, x is such that either*

$$F_1(x,\mu) = 0 \textit{ and } H(x,\mu) < 0$$

or

$$F_2(x,\mu) = 0 \textit{ and } H(x,\mu) > 0$$

Alternatively, we say that a point $y \in D$ is a virtual equilibrium *of (II.1) if either*

$$F_1(y,\mu) = 0 \textit{ but } H(y,\mu) > 0$$

or

$$F_2(y,\mu) = 0 \textit{ but } H(y,\mu) < 0$$

For some value of the system parameters, it is possible for an equilibrium to lie on the discontinuity boundary.

Definition II.2 *We say that a point $z \in D$ is a* boundary equilibrium *of (II.1) if, for a given μ,*

$$F_1(z,\mu) = F_2(z,\mu) = 0 \textit{ and } H(z,\mu) = 0$$

Note that under parameter variations the system might exhibit a boundary equilibrium for some value of its parameters μ. Without loss of generality let's assume that $x = 0$ is a boundary equilibrium for $\mu = 0$. We shall seek to unfold the bifurcation scenarios that can occur when μ is perturbed away from the origin. Namely, we shall seek to identify the possible branches of asymptotic solutions exhibited by the system of interest under parameter variations.

Definition II.3 *An admissible equilibrium $x^* = x^*(\mu)$, which we assume depends smoothly on μ, is said to undergo a* boundary equilibrium transition *at $\mu = \mu^*$ if, for $i = 1$ or $i = 2$,*

- $F_1(x^*,\mu^*) = 0$,
- $H(x^*,\mu^*) = 0$,
- $F_{1x}(x^*,\mu^*)$ *is invertible (or equivalently det*$(F_{1x}) \neq 0$*).*

While the first two conditions state that x^* is a boundary equilibrium when $\mu = \mu^*$, the third condition ensures that the branch of admissible equilibria undergoing the bifurcation is isolated.

B. Persistence or Annihilation

The existence of different types of bifurcation scenarios following this type of nonsmooth transitions was discussed in [18], [19] and illustrated through some one-dimensional and two-dimensional examples. It was shown, for example, that nonsmooth transitions of equilibria can be associated, in the simplest cases, to the persistence of the bifurcating equilibrium or its annihilation through a saddle-node like scenario. Namely, it was conjectured that a boundary equilibrium bifurcation can lead to the following simplest scenarios:

1) Persistence: under any parameter variation, the boundary equilibrium is turned into an admissible equilibrium lying either in region G_1 or region G_2. In terms of collision of equilibria with the boundary, this scenario describes how the only admissible equilibrium point x^- for $\mu < 0$ hits the boundary when $\mu = 0$ and turns smoothly into the admissible equilibrium x^+ for $\mu > 0$.
2) Nonsmooth Saddle-Node: the boundary equilibrium gives rise to a branch of admissible equilibria for some parameter variations or is annihilated. Here the two equilibria are both admissible for $\mu < 0$ turning into two virtual equilibria past the border-collision point (leaving the system with no regular equilibrium either in region G_1 or region G_2).

¿From a control perspective, it is clear that persistence is desirable as in the latter scenario the boundary equilibrium can disappear under perturbations. In this case the system might exhibit other asymptotic solutions such as limit cycles or aperiodic attractors as discussed later in the paper.

Here we wish to derive simple conditions to assess whether the boundary equilibrium will persist or not. The aim is to classify the simplest possible scenarios associated with a boundary equilibrium transition in n-dimensional nonsmooth continuous flows.

We will now give conditions to distinguish between these two fundamental cases in the case of n-dimensional continuous nonsmooth flows. Namely, in order for x^+ and x^- to be two admissible equilibria of the system, we must have

$$\begin{aligned} F_1(x^-,\mu) &= 0, \\ H(x^-,\mu) &:= \lambda^- < 0 \end{aligned} \tag{II.3}$$

and, using (II.2),

$$\begin{aligned} F_2(x^+,\mu) &= F_1(x^+,\mu) + G(x^+,\mu)H(x^+,\mu), \\ H(x^+,\mu) &:= \lambda^+ > 0 \end{aligned} \tag{II.4}$$

Now, linearizing about the the boundary equilibrium bifurcation point, $x = 0, \mu = 0$ we have:

$$Ax^- + B\mu = 0 \tag{II.5}$$
$$Cx^- + D\mu = \lambda^- \tag{II.6}$$

and

$$Ax^+ + B\mu + E\lambda^+ = 0 \tag{II.7}$$
$$Cx^+ + D\mu = \lambda^+ \tag{II.8}$$

where $A = F_{1x}, B = F_{1\mu}, C = H_x, D = H_\mu$ and $E = G$ all evaluated at $x = 0, \mu = 0$.

