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Convergent piecewise affine systems: analysis and design Part II: discontinuous case

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Abstract— **In this paper convergence properties of piecewise affine (PWA) systems with discontinuous right-hand sides are studied. It is shown that for discontinuous PWA systems existence of a common quadratic Lyapunov function is not sufficient for convergence. For discontinuous bimodal PWA systems necessary and sufficient conditions for quadratic convergence, i.e. convergence with a quadratic Lyapunov function, are derived.**

I. Introduction

Convergent systems are systems that have a globally asymptotically stable steady state solution which depends only on the input and does not depend on the initial conditions. This property plays an important role in many control problems including tracking, synchronization, observer design, the output regulation problem and performance analysis of nonlinear systems, see e.g. [1], [2], [3], [4], [5] and references therein. It is easy to see that a linear time-invariant system with a stable transfer function is convergent, so the properties of convergence and stability are closely related. However for nonlinear systems, there are many examples showing that a globally asymptotically stable system perturbed by an extra input can have more than two steady state solutions and thus it is not convergent.

Studies related to convergence systems were originated in the 1960-s, for a short survey see [6]. Recent results on smooth convergent systems can be found in [7]. In this paper we continue the previous study of the convergence properties of piecewise affine systems initiated in [8]. Piecewise affine systems recently attracted considerable attention, see e.g. [9], [10], [11] and references therein.

In the first part of our study [8], the case of piecewise affine systems with continuous right-hand sides was considered and conditions for quadratic convergence were derived in terms of Linear Matrix Inequalities. It turns out that for PWA systems with continuous right-hand sides the exponential convergence property follows from the existence of a common quadratic Lyapunov function for the linear parts of the system dynamics in every mode. The goal of this paper is to study the convergence property for a more general class of PWA systems which includes also systems with discontinuous right-hand sides.

The paper is organized as follows. In Section II we provide preliminaries on systems with discontinuous right-hand sides. In Section III definitions of (uniformly, exponentially) convergent systems are provided. Also, in this section we introduce the notion of quadratic convergence and show its relation to exponential convergence. In Section IV we first present a counterexample which shows that for discontinuous PWA systems existence of a common quadratic Lyapunov function is not sufficient for convergence. Then necessary and sufficient conditions for quadratic convergence for bimodal PWA systems with (possibly) discontinuous right-hand sides are presented.

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II. Preliminaries

In this paper we consider systems of the form

$$\dot{x} = f(x,t), \tag{1}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $f(x,t)$ is a possibly discontinuous vector field. It is assumed that $f(x,t)$ satisfies some mild regularity assumptions which guarantee the existence of solutions of the system in the sense of Filippov, see e.g. [12]. According to [12], one can construct a set-valued function $F(x,t)$ such that a solution of the differential inclusion

$$\dot{x} \in F(x,t)$$

is called a solution for system (1). By definition, the solution $x(t,t_0,x_0)$ with the initial condition $x(t_0,t_0,x_0) = x_0$ is an absolutely continuous function of time.

Consider a scalar continuously differentiable function $V(x)$. Define a time derivative of this function along solutions of system (1) as follows

$$\dot{V} := \frac{\partial V(x)}{\partial x}\dot{x}(t,t_0,x_0).$$

Since V is continuously differentiable and the solution $x(t,t_0,x_0)$ is an absolutely continuous function of time, the derivative $\dot{V}(x(t,t_0,x_0))$ exists almost everywhere in the maximal interval of existence $[t_0,\bar{T})$ of the solution $x(t,t_0,x_0)$.

For the function V we can also define its upper derivative along solutions of system (1) as follows

$$\dot{V}^*(x,t) = \sup_{\xi \in F(x,t)} \left(\frac{\partial V(x)}{\partial x}\xi \right).$$

Then for almost all $t \in [t_0, \bar{T})$ it follows that

$$\dot{V}(x(t, t_0, x_0)) \leq \dot{V}^*(x(t, t_0, x_0), t). \quad (2)$$

Remark 1 *Notice that in the domains of continuity of the function $f(x,t)$ the derivative of $V(x)$ along solutions of system (1) equals $\dot{V} = \frac{\partial V(x)}{\partial x} f(x,t)$. According to [12] p.155, for a* continuously differentiable *function $V(x)$ it holds that if the inequality*

$$\frac{\partial V(x)}{\partial x} f(x,t) \leq 0$$

is satisfied in the domains of continuity of the function $f(x,t)$, then the inequality $\dot{V}^*(x,t) \leq 0$ holds for all $(x,t) \in \mathbb{R}^{n+1}$.

