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Maximal Solution to Perturbed Algebraic Riccati Equations Arising in Markovian Jump Control Revisited

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Abstract— **In this paper we revisit the maximal solution problem studied in [7]. It is shown that, for the Markovian jump scenario, we can get rid of an inconvenient technical hypothesis used in [7] (originally introduced in [20]). This is achieved, essentially, via the mean square stability concept.**

I. INTRODUCTION

In this paper, we revisit the maximal solution problem studied in [7]. We consider a class of linearly perturbed algebraic Riccati equation (PARE) - given by (3) - which arises in linear quadratic control problems where system parameters vary according to a Markov chain that takes values in a finite state space. In this context the systems' dynamic corresponds the so-called Markov Jump Linear Systems (MJLSs) given by the following stochastic differential equation:

$$(A,B,\Lambda):\begin{cases}\dot{x}(t)=A_{\theta(t)}x(t)+B_{\theta(t)}u(t), & t\geq 0\\ x(0)=x_0, \quad \theta(0)=\theta_0.\end{cases} \tag{1}$$

In the above equation, $x(t)\in\mathbb{C}^n$ denotes the state vector, $u(t)\in\mathbb{C}^m$ the control input and $\{\theta(t),\ s\leq t\leq T\}$ a standard conservative Markov chain with infinitesimal matrix Λ and a finite state space $\mathcal{S}=\{1,2,...,N\}$. We consider (x_0,θ_0) an initial joint random variable with distribution ϑ_0. Randomness is introduced in the parameters by means of some correspondence $i\mapsto\eta_i$, for $\theta(t)=i$, η_i standing for the system matrices A_i or B_i.

MJLSs model physical systems that have their structures subject to abrupt changes. We can mention [8], [10], [11], [12], [13], [17], [19] and the references therein as a sample of works dealing with stability, optimal control, filtering, H_∞-control and Riccati differential equations. We mention also [2], [15] and [18] as works dealing with applications.

In this paper we show that we can get rid of an inconvenient technical hipothesis used in [7] (originally introduced in [20]) to derive the results on existence (and uniqueness) of maximal solution to the PARE obtained in [7] set to the Markovian jump scenario. More explicitely, instead of stability in the usual sense plus the above hipothesis, we use the concept of mean square stability (or equivalently, stochastic stability) - see [10] and [17], to obtain these results. The above hipothesis (Assumption 2.1 of [7]) assumes a certain operator to be a contraction and essentially, imposes the rate of transition between states of the Markov chain not to be too large.

We also provide a result in terms of a stabilizing maximal solution - instead of a strong maximal solution - to the PARE. It is noteworthy that this result cannot be derived via the technique of proof of [7].

II. NOTATIONS AND PRELIMINARIES

As usual, $\mathbb{C}^n$ stands for the complex n-space. We denote by $\mathcal{S}=\{1,2,...,N\}$ the state space of the Markov chain associated to a MJLS. We use the superscript * for conjugate transpose of a matrix. We call $\mathbb{M}(\mathbb{C}^m,\mathbb{C}^n)$ the normed linear space of all n by m complex matrices and, for simplicity, write $\mathbb{M}(\mathbb{C}^n)$ whenever $n=m$. The notation $L\geq 0$ is adopted if a self-adjoint matrix is nonnegative and we write $\mathbb{M}(\mathbb{C}^n)^+=\{L\in\mathbb{M}(\mathbb{C}^n); L=L^*\geq 0\}$. Furthermore, I_n stands for the identity operator in $\mathbb{M}(\mathbb{C}^n)$.

We denote by $\|\cdot\|$ the norm in $\mathbb{C}^n$ or the spectral induced norm in $\mathbb{M}(\mathbb{C}^n)$. We denote by $\mathcal{H}^{mnN}$ the linear space of all finite sequences of complex matrices $H=(H_1,...,H_N)$, $H_i\in\mathbb{M}(\mathbb{C}^m,\mathbb{C}^n)$ and write $\mathcal{H}^{nN}$ whenever $n=m$.

