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A Power-Based Perspective of Mechanical Systems

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Abstract— **This paper is concerned with the construction of a power-based modeling framework for a large class of mechanical systems. Mathematically this is formalized by proving that every standard mechanical system (with or without dissipation) can be written as a gradient vector field with respect to an indefinite metric. The form and existence of the corresponding potential function is shown to be the mechanical analogue of Brayton and Moser's mixed-potential function as originally derived for nonlinear electrical networks in the early sixties. In this way, several recently proposed analysis and control methods that use the mixed-potential function as a starting point can also be applied to mechanical systems.**

I. Introduction and Motivation

It is well-known that a large class of physical systems (e.g., mechanical, electrical, electro-mechanical, thermodynamical, etc.) admits, at least partially, a representation by the Euler-Lagrange or Hamiltonian equations of motion, see e.g. [1], [6], [8], [9], [10], and the references therein. A key aspect for both sets of equations is that the energy storage in the system plays a central role. For standard mechanical systems with n degrees of freedom, and locally represented by n generalized displacement coordinates $q = \mathrm{col}(q_1, \ldots, q_n) \in \mathbb{Q}$, the Euler-Lagrange (EL) equations of motion are given by[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\nabla_{\dot q}\mathcal{L}(q,\dot q)\Big) - \nabla_q \mathcal{L}(q,\dot q) = \tau, \tag{1}$$

where $\dot q = \mathrm{col}(\dot q_1, \ldots, \dot q_n) \in \mathbb{V}$ denote the generalized velocities, and $\mathcal{L} : \mathbb{Q} \times \mathbb{V} \to \mathbb{R}$ represents the Lagrangian which is defined by the difference between the kinetic co-energy and the potential energy. Usually the forces τ are decomposed into dissipative forces and generalized external forces.

The relation between the Euler-Lagrange equations and the Hamiltonian equations is classically established as follows. Defining the generalized momenta $p = \nabla_{\dot q}\mathcal{L}(q,\dot q)$, with $p = \mathrm{col}(p_1, \ldots, p_n) \in \mathbb{P}$, the equations of motion, as originally described by the set of second-order equations (1), can be written as a set of $2n$ first-order equations:

$$\dot q = \nabla_p \mathcal{H}(q,p), \quad \dot p = -\nabla_q \mathcal{H}(q,p) + \tau. \tag{2}$$

Here, $\mathcal{H} : \mathbb{Q} \times \mathbb{P} \to \mathbb{R}$ denotes the Hamiltonian which represents the sum of the kinetic and potential energy.

The relationship between (1) and (2) is graphically represented in the diagram shown in Fig. 1 (solid lines). Clearly, the diagram suggests that there exists a dual form of (1) in the sense that a mechanical system can be expressed in terms of

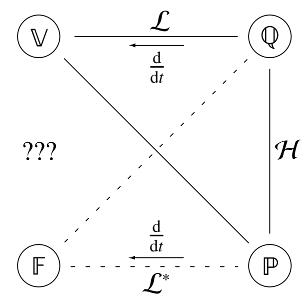

Fig. 1. Mechanical configuration space quadrangle: The symbols $\mathbb{Q}$, $\mathbb{P}$, $\mathbb{V}$ and $\mathbb{F}$ denote the spaces of the generalized displacements, momenta, velocities and forces. The solid and dashed diagonal lines represent the directions for the Legendre transformations of the Lagrangian and co-Lagrangian, respectively, in relation to the Hamiltonian; the question marks denote the fourth equation set to be explored in this paper. Notice that the relation between the spaces $\mathbb{Q}$ and $\mathbb{V}$, and similarly between $\mathbb{P}$ and $\mathbb{F}$, is the $\mathrm{d}/\mathrm{d}t$ operator.

a set of generalized momenta and its time-derivatives, which represent a set of generalized forces. Indeed, in [6] a description of the dynamics in the generalized momentum and force spaces $\mathbb{P}$ and $\mathbb{F}$, respectively, is called a co-Lagrangian system, where the Lagrangian $\mathcal{L}$ in (1) is replaced by its dual form $\mathcal{L}^* : \mathbb{P} \times \mathbb{F} \to \mathbb{R}$, representing the difference between the potential co-energy and the kinetic energy, while the forces τ are replaced by external velocities τ^*, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\nabla_{\dot p}\mathcal{L}^*(p,\dot p)\Big) - \nabla_p \mathcal{L}^*(p,\dot p) = \tau^*, \tag{3}$$

with $\dot p = \mathrm{col}(\dot p_1, \ldots, \dot p_n) \in \mathbb{F}$. Hence, the co-Lagrangian system (3) represents a velocity-balance equation.