III. Convergent systems

In this section we give definitions of convergent systems. These definitions extend the definition given in [13].

Definition 1 *System (1) is said to be*

- convergent *if there exists a solution $\bar{x}(t)$ satisfying the following conditions*
  (i) *$\bar{x}(t)$ is defined and bounded for all $t \in \mathbb{R}$,*
  (ii) *$\bar{x}(t)$ is globally asymptotically stable;*
- uniformly convergent *if it is convergent and $\bar{x}(t)$ is globally uniformly asymptotically stable.*
- exponentially convergent *if it is convergent and $\bar{x}(t)$ is globally exponentially stable.*

The solution $\bar{x}(t)$ is called a *steady-state solution*. As follows from the definition of convergence, any solution of a convergent system "forgets" its initial condition and converges to some steady-state solution which is independent of the initial condition. In general, the steady-state solution $\bar{x}(t)$ may be non-unique. But for any two steady-state solutions $\bar{x}_1(t)$ and $\bar{x}_2(t)$ it holds that $|\bar{x}_1(t) - \bar{x}_2(t)| \to 0$ as $t \to +\infty$. At the same time, for *uniformly* convergent systems the steady-state solution is unique, as formulated below [8].

Property 1 *If system (1) is uniformly convergent, then the steady-state solution $\bar{x}(t)$ is the only solution defined and bounded for all $t \in \mathbb{R}$.*

Remark 2 *In the original definition of convergent systems given in [13], the steady-state solution $\bar{x}(t)$ is required to be unique. In Definition 1 this requirement of uniqueness is omitted, since for the practically important case of uniform convergence uniqueness of the steady-state solution can be proved as a corollary to the definition of the uniform convergence.*

In systems theory, time dependency of the right-hand side of system (1) is usually due to some input. This input may represent, for example, a disturbance or a feedforward control signal. Below we will consider convergence properties for systems with inputs. So, instead of systems of the form (1), we consider systems

$$\dot{x} = f(x, w), \quad (3)$$

with state $x \in \mathbb{R}^n$ and input $w \in \mathbb{R}^m$. In the sequel we will consider the class $\overline{\mathbb{PC}}_m$ of piecewise continuous inputs $w(t) : \mathbb{R} \to \mathbb{R}^m$ which are bounded for all $t \in \mathbb{R}$. We assume that the function $f(x,w)$ is bounded on any compact set of (x,w) and the set of discontinuity points of the function $f(x,w)$ has measure zero. Under these assumptions on $f(x,w)$, for any input $w \in \overline{\mathbb{PC}}_m$ the differential equation $\dot{x} = f(x, w(t))$ has well-defined solutions in the sense of Filippov.

Below we define the convergence property for systems with inputs.

Definition 2 *System (3) is said to be* (uniformly, exponentially) convergent *if it is (uniformly, exponentially) convergent for every input $w \in \overline{\mathbb{PC}}_m$. In order to emphasize the dependency on the input $w(t)$, the steady-state solution is denoted by $\bar{x}_w(t)$.*

The (uniform, exponential) convergence property is an extension of stability properties of asymptotically stable LTI systems. Therefore, convergent systems enjoy various properties which are encountered in asymptotically stable LTI systems, but which are not usually met in general asymptotically stable nonlinear systems, see [4]. As an illustration, we provide a statement which summarizes some properties of uniformly convergent systems excited by periodic or constant inputs.

Property 2 ([13]) *Suppose system (3) with a given input $w(t)$ is uniformly convergent. If the input $w(t)$ is constant, the corresponding steady-state solution $\bar{x}_w(t)$ is also constant; if the input $w(t)$ is periodic with period T, then the corresponding steady-state solution $\bar{x}_w(t)$ is also periodic with the same period T.*

Below we give an important technical definition of quadratic convergence.