We define the nonnegative set $\mathcal{H}^{nN+}=\{H\in\mathcal{H}^{nN}, H_i\in\mathbb{M}(\mathbb{C}^n)^+,\ i\in\mathcal{S}\}$, the strictly positive set $\breve{\mathcal{H}}^{nN+}=\{H\in\mathcal{H}^{nN+},\ H_i>\alpha_H I$ for some $\alpha_H>0,\ i\in\mathcal{S}\}$ and the set $\mathcal{H}^{n*}=\{H\in\mathcal{H}^{nN},\ H_i^*=H_i,\ i\in\mathcal{S}\}$. For $H=(H_1,...,H_N)$ and $L=(L_1,...,L_N)$ in $\mathcal{H}^{nN*}$, we say that $H\leqslant L$ if $H_i\leqslant L_i$ for each i in $\mathcal{S}$ and, for H, $L\in\mathcal{H}^{nN+}$, we have that $H\leqslant L\Rightarrow\|H\|\leqslant\|L\|$. For $C=(C_1,...,C_N)\in\mathcal{H}^{nN}$, we denote $C^*=(C_1^*,...,C_N^*)\in\mathcal{H}^{nN}$ and $C^{-1}=(C_1^{-1},...,C_N^{-1})\in\mathcal{H}^{nN}$ whenever C_i^{-1}, $i\in\mathcal{S}$, are invertible.

We denote by $\sigma(L)$ the spectrum of L, L being an arbitrary linear transformation defined on $\mathcal{H}^{nN}$ into itself.

We define the product of an element $A\in\mathcal{H}^{mnN}$ by another element $B\in\mathcal{H}^{qmN}$ by

$$AB=(A_1B_1,...,A_NB_N), \tag{2}$$

which belongs to $\mathcal{H}^{qnN}$. Clearly, $\mathcal{H}^{nN}$ equipped with (2) is a Banach algebra with identity $(I_n,...,I_n)$. Finally, we denote by E the expectation operator.

III. PROBLEM STATEMENT

We consider the perturbed algebraic Riccati equation (PARE) as given in [7] which, set to the Markovian jump scenario, reads as follows:

$$(A+\frac{1}{2}\lambda I)^*S+S(A+\frac{1}{2}\lambda I)+\chi(S)+Q-SBR^{-1}B^*S=0 \tag{3}$$

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with $\lambda = (\lambda_{11}, ..., \lambda_{NN})$, $I = (I_n, ..., I_n) \in \mathcal{H}^{nN}$, $A = (A_1, ..., A_N) \in \mathcal{H}^{nN}$, $B = (B_1, ..., B_2) \in \mathcal{H}^{mnN}$, $Q = (Q_1, ..., Q_N) \in \mathcal{H}^{nN+}$ and $R = (R_1, ..., R_N) \in \breve{\mathcal{H}}^{mN+}$. Multiplication in (3) goes as (2). We specify $\chi = (\chi_1, ..., \chi_N) : \mathcal{H}^{nN} \rightarrow \mathcal{H}^{nN}$ as it appears in problems involving MJLSs (see, e.g., [11] for the infinite countable case and references therein):

$$\chi_i(H) = \sum_{j=1, j\neq i}^{N} \lambda_{ij} H_j, \ i = 1, ..., N. \tag{4}$$

Viewing H as a column of N matrices, $\chi H = ((\Lambda - diag(\lambda_{ii})) \otimes I_n)H$, where $\Lambda = [\lambda_{ij}]_{i,j\in\mathcal{S}}$ is the infinitesimal matrix of a standard conservative Markov chain $\{\theta\}$ with values in $\mathcal{S}$, $\lambda_{ij} \geq 0$, $i \neq j$, $0 < -\lambda_{ii} = \sum_{j=1, j\neq i}^{N} \lambda_{ij} < \infty$, $i = 1, ..., N$. Note that χ is a positive operator in that it maps $\mathcal{H}^{nN*}$ into $\mathcal{H}^{nN*}$ and $\mathcal{H}^{nN+}$ into $\mathcal{H}^{nN+}$ and is responsible for the interconnection among the lines of (3).

We say that S is a solution to (3) if it belongs to $\mathcal{H}^{nN*}$ and satisfies (3).