Then, from (II.5) we get:

$$x^- = -A^{-1}B\mu$$

and, substituting in (II.6), we obtain

$$\lambda^- = (D - CA^{-1}B)\mu. \tag{II.9}$$

Moreover from (II.7) we can write

$$x^+ = -A^{-1}B\mu - A^{-1}E\lambda^+$$

that substituted in (II.8) yields

$$\lambda^+ = \frac{D - CA^{-1}B}{1 + CA^{-1}E}\mu. \tag{II.10}$$

Finally, substituting (II.9) into (II.10), we get

$$\lambda^+ = \frac{1}{1 + CA^{-1}E}\lambda^- \tag{II.11}$$

Hence under variations of μ, the two equilibria, x^- and x^+, will be both admissible for the same value of μ (i.e. $\lambda^+ > 0$ and $\lambda^- < 0$) if $1 + CA^{-1}E < 0$ while they will be existing for opposite values of μ otherwise.

Therefore we can state the following theorem.

Theorem II.4 *For the systems of interest, assuming*

$$det(A) \neq 0 \tag{II.12}$$
$$D - CA^{-1}B \neq 0 \tag{II.13}$$
$$1 + CA^{-1}E \neq 0 \tag{II.14}$$

- *a* Persistence *scenario is observed at the boundary equilibrium bifurcation point if*
$$1 + CA^{-1}E > 0; \tag{II.15}$$
- *a* Nonsmooth Saddle-Node *is instead observed if*
$$1 + CA^{-1}E < 0; \tag{II.16}$$

Note that the asymptotic stability of the admissible equilibria existing on one or both sides of the bifurcation point can be assessed by looking at the eigenvalues of matrices A and $A + EC$.

III. Hopf-like transition

An important issue is to assess what happens to trajectories of a piecewise smooth continuous systems if under perturbations a stable boundary equilibrium undergoes a nonsmooth saddle node or becomes unstable. Here we wish to prove the following result.

Theorem III.1 *For the systems of interest if:*

1) *$A + EC$ is unstable and $1 + CA^{-1}E > 0$*
2) *the boundary equilibrium is asymptotically stable for $\mu = 0$*

then under perturbations of μ the system will exhibit in the simplest case a family of stable limit cycles originating locally from the boundary equilibrium point.

In fact, under these hypotheses, according to Theorem II.4, parameter perturbations will cause the boundary equilibrium to persist as $1 + CA^{-1}E > 0$. Moreover, as $A + EC$ is unstable, at least one of the equilibria existing for opposite values of μ will be unstable. Now, as the boundary equilibrium is assumed to be asymptotically stable for $\mu = 0$, by continuity with respect to parameter variations, for $\mu \neq 0$ a local neighborhood, say $\mathcal{B}(0)$, of the boundary equilibrium exists where trajectories continue to be attracted to. As under variations of μ, under the hypotheses stated, no stable equilibrium can exist (at least when μ is varied in one direction), then $\mathcal{B}(0)$ must contain some other attracting invariant set. In the simplest case, this will be a limit cycle (it must be in the case of two-dimensional systems). Note that in general $\mathcal{B}(0)$ might also contain a strange attractor in the case of higher-dimensional systems.

The problem of assessing the asymptotic stability of a piecewise linear system is still an open problem and no general conditions can be given to guarantee for the boundary equilibrium of a given system to be asymptotically stable. It is shown in [20] that this problem is computationally challenging in general. In what follows, we review available stability results for some special cases.

Consider the piecewise linear system

$$\dot{x} = \begin{cases} A^- x + B\mu & \text{if } Cx \leq 0 \\ A^+ x + B\mu & \text{if } Cx \geq 0 \end{cases} \tag{III.1}$$

where $A^\pm \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. We assume that the overall vector field is continuous across the hyperplane $\{x \mid c^T x = 0\}$. This means that

$$A^- - A^+ = EC \tag{III.2}$$

for some $E \in \mathbb{R}^n$. For the planar case, i.e. $n = 2$, a complete proof is available where it was shown that, when $\mu = 0$, the origin of a piecewise linear system (III.1) is asymptotically stable.