Definition 3 *System (3) is called quadratically convergent if there exists a positive definite matrix $P = P^T > 0$ and a constant $\alpha > 0$ such that for any input $w \in \overline{\mathbb{PC}}_m$, the function $V(x_1, x_2) = 1/2(x_1 - x_2)^T P (x_1 - x_2)$ satisfies*

$$\dot{V}^*(x_1, x_2, t) \leq -2\alpha V(x_1, x_2), \quad (4)$$

where $\dot{V}^*(x_1, x_2, t)$ is the upper derivative of the function $V(x_1, x_2)$ along any two solutions of the corresponding differential inclusion $\dot{x} \in F(x, w(t))$, i.e.

$$\dot{V}^*(x_1, x_2, t) = \sup_{\xi_1 \in F(x_1, w(t))} \left(\frac{\partial V}{\partial x_1}(x_1, x_2)\xi_1 \right) + \sup_{\xi_2 \in F(x_2, w(t))} \left(\frac{\partial V}{\partial x_2}(x_1, x_2)\xi_2 \right).$$

Quadratic convergence is a useful tool for establishing exponential convergence, as follows from the next lemma.

Lemma 1 *If system (3) is quadratically convergent, then it is exponentially convergent.*

Proof: Consider the system

$$\dot{x} = f(x, w(t)), \tag{5}$$

where $w(t)$ is some bounded piecewise-continuous input. First, we show the existence of a solution $\bar{x}_w(t)$ of system (5) which is defined and bounded on the whole time axis $(-\infty, +\infty)$. The existence of such $\bar{x}_w(t)$ will be shown using the following lemma.

Lemma 2 ([14]) *Consider system (5) with a given input $w(t)$ defined for all $t \in \mathbb{R}$. Let $\mathcal{D} \subset \mathbb{R}^n$ be a compact set which is positively invariant with respect to system (5). Then there is at least one solution $\bar{x}(t)$ satisfying $\bar{x}(t) \in \mathcal{D}$ for all $t \in (-\infty, +\infty)$.*

In order to apply this lemma, we need to prove the existence of a compact positively invariant set $\mathcal{D}$. Consider the function $W(x) := 1/2x^T Px$. The upper derivative of this function along solutions of system (5) satisfies

$$\begin{aligned} \dot{W}^*(x,t) &= \sup_{\xi \in F(x,w(t))} x^T P\xi \leq \sup_{\xi \in F(x,w(t))} x^T P\xi \\ &\quad - \inf_{\xi_1 \in F(0,w(t))} x^T P\xi_1 + \sup_{\xi_2 \in F(0,w(t))} x^T P\xi_2. \end{aligned}$$

Notice that for the function $V(x_1, x_2)$ from the definition of quadratic stability it holds that

$$\begin{aligned} \dot{V}^*(x,0,t) &= \sup_{\xi \in F(x,w(t))} x^T P\xi + \sup_{\xi_1 \in F(0,w(t))} (-x^T P\xi_1) \\ &= \sup_{\xi \in F(x,w(t))} x^T P\xi - \inf_{\xi_1 \in F(0,w(t))} x^T P\xi_1. \end{aligned}$$

Therefore,

$$\dot{W}^*(x,t) \leq \dot{V}^*(x,0,t) + \sup_{\xi_2 \in F(0,w(t))} |x^T P\xi_2|. \tag{6}$$

By the quadratic convergence property it holds that

$$\dot{V}^*(x,0,t) \leq -2\alpha V(x,0) = -\alpha |x|_P^2, \tag{7}$$

where $|x|_P^2 = x^T Px$. At the same time, by the Cauchy inequality it holds that $|x^T P\xi_2| \leq |x|_P |\xi_2|_P$. Hence

$$\sup_{\xi_2 \in F(0,w(t))} |x^T P\xi_2| \leq |x|_P \sup_{\xi_2 \in F(0,w(t))} |\xi_2|_P. \tag{8}$$

Recall that the input $w(t)$ is bounded, i.e. $|w(t)| \leq R$ for all $t \in \mathbb{R}$, for some $R > 0$. By the assumption on the right-hand side of system (3) (see Section III), the function $f(x,w)$ takes bounded values on any compact set of (x,w). Therefore the set $\{\xi \in \mathbb{R}^n : \ \xi \in F(0,w), \ |w| \leq R\}$ is bounded. Therefore, for some constant $\bar{c} > 0$ it holds that

$$\sup_{\xi_2 \in F(0,w(t))} |\xi_2|_P \leq \sup_{\substack{\xi_2 \in F(0,w) \\ |w| \leq R}} |\xi_2|_P \leq \bar{c}. \tag{9}$$