Alternatively, we may write (3) as the following finite set of the coupled perturbed Riccati equations:

$$\begin{aligned}&(A_i + \frac{1}{2}\lambda_{ii} I_n)^* S_i + S_i(A_i + \frac{1}{2}\lambda_{ii} I_n) + \chi_i(S)\\ &+Q_i - S_i B_i R_i^{-1} B_i^* S_i = 0, \quad i \in \mathcal{S}.\end{aligned} \tag{5}$$

Essentially, our problem is showing that there is a (unique) maximal solution, say $\bar{S}$, to (3) (i.e., there is $\bar{S} > S$ for every solution $S \neq \bar{S}$) and that $\bar{S}$ is a strong solution (as termed in Definition 6).

Remark 1: We may view $A, B, ...$in (3) as diagonal matrices of the type

$$C = diag\,(C_i) \equiv \begin{bmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_N \end{bmatrix} \tag{6}$$

where product is defined as the usual matrix product and spaces $\mathcal{H}^{\cdot}$ are now matrices spaces of the above type. Note that if we replace the spaces $\mathcal{H}^{mnN}$ and $\mathcal{H}^{nN}$ by $\mathbb{M}(\mathbb{C}^{mN}, \mathbb{C}^{nN})$ and $\mathbb{M}(\mathbb{C}^{nN})$, and allow solutions to (3) in $\mathbb{M}(\mathbb{C}^{nN})$, these would be equivalence classes, each with a representative solution S shaped as (6). Indeed, if we allow the solution space to be $\mathbb{M}(\mathbb{C}^{nN})$ and $\hat{S} \in \mathbb{M}(\mathbb{C}^{nN})$ is a solution to (3), then inspection of (3) clearly shows us that so is $diag(\hat{S}_i)$ where $\hat{S}_i$ are $n \times n$ block diagonal matrices of $\hat{S}$. Since the space of all $n \times n$ matrices shaped as (6) and $\mathcal{H}^{nN}$ are isomorfic, we can view S as belonging to $\mathcal{H}^{nN}$.

IV. RELATED STABILITY ASPECTS

Let the operator $\Gamma = (\Gamma_1, ..., \Gamma_N) : \mathcal{H}^{nN} \rightarrow \mathcal{H}^{nN}$ be such that, for $U = (U_1, ..., U_N)$,

$$\Gamma_i(U) = \sum_{j=1, j\neq i}^{N} \lambda_{ji} U_j, \quad i \in \mathcal{S}.$$

For $A \in \mathcal{H}^{nN}$, $B \in \mathcal{H}^{mnN}$ and $K \in \mathcal{H}^{nmN}$, define $\mathcal{D} = (\mathcal{D}_1, ..., \mathcal{D}_N) : \mathcal{H}^{nN} \rightarrow \mathcal{H}^{nN}$, such that, for every $W = (W_1, ..., W_N)$ and each $i \in \mathcal{S}$,

$$\begin{aligned}&\mathcal{D}_i(W) = (A_i + \tfrac{1}{2}\lambda_{ii} I_n - B_i K_i) W_i\\ &+W_i(A_i + \tfrac{1}{2}\lambda_{ii} I_n - B_i K_i)^* + \Gamma_i(W).\end{aligned} \tag{7}$$

In view of (2) and (7), $\mathcal{D}$ also writes

$$\begin{aligned}&\mathcal{D}(W) = (A + \tfrac{1}{2}\lambda I - BK)W\\ &+W(A + \tfrac{1}{2}\lambda I - BK)^* + \Gamma(W),\end{aligned} \tag{8}$$

In order to emphasize the structural aspect, which associate the equation to the relevant parameters, we denote by (A, B, Γ) the N-dimensional differential equation

$$\dot{W}(t) = \mathcal{D}(W(t)), \quad t \geq 0. \tag{9}$$

This equation describes the behavior of a version of the state correlation matrix running in MJLSs. We define the following concept of stability, preserving the nomenclature of the MJLS scenario:

Definition 2 (Mean Square Stability for (9)): We say that (A, B, Γ) is mean square stabilizable (MSS) if there exists (a stabilizing) $K \in \mathcal{H}^{nmN}$ such that, for any $W(0) \in \mathcal{H}^{nN}$,

$$\|W(t)\| \rightarrow 0 \ as \ t \rightarrow \infty, \tag{10}$$

where $W(t) \in \mathcal{H}^{nN}$ is given by (9).