Proposition III.2 ([21]) *Consider the system* (III.1) *with $n = 2$. Assume that the pair (C, A^-) is observable. The following statements hold for $\mu = 0$.*

1) *The origin is asymptotically stable if and only if*
   a) *neither A^- nor A^+ has a real nonnegative eigenvalue, and*
   b) *if both A^- and A^+ have nonreal eigenvalues then $\sigma^-/\omega^- + \sigma^+/\omega^+ < 0$ where $\sigma^\pm \pm i\omega^\pm$ $(\omega^\pm > 0)$ are the eigenvalues of $A^\pm$.*
2) *The system* (III.1) *has a nonconstant periodic solution if and only if both A^- and A^+ have nonreal eigenvalues, and $\sigma^-/\omega^- + \sigma^+/\omega^+ = 0$ where $\sigma^\pm \pm i\omega^\pm$ $(\omega^\pm > 0)$ are the eigenvalues of $A^\pm$. Moreover, if there is one periodic solution, then all other solutions are also periodic. And, $\pi/\omega^- + \pi/\omega^+$ is the period of any solution.*

For planar systems, it is also possible to show that the amplitude of the limit cycle originate under parameter variation scales linearly with the parameter perturbation (see [22] for further details).

In higher dimensions, the problem becomes considerably more difficult. An interesting phenomena that occurs in higher dimensions is that even though both $A^\pm$ are Hurwitz matrices (i.e. matrices for which all the eigenvalues lie in the open left half plane), the overall system can exhibit instability. Such an example (see [23]) can be obtained by taking $\mu = 0$,

$$A^- = \begin{bmatrix} -1 & -1 & 0 \\ 1.28 & 0 & -1 \\ -0.624 & 0 & 0 \end{bmatrix} \quad A^+ = \begin{bmatrix} -3.2 & -1 & 0 \\ 25.61 & 0 & -1 \\ -75.03 & 0 & 0 \end{bmatrix} \tag{III.3}$$

and $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

Carmona et. al. [23] study the stability of the origin for $n = 3$. With the help of the notion of invariant cones, they reach the following result.

Proposition III.3 ([23]) *Consider the system* (III.1) *with $n = 2$. Assume that the pair (C, A^-) is observable. Let $A^\pm$ and c be given by*

$$A^\pm = \begin{bmatrix} t^\pm & -1 & 0 \\ m^\pm & 0 & -1 \\ d^\pm & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \tag{III.4}$$

in the so-called observability canonical form. Suppose that the eigenvalues of the matrices $A^\pm$ are $\lambda^\pm \in \mathbb{R}$ and $\sigma^\pm \pm i\omega^\pm$ where $\omega^\pm > 0$. Also suppose that

$$(\sigma^- - \lambda^-)(\sigma^+ - \lambda^+) < 0 \tag{III.5}$$

and

$$(t^+ - t^-)(\sigma^+ - \lambda^+) \leq 0. \tag{III.6}$$

Then, the origin is an asymptotically stable equilibrium point if, and only if, $\lambda^\pm$ are both negative.

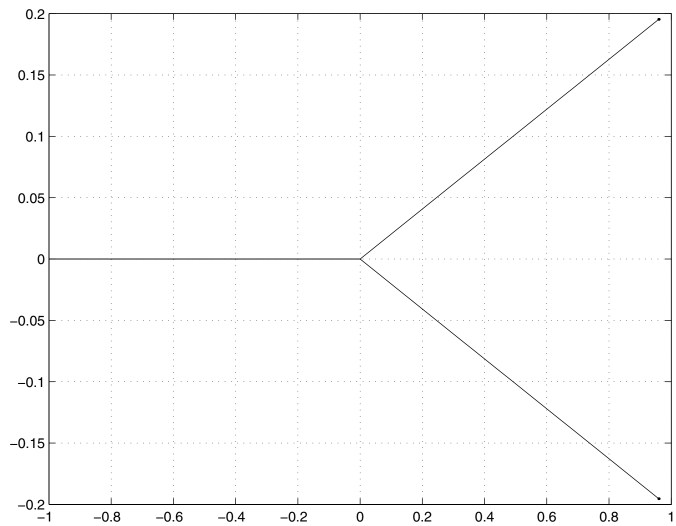

Fig. 1. Bifurcation diagram of system (III.7). The lines for $\mu > 0$ represent the Poincaré intersections of the system limit cycle and are therefore representative of the amplitude of the limit cycles generated at the boundary equilibrium transition.