Combining inequalities (6)- (9) we obtain

$$\dot{W}^*(x,t) \leq |x|_P(-\alpha |x|_P + \bar{c}). \tag{10}$$

Hence, $\dot{W}^*(x,t) \leq 0$ for all $t \in \mathbb{R}$ and all x satisfying $|x|_P \geq \bar{c}/\alpha$. Taking into account the relation between the derivative and upper derivative of $W(x)$ along solutions $x(t)$ of system (5) (see (2)), we obtain

$$\dot{W}(x(t)) \leq 0$$

for almost all t such that $|x(t)|_P \geq \bar{c}/\alpha$. This implies that the set $\mathcal{D} := \{x : \ |x|_P \leq \bar{c}/\alpha\}$ is compact and positively invariant. By Lemma 2 there exists a solution $\bar{x}_w(t)$ which satisfies $\bar{x}_w(t) \in \mathcal{D}$ for all $t \in \mathbb{R}$.

Next, we need to show global exponential stability of $\bar{x}_w(t)$. By the quadratic convergence property it holds that

$$\dot{V}^*(x, \bar{x}_w(t), t) \leq -2\alpha V(x, \bar{x}_w(t)).$$

Consider some solution $x(t) := x(t, t_0, x_0)$ of system (5). Recall that $\dot{V}(x(t), \bar{x}_w(t)) \leq \dot{V}^*(x(t), \bar{x}_w(t), t)$ for almost all t (see Section II). Therefore,

$$\dot{V}(x(t), \bar{x}_w(t)) \leq -2\alpha V(x(t), \bar{x}_w(t))$$

for almost all $t \geq t_0$. Since $V(x_1, x_2)$ is a quadratic form with respect to the difference $(x_1 - x_2)$, the last inequality implies

$$|x(t) - \bar{x}_w(t)| \leq Ce^{-\alpha(t-t_0)}|x(t_0) - \bar{x}_w(t_0)|,$$

where the number $C > 0$ depends only on the matrix P. ■

Remark 3 *As follows from Remark 1 (Section II), inequality (4) is equivalent to the inequality*

$$\begin{aligned} &(x_1 - x_2)^T P(f(x_1, w) - f(x_2, w)) \\ &\qquad \leq -\alpha (x_1 - x_2)^T P(x_1 - x_2) \end{aligned} \tag{11}$$

for all $w \in \mathbb{R}^m$ and all x_1 and x_2 from the continuity domain of the function $f(x,w)$.

IV. Discontinuous PWA systems

In this section we study convergence properties for PWA systems with possibly discontinuous right-hand sides.

Consider the state space $\mathbb{R}^n$ divided into polyhedral cells Λ_i, $i = 1, \ldots, l$, by hyperplanes given by equations of the form $H_j^T z + h_j = 0$, for some $H_j \in \mathbb{R}^n$ and $h_j \in \mathbb{R}$, $j = 1, \ldots, k$. We will consider piecewise-affine systems of the form

$$\dot{x} = A_i x + b_i + Dw, \quad \text{for } \ x \in \Lambda_i, \ i = 1, \ldots, l. \tag{12}$$

Here $A_i \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{n \times m}$ and $b_i \in \mathbb{R}^n$, $i = 1, \ldots, l$, are constant matrices and vectors, respectively. The vector $x \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^m$ is the input. The hyperplanes $H_j^T z + h_j = 0$, $j = 1, \ldots, k$, are the switching surfaces. Before proceeding with the case of general (discontinuous) PWA systems, we review a result from [8] on sufficient conditions for quadratic convergence for PWA systems with continuousos right-hand sides.

Theorem 1 ([8]) *Consider system (12). Suppose the right-hand side of system (12) is continuous and there exists a positive definite matrix $P = P^T > 0$ such that*

$$PA_i + A_i^T P < 0, \quad i = 1, \ldots, l. \tag{13}$$

Then system (12) is quadratically convergent.

Remark 4 *In fact, in this theorem it is shown that for a* continuous *piecewise-affine vector-field $f(x, w)$ of the form*

$$f(x, w) = A_i x + b_i + Dw, \quad \textit{for } x \in \Lambda_i, \; i = 1, \ldots, l,$$

condition (13) is equivalent to the inequality

$$(x_1 - x_2)^T P(f(x_1, w) - f(x_2, w)) \\ \leq -\alpha(x_1 - x_2)^T P(x_1 - x_2) \quad (14)$$

for some $\alpha > 0$ and all $w \in \mathbb{R}^m$ and all $x_1, x_2 \in \mathbb{R}^n$.