The intimate relation of (9) with MJLSs can be ilustrated by the fact that the Definition (2) and the Definition (12) of the Appendix are equivalent.

Remark 3: It is worth noticing that Γ, whose matrix representation is $((\Lambda - \ diag(\lambda_{ii})) \otimes I_n)^*$, is the adjoint operator of χ in the Hilbert space of finite sequences of matrices (N matrices) in $\mathbb{M}(\mathbb{C}^n)$ with norm $\|H\|_2^2 = \sum_{j=1}^{N} \|H_j\|^2$.

Now an equivalence lemma.

Lemma 4: The following assertions are equivalent.

(a) (A, B, Γ) is mean square stabilizable (MSS) with stabilizing K.

(b) Given any $V \in \breve{\mathcal{H}}^{nN+}$, there is $S \in \breve{\mathcal{H}}^{nN+}$, unique in $\mathcal{H}^{nN*}$, satisfying the countably infinite set of perturbed coupled Lyapunov equations given by

$$\begin{aligned}&(A_i + \frac{1}{2}\lambda_{ii} I_n - B_i K_i)^* S_i + S_i(A_i + \frac{1}{2}\lambda_{ii} I_n - B_i K_i)\\ &+\chi_i(S) + V_i = 0, \quad i \in \mathcal{S},\end{aligned} \tag{11}$$

or equivalently

$$\begin{aligned}&(A + \frac{1}{2}\lambda I - BK)^* S + S(A + \frac{1}{2}\lambda I - BK)\\ &+\chi(S) + V = 0.\end{aligned} \tag{12}$$

(c) Given some $V \in \breve{\mathcal{H}}_{\infty}^{n^+}$, there is $S \in \breve{\mathcal{H}}_{\infty}^{n^+}$ satisfying (11) or equivalently (12).

(d) $\sup\{ \ Re\lambda : \lambda \in \sigma(\mathcal{D})\} < 0$.

Proof: It follows from a particularization of the proofs of Lemmas 4.3 and 6.6 of [11] and Theorems 7 and 8 of [12] to the finite dimensional case, also reminding that in this (finite dimensional) case the concepts of stochastic stability defined in [11] and [12] and that of mean square stability are equivalent (see [10]). ■

This motivates the following definitions.

Definition 5: S is a stabilizing solution to the PARE if S belongs to $\mathcal{H}^{nN+}$, is a solution to the PARE and $K = R^{-1}B^*S$ stabilizes (A, B, Γ).

Definition 6: S is a strong solution to the PARE if S belongs to $\mathcal{H}^{nN+}$, is a solution to the PARE and

$$\sup\{Re\lambda : \lambda \in \sigma(\mathcal{D})\} \leq 0,$$

with $\mathcal{D}$ equipped with $K = R^{-1}B^*S$.

V. RESULTS

We start with a preparatory Lemma.

Lemma 7: Let (A, B, Γ) be MSS with stabilizing $K^1 \in \mathcal{H}^{nmN}$ and consider the sequences $(K^j)_{j\in\mathbb{N}}$ and $(S^j)_{j\in\mathbb{N}}$ as follows. For given K^j, S^j satisfies

$$\begin{aligned}&(A + \tfrac{1}{2}\lambda I - BK^j)^*S^j + S^j(A + \tfrac{1}{2}\lambda I - BK^j)\\ &+K^{j*}RK^j = -\chi(S^j) - Q\end{aligned} \tag{13}$$

and

$$K^{j+1} = R^{-1}B^*S^j, \quad j \in \mathbb{N}. \tag{14}$$

Also define the sequence $(\mathcal{D}^j)_{j\in\mathbb{N}}$ of operators with

$$\begin{aligned}&\mathcal{D}^j \text{ as in the right hand side of}\\ &(8) \text{ equipped with } K = K^j.\end{aligned} \tag{15}$$

Then $(S^j)_{j\in\mathbb{N}}$ exists in $\breve{\mathcal{H}}^{nN+}$, is non-increasing and uniquely defined in $\mathcal{H}^{nN*}$ and, for each j, $S^j \geq \hat{S}$, $\hat{S} \in \mathcal{H}^{nN*}$ being an arbitrary solution to (3). Also, $K^j \in \mathcal{H}^{nmN}$, $j \in \mathbb{N}$, are stabilizing for (A, B, Γ), or equivalently, $\sup\{Re\lambda : \lambda \in \sigma(\mathcal{D}^j)\} < 0 \quad \forall j \in \mathbb{N}$.