Using this result, we can construct a three-dimensional piecewise linear example where, under parameter variations, the boundary equilibrium at the origin becomes unstable giving rise to a family of stable limit cycles whose amplitude scales linearly with the parameter perturbation. Specifically, Fig. 1 shows the bifurcation diagram of a three-dimensional system of the form (III.1) where

$$A^- = \begin{bmatrix} -5 & 1 & 0 \\ -9 & 0 & 1 \\ -5 & 0 & 0 \end{bmatrix} \quad A^+ = \begin{bmatrix} -5 & 1 & 0 \\ -12 & 0 & 1 \\ -14 & 0 & 0 \end{bmatrix} \tag{III.7}$$

and

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{III.8}$$

Note that $A^+ = A^- + EC$ with $E = \begin{bmatrix} 0 & -3 & -9 \end{bmatrix}^T$. As predicted by Proposition III.3, since $\lambda^- = -1$, $\sigma^- \pm j\omega^- = -2 \pm j$ and $\lambda^+ = -7$, $\sigma^+ \pm j\omega^+ = 1 \pm j$, in this case the boundary equilibrium at the origin is asymptotically stable when $\mu = 0$. Moreover, we have

$$\frac{1}{1 + C\,(A^-)^{-1}\,E} = 0.3571 > 0.$$

Hence, according to Theorem II.4 under variations of μ we expect a branch of stable equilibria for $\mu < 0$ turning into a branch of unstable equilibria for $\mu > 0$ and, as expected from Theorem III.1, a family of stable limit cycles is observed locally to the boundary equilibrium transition (see Fig. 1).

IV. Control Synthesis

We want to discuss a straightforward but effective use of the results presented above for the synthesis of an innovative switching control strategy aimed at generating a stable limit cycle of a given amplitude.

For instance, let $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^3$ be some SISO system we wish to control and assume the problem is to design a controller $u(t)$ such that the state evolves along a stable periodic solution of amplitude M. Then, according to the results presented above the following switching control strategy will be an effective way of solving the problem.

Specifically, assume a matrix K exists such that a piecewise linear dynamical systems of the form (III.1) with matrices $A^- := A$, $A^+ := A - BK$, $C := K$ fulfills the hypothesis of theorem III.1, i.e.

1) $A - BK$ is an unstable matrix with a pair of complex conjugate eigenvalues;
2) $1 - KA^{-1}B > 0$;
3) the boundary equilibrium at the origin is asymptotically stable for $\mu = 0$

then u can be chosen to be the following piecewise linear feedback law:

$$u(t) = \begin{cases} \hat{\mu}, & \text{if } Kx < 0 \\ -Kx + \hat{\mu}, & \text{if } Kx > 0 \end{cases} \qquad \text{(IV.1)}$$

with $\hat{\mu}$ chosen appropriately to obtain a periodic solution of the desired amplitude (by trial-and-error for example).

Indeed under the action of such a controller, the closed loop system becomes

$$\dot{x} = \begin{cases} Ax + B\hat{\mu}, & \text{if } Kx < 0 \\ (A - BK)x + B\hat{\mu}, & \text{if } Kx > 0 \end{cases}$$

fulfilling the hypothesis of Theorem III.1.

The detailed investigation of this control strategy is beyond the scope of this paper and will be detailed elsewhere. In what follow we give a representative example to better illustrate the strategy outline above.

A. A representative example

Let $\dot{x} = Ax+Bu$ be the plant we wish to control. Assume (A, B) to be controllable. For the sake of clarity, the matrices A and B are chosen as those of the example in Sec. III; namely:

$$A = \begin{bmatrix} -5 & 1 & 0 \\ -9 & 0 & 1 \\ -5 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then according to the strategy outlined we choose u as in (IV.1) where the gain matrix $K = [k_1\ k_2\ k_3]$ is chosen so that

- the eigenvalues of $A - BK$ are placed at $-7, 1 \pm j$ as those of matrix A^+ in Sec. III;
- $1 - KA^{-1}B > 0$
- the origin is an asymptotically stable equilibrium for $\hat{\mu} = 0$.