Based on the result of Theorem 1, one can conjecture that a discontinuous piecewise affine system (12) is also convergent provided there is a common quadratic Lyapunov function for the linear parts of the system dynamics $A_i x$. However this is not the case as one can see from the following simple example. Suppose that the system dynamics is governed by the following scalar differential equation with discontinuous right-hand side:

$$\dot{x} = a(x), \quad x \in \mathbb{R}^1,$$

where the function $a(x)$ is depicted schematically on Fig. 1. It is seen that the system belongs to the class of piecewise affine systems and in each region the dynamics is linear. Moreover, it is not difficult to see that the system is globally asymptotically stable with common quadratic Lyapunov function $V = x^2$.

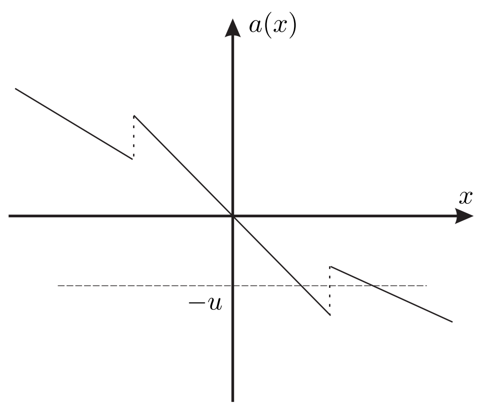

Fig. 1. Piecewise affine characteristics $a(x)$.

Now suppose that the dynamics of the system is modified with an additive input signal, that can be either disturbance or reference signal:

$$\dot{x} = a(x) + u(t), \quad x \in \mathbb{R}^1.$$

It is clear from the picture that for some input signals (e.g. constant) the dynamics of the system can depend on the initial conditions (one can take such a constant input signal that the system has two asymptotically stable equilibria), or, in other words, the system is not convergent. This simple example illustrates that even the existence of common Lyapunov function for each mode of a piecewise affine system is not sufficient to guarantee its convergence. Moreover, this example shows that the continuity conditions play an important role for the convergence of PWA systems and we have to be careful when analyzing convergence for discontinuous PWA systems. In fact, for bimodal piecewise-affine systems the existence of a common Lyapunov function and the conditions similar to the continuity requirements are even necessary and sufficient for the quadratic convergence, as follows from the result presented hereafter.

Consider the bimodal system

$$\dot{x} = \begin{cases} A_1 x + b_1 + Dw, & \text{for } H^T x \geq 0 \\ A_2 x + b_2 + Dw, & \text{for } H^T x < 0, \end{cases} \quad (15)$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ and A_i, b_i, $i = 1, 2$, and D are matrices of the appropriate dimensions. The switching plane is determined by the constant vector $H \in \mathbb{R}^n$. Denote $\Delta A := A_1 - A_2$, $\Delta b := b_1 - b_2$.

Theorem 2 *Consider system (15). The following statements are equivalent:*

(i) *System (15) is quadratically convergent.*

(ii) *There exist a positive definite matrix $P = P^T > 0$ and constants $\beta > 0$ and $\gamma \geq 0$ satisfying the following LMI*

$$\begin{pmatrix} PA_1 + A_1^T P + \beta I & P\Delta A - \frac{1}{2} H H^T \\ \Delta A^T P - \frac{1}{2} H H^T & -H H^T \end{pmatrix} \leq 0, \quad (16)$$

$$P \Delta b = -\gamma H. \quad (17)$$

(iii) *There exist a positive definite matrix $P = P^T > 0$, a number $\gamma \in \{0, 1\}$ and a vector $G \in \mathbb{R}^n$ such that*

$$PA_i + A_i^T P < 0, \quad i = 1, 2, \quad (18)$$

$$\Delta A = G H^T, \quad (19)$$

$$P \Delta b = -\gamma H. \quad (20)$$

Proof: The theorem will be proved in the following order: (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii). According to Remark 3, quadratic convergence of system (15) implies that there exists a positive definite matrix $\bar{P} = \bar{P}^T > 0$ and a number $\alpha > 0$ such that for any x_1 and x_2 satisfying the inequalities $H^T x_1 > 0$ and $H^T x_2 < 0$ it holds that

$$(x_1 - x_2)^T \bar{P}(A_1 x_1 + b_1 - A_2 x_2 - b_2) \\ \leq -\alpha(x_1 - x_2)^T \bar{P}(x_1 - x_2). \quad (21)$$