Proof: Let us first write (3) as the following system:

$$\begin{aligned}&(A + \tfrac{1}{2}\lambda I - BK)^*S + S(A + \tfrac{1}{2}\lambda I - BK)\\ &+K^*RK = -\chi(S) - Q\end{aligned} \tag{16}$$

$$K = R^{-1}B^*S. \tag{17}$$

An essential property we use here is that, for arbitrarily fixed $S \in \mathcal{H}^{nN*}$, $K_0 = R^{-1}B^*S$ is a point of minimum, in that, for every $K \in \mathcal{H}^{nmN}$

$$\begin{aligned}&(A + \frac{1}{2}\lambda I - BK)^*S \\ &+ S(A + \frac{1}{2}\lambda I - BK) + K^*RK - U\\ &= (A + \frac{1}{2}\lambda I - BK_0)^*S\\ &+ S(A + \frac{1}{2}\lambda I - BK_0) + K_0^*RK_0,\end{aligned} \tag{18}$$

where $U := (K - K_0)^*R(K - K_0) \in \breve{\mathcal{H}}^{nN+}$, $K \neq K_0$.

We use induction to proceed with the proof. So, for arbitrary $j \in \mathbb{N}$, assume that an element $K^j \in \mathcal{H}^{nmN}$ stabilizes (A, B, Γ). Note that $V^j := K^{j*}RK^j + Q$ belongs to $\breve{\mathcal{H}}^{nN+}$. Then, from Lemma 4 ($(a) \Rightarrow (b)$), there exists $S^j \in \breve{\mathcal{H}}^{nN+}$, unique in $\mathcal{H}^{nN*}$, that satisfies

$$\begin{aligned}&(A + \tfrac{1}{2}\lambda I - BK^j)^*S^j\\ &+S^j(A + \tfrac{1}{2}\lambda I - BK^j) + \chi(S^j) + V^j = 0,\end{aligned} \tag{19}$$

or else, (13). Let us now show that $S^j \geq \hat{S}$, $\hat{S} \in \mathcal{H}^{nN*}$ being an arbitrary solution to (3) or, equivalently, system (16)/(17) with K replaced by $\hat{K}$. From the minimum property given by (18),

$$\begin{aligned}&(A + \tfrac{1}{2}\lambda I - BK^j)^*\hat{S} + \hat{S}(A + \tfrac{1}{2}\lambda I - BK^j)\\ &+K^{j*}RK^j - (K^j - \hat{K})^*R(K^j - \hat{K})\\ &= -\chi(\hat{S}) - Q\end{aligned} \tag{20}$$

Subtracting (20) from (13), we have that

$$\begin{aligned}&(A + \tfrac{1}{2}\lambda I - BK^j)^*\phi^j + \phi^j(A + \tfrac{1}{2}\lambda I - BK^j)\\ &+\chi(\phi^j) + Y^j = 0,\end{aligned} \tag{21}$$

where $\phi^j := S^j - \hat{S}$ and $Y^j := (K^j - \hat{K})^*R(K^j - \hat{K})$. Now, Y^j belongs to $\breve{\mathcal{H}}^{nN+}$ if $K^j \neq \hat{K}$ and by assumption K^j stabilizes (A, B, Γ). Then the solution of (21) is unique and belongs to $\breve{\mathcal{H}}^{nN+}$, as Lemma 4 ($(a) \Rightarrow (b)$) shows. Hence this is the case of $\phi^j := S^j - \hat{S}$. Since both S^j and $\hat{S}$ belong to $\mathcal{H}^{nN*}$, it follows that $S^j \geq \hat{S}$. If $K^j = \hat{K}$, then $S^j = \hat{S}$.