Using a pole placement strategy $K = [114\ \ -22\ \ 0]$ was found to fulfill all of the conditions above.

Simulations showing the phase space behaviour of the closed loop system and its bifurcation diagram $\hat{\mu}$ is varied are shown in Figs. 2 and 3. As expected, we observe a family of stable limit cycles for $\hat{\mu} > 0$ with amplitude increasing linearly with $\hat{\mu}$.

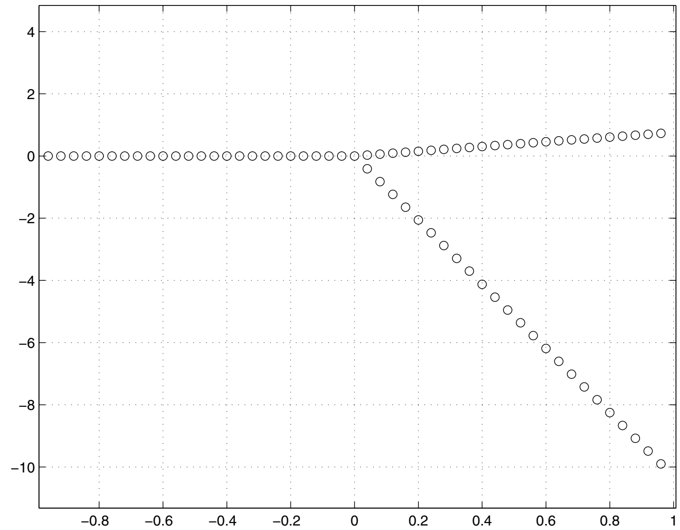

Fig. 2. Bifurcation diagram of the closed loop system under the action of the controller described in Sec. V. The lines for $\hat{\mu} > 0$ represent the x_2 components of the Poincaré intersections on the switching line $Kx = 0$ of the system limit cycle and are therefore representative of the amplitude of the limit cycles generated at the boundary equilibrium transition.

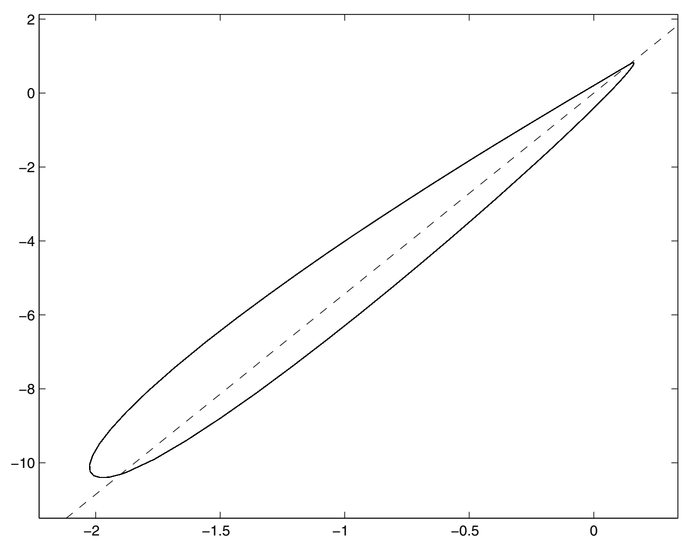

Fig. 3. Representative phase space (x_1, x_2) projection of the limit cycle exhibited by the closed loop piecewise linear system for $\hat{\mu} = 1$. The dashed line is the switching line $Kx = 0$.

V. Conclusions

We have studied the structural stability of so-called boundary equilibria in piecewise smooth continuous systems. We observed that, under parameter variations, such equilibria can undergo two major types of transitions leading to their persistence or annihilation. After deriving analytical conditions to classify each of these scenarios, we discussed the case of Hopf-like transitions. Namely, when the boundary equilibrium turns into an unstable equilibrium or disappears at a boundary equilibrium transition, the possibility is discussed of a family of stable limit cycles (or strange attractors) existing for local parameter variations. Recent stability results for the stability of piecewise linear continuous systems are used to construct a three-dimensional example where such a scenario is observed. The possible use of the results presented for control system design are briefly discussed. Further work in this direction is currently under investigation.

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