By denoting $e := x_1 - x_2$ and taking into account the fact that $-\alpha \bar{P} \leq -\bar{\beta} I$ for some $\bar{\beta} > 0$ and I being the identity matrix, we conclude that inequality (21) implies

$$e^T \bar{P}(A_1 e + \Delta A x_2 + \Delta b) \leq -\bar{\beta}|e|^2 \quad (22)$$

for all e and x_2 from the set $\Omega_1 := \{(e, x_2) : \; H^T x_2 < 0, \; H^T e + H^T x_2 > 0\}$. Let us show that inequality (22) yields

$$e^T \bar{P}(A_1 e + \Delta A x_2) + \beta |e|^2 \leq 0 \quad (23)$$

$$e^T \bar{P} \Delta b \leq 0 \quad (24)$$

for all $(e, x_2) \in \Omega_1$. Consider some point $(e, x_2) \in \Omega_1$. Then for all $\lambda > 0$ it holds that $(\lambda e, \lambda x_2) \in \Omega_1$. As follows from inequality (22), this yields

$$\lambda^2(e^T \bar{P}(A_1 e + \Delta A x_2) + \bar{\beta}|e|^2) + \lambda e^T \bar{P} \Delta b \leq 0$$

for all $\lambda > 0$. One can easily check that this inequality is satisfied for all $\lambda > 0$ iff the inequalities (23) and (24) hold. Due to arbitrary choice of $(e, x_2) \in \Omega_1$, we conclude that inequalities (23) and (24) are satisfied for all $(e, x_2) \in \Omega_1$.

Repeating the same steps as in the first part of the proof, but this time for points x_1 and x_2 satisfying $H^T x_1 < 0$ and $H^T x_2 > 0$, we conclude that the inequality

$$e^T \bar{P}(A_1 e - \Delta A x_1) + \bar{\beta}|e|^2 \leq 0 \tag{25}$$

holds for all $(e, x_1) \in \Omega_2$, where $\Omega_2 := \{(e, x_1) : \ H^T x_1 < 0,\ -H^T e + H^T x_1 > 0\}$. By denoting $\tilde{x}_1 := -x_1$, we obtain that

$$e^T \bar{P}(A_1 e + \Delta A \tilde{x}_1) + \bar{\beta}|e|^2 \leq 0 \tag{26}$$

holds for all $(e, \tilde{x}_1) \in \tilde{\Omega}_2$, where $\tilde{\Omega}_2 := \{(e, \tilde{x}_1) : \ H^T \tilde{x}_1 > 0,\ H^T e + H^T \tilde{x}_1 < 0\}$. Now we can show that (16) is feasible.

Combining inequalities (23) and (26) we obtain that the quadratic form $\mathcal{F}(e, \xi) := e^T \bar{P}(A_1 e + \Delta A \xi) + \bar{\beta}|e|^2$ satisfies

$$\mathcal{F}(e, \xi) \leq 0 \quad \text{for } (e, \xi) : \ \mathcal{G}(e, \xi) < 0, \tag{27}$$

where $\mathcal{G}(e, \xi) := \xi^T H(H^T e + H^T \xi)$. Due to continuity of $\mathcal{F}$ and non-strict inequality for $\mathcal{F}$ in (27), the last inequality is equivalent to

$$\mathcal{F}(e, \xi) \leq 0 \quad \text{for } (e, \xi) : \ \mathcal{G}(e, \xi) \leq 0. \tag{28}$$

Applying the S-procedure, see e.g. [15], [16], we obtain that the conditional inequality (28) is equivalent to the unconditional inequality

$$\mathcal{F}(e, \xi) - \tau \mathcal{G}(e, \xi) \leq 0 \tag{29}$$

for some $\tau \geq 0$ and all $(e, \xi) \in \mathbb{R}^{2n}$. The equivalence holds because the S-procedure is lossless in case of one quadratic constraint, see e.g. [15]. Notice that since the quadratic form $\mathcal{F}(e, \xi)$ is not negative semidefinite, $\tau \neq 0$ (otherwise the equivalence between (28) and (29) does not hold). Notice that inequality (29) is equivalent to the following LMI

$$\begin{pmatrix} \bar{P}A_1 + A_1^T \bar{P} + 2\bar{\beta}I & \bar{P}\Delta A - \tau H H^T \\ \Delta A^T \bar{P} - \tau H H^T & -2\tau H H^T \end{pmatrix} \leq 0. \tag{30}$$

Since $\tau > 0$, this inequality is equivalent to (16) with $P := \bar{P}/(2\tau)$ and $\beta := \bar{\beta}/\tau$.