Again from the minimum property, $K^{j+1} = R^{-1}B^*S^j$ minimizes the left hand side of (13), so we have that

$$\begin{aligned}&(A + \tfrac{1}{2}\lambda I - BK^{j+1})^*S^j + S^j(A + \tfrac{1}{2}\lambda I - BK^{j+1})\\ &+K^{j+1*}RK^{j+1} + \chi(S^j) + Z^j = 0,\end{aligned} \tag{22}$$

$Z^j := (K^{j+1} - K^j)^*R(K^{j+1} - K^j) + Q \in \breve{\mathcal{H}}^{nN+}$ if $K^{j+1} \neq K^j$. Hence, for some element in $\breve{\mathcal{H}}^{nN+}$ (Z^j), there is an element in $\breve{\mathcal{H}}^{nN+}$ (S^j) that satisfies (22). From Lemma 4 ($(c) \Rightarrow (a)$), K^{j+1} stabilizes (A, B, Γ). If $K^{j+1} = K^j$, the same conclusion is obvious.

For the step $j + 1$, we note as before that $V^{j+1} := K^{j+1*}RK^{j+1} + Q$ belongs to $\breve{\mathcal{H}}^{nN+}$ and K^{j+1} stabilizes $(\hat{A}, B, \Gamma)$ so that, from Lemma 4 ($(a) \Rightarrow (b)$), there exists a unique $S^{j+1} \in \breve{\mathcal{H}}^{nN+}$ that satisfies

$$\begin{aligned}&(A+\frac{1}{2}\lambda I-BK^{j+1})^*S^{j+1}+S^{j+1}(A+\frac{1}{2}\lambda I-BK^{j+1})\\ &+\chi(S^{j+1}) + V^{j+1} = 0,\end{aligned}$$

or else,

$$\begin{aligned}&(A+\tfrac{1}{2}\lambda I-BK^{j+1})^*S^{j+1} + S^{j+1}(A+\tfrac{1}{2}\lambda I-BK^{j+1})\\ &+K^{j+1*}RK^{j+1} = -\chi(S^{j+1}) - Q.\end{aligned} \tag{23}$$

Subtracting (23) from (22) we obtain

$$\begin{aligned}&(A + \tfrac{1}{2}\lambda I - BK^{j+1})^*\Delta^j\\ &+\Delta^j(A + \tfrac{1}{2}\lambda I - BK^{j+1}) + \chi(\Delta^j) + U^j = 0,\end{aligned} \tag{24}$$

where $U^j := (K^{j+1} - K^j)^*R(K^{j+1} - K^j)$ and Δ^j exists and is defined by $\Delta^j := S^j - S^{j+1}$. Now, U^j belongs to $\breve{\mathcal{H}}^{nN+}$

if $K^{j+1} \neq K^j$ and K^{j+1} stabilizes (A, B, Γ), so that, using Lemma 4 ($(a) \Rightarrow (b)$), the solution of (24) is unique and belongs to $\breve{\mathcal{H}}^{nN+}$. Hence this is the case of Δ^j. Since both S^j and S^{j+1} belong to $\mathcal{H}^{nN*}$, it follows that $S^j \geq S^{j+1}$. Clearly, if $K^{j+1} = K^j$,then $S^j = S^{j+1}$. This completes the induction. Furthermore, once K^j stabilize (A, B, Γ), we have from Lemma (4) ($(a) \Rightarrow (d)$) that $\sup\{ Re\lambda : \lambda \in \sigma(\mathcal{D}^j)\} < 0 \; \forall \; j \in \mathbb{N}$. ■

The theorem that follows recovers the result of [7] set to the Markovian jump scenario, now free from the inconvenient contraction assumption used in [7] and [20].

Theorem 8: Let (A, B, Γ) be MSS. Then the PARE (3) has a (unique) maximal solution, say $\bar{S}$, in the set of all solutions in $\mathcal{H}^{nN*}$, and $\bar{S} \in \mathcal{H}^{nN+}$. Moreover, $\bar{S}$ is a strong solution to the PARE.

Proof: We show that $\{S^j\}_{j\in\mathbb{N}}$ as given in Lemma 7 converges in the sense of the right hand side of (25) to the maximal solution of (3) or equivalently, system (16)/(17), and that the other assertions of the theorem follows. From Lemma 7, $\{S^j\}_{j\in\mathbb{N}}$ is non-increasing and bounded from below by zero. So, from a finite dimensional result on nonnegative matrices, there exists the limit in $\mathbb{M}(\mathbb{C}^n)$

$$\bar{S}_i := \lim_{j\to\infty} S_i^j, \quad \forall i \in \mathcal{S}. \tag{25}$$

Let $\hat{S} = (\hat{S}_1, ..., \hat{S}_N) \in \mathcal{H}^{nN*}$ be an arbitrary solution to (3). Then, from (25) and the fact that $S_i^j \geq \hat{S}_i \; \forall i, j$ (recall, from Lemma 7, that $S^j \geq \hat{S}$), it follows that $\bar{S}_i \geq \hat{S}_i \; \forall i$. Lemma 7 also gives us that S^j belongs to $\mathcal{H}^{nN+}$ and so $\mathcal{H}^{nN+} \ni \bar{S} \geq \hat{S}$. Now, from (14), there exists the limit

$$\bar{K}_i := \lim_{j\to\infty} K_i^{j+1} = \lim_{j\to\infty} R_i^{-1} B_i^* S_i^j = R_i^{-1} B_i^* \bar{S}_i \tag{26}$$

and we define $\bar{K} := (\bar{K}_1, ..., \bar{K}_N) \in \mathcal{H}^{nmN}$. Also,

$$\chi_i(S^j) := \sum_{r=1, r\neq i}^{N} \lambda_{ir} S_r^j \to \sum_{r=1, r\neq i}^{N} \lambda_{ir} \bar{S}_r =: \chi_i(\bar{S}) \;\text{ as } j \to \infty. \tag{27}$$

Passing (13) to the limit for an arbitrarily fixed entry $i \in \mathcal{S}$, we obtain

$$\begin{aligned} &(A_i + \tfrac{1}{2}\lambda_{ii} I_n - B_i \bar{K}_i)^* \bar{S}_i \\ &+ \bar{S}_i (A_i + \tfrac{1}{2}\lambda_{ii} I_n - B_i \bar{K}_i) \\ &+ \bar{K}_i^* R_i \bar{K}_i = -\chi_i(\bar{S}) - Q_i \end{aligned}$$

or else,

$$\begin{aligned} &(A + \tfrac{1}{2}\lambda I - B\bar{K})^* \bar{S} + \bar{S}(A + \tfrac{1}{2}\lambda I - B\bar{K}) \\ &+ \bar{K}^* R \bar{K} = -\chi(\bar{S}) - Q \end{aligned}$$

So, $\bar{S}$ satisfies (16)/(17).

Now, define $\bar{\mathcal{D}}$ as in the right hand side of (8) equipped with $\bar{K} = R^{-1} B^* \bar{S}$ instead of K. The space of all linear operators defined on $\mathcal{H}^{nN}$ into itself with the norm topology (generated by the uniform induced norm) - to which $\bar{\mathcal{D}}$ and $\mathcal{D}^j$ belong, is a finite dimensional Banach algebra and so the spectrum - the set valued function $\sigma(\cdot)$ defined on this space into $\mathbb{C}$, is continuous everywhere (see, e.g., [3] and [16, pg 56]), in particular at $\bar{\mathcal{D}}$. Also, from (26), $\bar{K}_i = \lim_{j\to\infty} K_i^j$ for each i which, in the finite dimensional case, suffices to insure the convergence $\mathcal{D}^j \to \bar{\mathcal{D}}$ in the norm topology. Hence, since every $\mathcal{D}^j$ is stable (in the sense of Lemma 4 (d)), it follows from the continuity of the spectrum that $\sup \{Re\lambda : \lambda \in \sigma(\bar{\mathcal{D}})\} \leq 0$. ■

The corollary that follows provides a result in terms of a stabilizing solution, instead of a strong solution, to the PARE. It is worth mentioning that this result cannot be derived using the technique of [7].