It remains to show that inequality (17) holds for the presented P and some $\gamma \geq 0$. To this end, consider inequality (24), which holds for all $(e, x_2) \in \Omega_1$. Notice that for all e satisfying $H^T e > 0$ there exists x_2 such that $(e, x_2) \in \Omega_1$. Therefore, $e^T \bar{P} \Delta b \leq 0$ for all e satisfying $H^T e > 0$. One can easily check that this is possible iff $\bar{P}\Delta b = -\bar{\gamma} H$ for some $\bar{\gamma} \geq 0$. After dividing both sides of the obtained equation by 2τ, we obtain (17) with $P = \bar{P}/(2\tau)$ and $\gamma := \bar{\gamma}/(2\tau)$. This finishes the proof of implication (i)⇒(ii).

(ii)⇒(iii) First, we will show that conditions (18)-(20) hold for some matrix $P = P^T > 0$, vector $G \in \mathbb{R}^n$ and some $\gamma \geq 0$. If $\gamma = 0$ this proves this implication. If $\gamma > 0$, then by dividing (18) and (20) by γ we obtain that relations (18) and (20) hold for $\tilde{P} := P/\gamma$ and $\tilde{\gamma} = 1$. This proves the remaining part of the implication.

Let us show that conditions (18)-(20) hold for some matrix $P = P^T > 0$, vector $G \in \mathbb{R}^n$ and some $\gamma \geq 0$. We only need to show (18) and (19), since (20) coincides with (17). One can easily see that inequality (16) implies $PA_1 + A_1^T P \leq -\beta I < 0$. Next we show that inequality $PA_2 + A_2^T P \leq -\beta I < 0$ holds. Denote the matrix in (16) by M. The inequality (16) yields

$$\begin{pmatrix} x \\ -x \end{pmatrix}^T M \begin{pmatrix} x \\ -x \end{pmatrix} \leq 0 \tag{31}$$

for all $x \in \mathbb{R}^n$. After elaborating the left-hand side of (31) we obtain $x^T(PA_2 + A_2^T P + \beta I)x \leq 0$ for all $x \in \mathbb{R}^n$. Hence, we have shown (18). Let us show that (19) holds for some $G \in \mathbb{R}^n$. This is done in the same way as in [2]. Suppose $\chi \in \ker(H^T)$. From the structure of the matrix M we obtain

$$\begin{pmatrix} 0 \\ \chi \end{pmatrix}^T M \begin{pmatrix} 0 \\ \chi \end{pmatrix} = 0.$$

Since $M = M^T \leq 0$, this equality implies $M(0, \chi^T)^T = 0$. Taking into account the structure of M, we obtain that $P\Delta A \chi = 0$. Since P is non-degenerate, we conclude that $\Delta A \chi = 0$. Thus we have shown that $\ker(H^T) \subset \ker(\Delta A)$. This relation, in turn, implies the existence of a vector $G \in \mathbb{R}$ such that $\Delta A = G H^T$. This concludes the proof of the implication (ii)⇒(iii).

(iii)⇒(i) Let us write the system (15) in the following form

$$\dot{x} = f(x, w) + b(x), \tag{32}$$

where

$$f(x, w) := \begin{cases} A_1 x + D w, & \text{for } \ H^T x \geq 0 \\ A_2 x + D w, & \text{for } \ H^T x < 0, \end{cases} \tag{33}$$

$$b(x) := \begin{cases} b_1, & \text{for } \ H^T x \geq 0 \\ b_2, & \text{for } \ H^T x < 0. \end{cases} \tag{34}$$

As follows from Remark 3, for quadratic convergence of system (32) it is sufficient that, for some matrix $P = P^T > 0$ and scalar $\alpha > 0$, the inequality

$$\begin{aligned} (x_1 - x_2)^T P(f(x_1, w) + b(x_1) - f(x_2, w) - b(x_2)) \\ \leq -\alpha (x_1 - x_2)^T P (x_1 - x_2) \end{aligned} \tag{35}$$

holds for all x_1 and x_2 such that $H^T x_1 \neq 0$ and $H^T x_2 \neq 0$, i.e. in the continuity points of the right-hand side of system (32). The vector-field $f(x, w)$ is piecewise affine. Moreover, one can easily check that condition (19) implies continuity of $f(x, w)$ (see [8], Lemma 1). Since the matrices A_1 and