Corollary 9: Suppose, in Theorem 8, that Q and some solution to the PARE belong to $\breve{\mathcal{H}}^{nN+}$. Then $\bar{S} \in \breve{\mathcal{H}}^{nN+}$ is a stabilizing solution to the PARE, or else, $\sup\{ Re\lambda : \lambda \in \sigma(\bar{\mathcal{D}})\} < 0$.

Proof: Taking arbitrarily the i^{th} line of (22), passing it to the limit as $j \to \infty$ and using (27), we obtain

$$\begin{aligned} &(A_i + \tfrac{1}{2}\lambda_{ii} I_n - B_i \bar{K}_i)^* \bar{S}_i \\ &+ \bar{S}_i (A_i + \tfrac{1}{2}\lambda_{ii} I_n - B_i \bar{K}_i) \\ &+ \bar{K}_i^* R_i \bar{K}_i + \chi_i(\bar{S}) + Q_i = 0, \end{aligned} \tag{28}$$

So, for some element in $\breve{\mathcal{H}}^{nN+}$ (Q), there is an element in $\breve{\mathcal{H}}^{nN+}$ $(\bar{S})$ that satisfies (28). From Lemma 4 ($(c) \Rightarrow (a)$) and (26), $\bar{K} = R^{-1} B^* \bar{S}$ stabilizes $(\hat{A}, B, \Gamma)$. Lemma 4 ($(a) \Leftrightarrow (d)$) completes the result. ■

We replicate below Lemma 4.1 of [13] due to its liaison to the subject matter dealt with in this part of the section.

Lemma 10: The PARE has at most one stabilizing solution. If it exists, it is the maximal solution in the set of all solutions in $\mathcal{H}^{nN*}$.

Remark 11: Spectral continuity analysis (see, e.g., [3], [16, pp 56] and references therein) bases its results on the following definitions of $\limsup$ and $\liminf$:

For a given sequence $(\alpha_n)_{j\in\mathbb{N}}$ in $\mathbb{C}$ of non-empty compact sets:

(i) $\limsup_n \alpha_n := \{\lambda \in \mathbb{C} : \lambda = \lim_{k\to\infty} \lambda_{n_k}$ where $\lambda_{n_k} \in \alpha_{n_k}$ and $n_{k+1} > n_k\}$,

(ii) $\liminf_n \alpha_n := \{\lambda \in \mathbb{C} : \lambda = \lim_{n\to\infty} \lambda_n$ where $\lambda_n \in \alpha_n\}$ and

(iii) $\lim_{n\to\infty} \alpha_n := \limsup_n \alpha_n = \liminf_n \alpha_n$, whenever the second equality holds

The sets $\limsup_n \alpha_n$ and $\liminf_n \alpha_n$ defined above slightly differ from $\limsup_n \alpha_n = \cap_{n\geq 1} \cup_{k\geq n} \alpha_k$ and $\liminf_n \alpha_n = \cup_{n\geq 1} \cap_{k\geq n} \alpha_k$, respectively. In fact, the former sets are the closure of the latter. This difference is perceptible in both finite (when only the point spectrum is present) and infinite dimensional cases. This is the reason for obtaining in theorem 8 "a strong solution" instead of "a stabilizing solution".

VI. APPENDIX

A. Support to Section IV

Definition 12 (Mean Square Stability for MJLSs): We say that $(A, B, \Lambda))$ is mean square stabilizable (MSS) if there exists (a stabilizing) $K \in \mathcal{H}^{nmN}$ such that, for any joint initial distribution ϑ_0,

$$\int_0^\infty E[\|x(t)\|^2]dt < \infty, \tag{29}$$

where $x(t)$ is given by (1) with $u(t) = -K_{\theta(t)}x(t)$, or else, by

$$(A, B, \Lambda) : \begin{cases} \dot{x}(t) = (A_{\theta(t)} - B_{\theta(t)}K_{\theta(t)})x(t), & t \geq 0 \\ x(0) = x_0, \quad \theta(0) = \theta_0. \end{cases} \tag{30}$$

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