A_2 satisfy (18) for some $P = P^T > 0$, then by Theorem 1 (see Remark 4) the inequality

$$\begin{aligned}(x_1 - x_2)^T P(f(x_1, w) - f(x_2, w)) \\ \leq -\alpha(x_1 - x_2)^T P(x_1 - x_2)\end{aligned} \tag{36}$$

holds for all x_1 and $x_2 \in \mathbb{R}^n$. Hence,

$$\begin{aligned}(x_1 - x_2)^T P(f(x_1, w) + b(x_1) - f(x_2, w) - b(x_2)) \\ \leq -\alpha(x_1 - x_2)^T P(x_1 - x_2) \\ +(x_1 - x_2)^T P(b(x_1) - b(x_2)).\end{aligned} \tag{37}$$

It remains to show that

$$(x_1 - x_2)^T P(b(x_1) - b(x_2)) \leq 0 \tag{38}$$

for all x_1 and x_2 such that $H^T x_i \neq 0$, $i = 1, 2$. If x_1 and x_2 belong to the same cell, i.e. either $H^T x_i > 0$, $i = 1, 2$ or $H^T x_i < 0$, $i = 1, 2$, then $b(x_1) = b(x_2)$ and, therefore, the left-hand side of (38) equals zero. If $H^T x_1 > 0$ and $H^T x_2 < 0$, then $b(x_1) - b(x_2) = b_1 - b_2 = \Delta b$. Taking into account equality (20), we see that the left-hand side of (38) satisfies

$$\begin{aligned}(x_1 - x_2)^T P\Delta b &= -\gamma(x_1 - x_2)^T H \\ &= -\gamma(H^T x_1 - H^T x_2) \leq 0.\end{aligned}$$

In the same way inequality (38) is proven for all x_1 and x_2 satisfying $H^T x_1 < 0$ and $H^T x_2 > 0$. Thus, we have shown that inequality (38) holds for all x_1 and x_2 such that $H^T x_i \neq 0$, $i = 1, 2$. Inequalities (38) and (37) jointly imply (35). This completes the proof of the implication (iii)⇒(i). ■

Remark 5 *In part (iii) of Theorem 2 there are two options:* $\gamma = 0$ *and* $\gamma = 1$. *For the case* $\gamma = 0$ *condition (20) yields* $\Delta b = 0$. *This, together with condition (19), implies that the right-hand side of system (15) is continuous (see [8], Lemma 1). In the case of* $\gamma = 1$, *we see that discontinuity may occur only due to the affine terms* b_i. *In this case conditions (18) and (20) mean that the two linear systems* $(A_1, \Delta b, H^T)$ *and* $(A_2, \Delta b, H^T)$ *with the state matrices* A_1, A_2, *input matrix* Δb *and output matrix* H^T *are simultaneously strictly passive with the same quadratic storage function* $V(x) = x^T P x$.

V. Conclusions

In this paper we have continued our studies of convergence properties of piecewise affine systems started in [8]. In [8] it has been shown that for a PWA system with a continuous right-hand side, the existence of a common quadratic Lyapunov function for linear parts of the system dynamics in each mode is sufficient for exponential convergence. For PWA systems with discontinuous right-hand sides this is not true, as has been demonstrated by a counterexample presented in this paper. Therefore, the case of discontinuous PWA systems requires separate treatment. In order to study convergence properties of discontinuous PWA systems, we have introduced the notion of quadratic convergence, i.e. convergence with a quadratic Lyapunov function. This quadratic convergence serves as a useful tool for establishing the exponential convergence. For discontinuous bimodal PWA systems we have presented necessary and sufficient conditions for the quadratic convergence. According to this result, a discontinuous bimodal PWA is quadratically convergent iff the discontinuity occurs only due to affine terms and, in addition to that, two certain linear systems, related to the PWA system dynamics in each mode, are simultaneously strictly passive with the same quadratic storage function. The obtained results provide tools for studying convergence properties for hybrid systems. They can be used, for example in observer design for discontinuous hybrid systems.

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