So far we have considered three possible representations describing the dynamics of a standard mechanical system. The underlying relationship between the three sets of equations is the existence of the Legendre transformations between $\mathbb{Q}$, $\mathbb{V}$, $\mathbb{P}$ and $\mathbb{F}$. Furthermore, the quadrangle of Fig. 1 also suggest a fourth equation set. Intuitively, at this point, one could be tempted to call a dynamic description on the spaces $\mathbb{V}$ and $\mathbb{F}$ the *co-Hamiltonian* equations of motion. Starting from the Hamiltonian equation set, if both the Legendre transformations of $\mathbb{Q} \to \mathbb{F}$ and $\mathbb{P} \to \mathbb{V}$ are considered simultaneously, one obtains $\mathcal{H}^* : \mathbb{F} \times \mathbb{V} \to \mathbb{R}$ which appears to be a bona-fide co-Hamiltonian candidate. Hence, based on the latter observation, and in comparison to (2), this would suggest that the 'co-Hamiltonian' equation set, should read:

$$\dot v = \nabla_f \mathcal{H}^*(v,f), \quad \dot f = -\nabla_v \mathcal{H}^*(v,f) + \phi,$$

where $v = \dot q$, and $f = \dot p$. However, the latter set of equations is *not* correctly describing the dynamics since the units of $\dot v$ and $\dot f$ do not coincide with the units of $\nabla_f \mathcal{H}^*(v,f)$ and

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[1] By ∇_x we denote the partial derivative operator $\frac{\partial}{\partial x}$. When clear from the context the subscript will be omitted.

$\nabla_v\mathcal{H}^*(v,f)$, respectively.[2] Furthermore, it is not clear how the external signals, represented by ϕ, relate to the original external signals τ and/or τ^*. Thus, the existence, form and meaning of the fourth description remains to be clarified.

In this paper, it is our objective to identify the fourth equation set (indicated by the question marks) in the quadrangle of Fig. 1, and to formally complete the relationships between the different sets of equations. It will be shown that the fourth equation set constitutes a mechanical analogue of the Brayton-Moser equations [2]. These equations constitute a gradient system with respect to an indefinite metric defined by the dynamic part of the network (capacitors and inductors), and a mixed-potential function which describes the static part (interconnection, resistors, and sources) of the network and has the units of power. Besides the completion of the quadrangle, the mechanical analogue of the Brayton-Moser equations can be useful for other features like:

- Stability analysis along the lines of [2]. The mixed-potential (and thus the power in the system) can be used to construct Lyapunov-type functions to prove stability under certain conditions — even in cases that the system contains (regions of) negative resistance! Additionally, the stability criteria stemming from this method can be used to find lower bounds on the control parameters when applying Passivity-Based Control[3], see e.g., [5] for some recent results in the field of electronic power converter control.
- Definition of new passivity properties along the lines of [3]. This includes the definition of alternative conjugated port-variables (inputs and outputs) with respect to an alternative storage function (i.e., the mixed-potential).
- The notion of the aforementioned new passivity properties have led to the paradigm of Power Shaping stabilization. Some recent applications to nonlinear RLC circuits have been reported in [7]. The Power-Shaping method is based on a particular selection of the input signals (the controls) as to shape the power flow (read: the mixed-potential).
- The Brayton-Moser equations seem to be a natural equation set in relation with bond-graph theory since the canonical state variables live in the flow and effort spaces.

Although there exists a widely accepted standard analogy between simple mechanical and electrical system elements, like for example, the 'spring-capacitor' and the 'mass-inductor' analogy, the existence of a well-defined analogy for more general mechanical systems is not straightforward. One of the main reasons for making such analogy difficult is the presence of the Coriolis and centrifugal forces, which do not appear as such in the electrical domain. Another difficulty is that, in contrast to electrical networks, mechanical systems are in general not nodical. Hence, a mechanical system can not always be considered as an interconnected graph. For these reasons, we can, in general, not equate the dynamics of a mechanical system mutatis-mutandis along the lines of [2]. For that, a more dedicated analysis is needed and a dedicated transformation algorithm that goes beyond the Legendre transformation needs to be developed.

The structure of the paper is as follows. Section II discusses the original form of the Brayton and Moser equations. In Section III, a lemma will be introduced which forms the key behind the main results presented in Section IV. The theory is exemplified using a well-known nonlinear mechanical system. The role of dissipative forces and velocities is studied in Section V. Finally, in Section VI, possible extensions of the theory and future research will be discussed.

II. The Brayton-Moser Equations

The Brayton-Moser (BM) equations as originally developed for a large class of nonlinear electrical RLC networks take the special gradient form

$$\underbrace{\begin{pmatrix} C(u) & 0 \\ 0 & -L(i) \end{pmatrix}}_{Q^e(u,i)} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ i \end{pmatrix} = \begin{pmatrix} \nabla_u \mathcal{P}^e(u,i) \\ \nabla_i \mathcal{P}^e(u,i) \end{pmatrix}, \tag{4}$$

where $u \in \mathbb{U}$ represents the voltages across the incremental capacitors $C(u)$, $i \in \mathbb{I}$ represent the currents through the incremental inductors $L(i)$, and $\mathcal{P}^e : \mathbb{U}\times\mathbb{I} \to \mathbb{R}$ is called the *mixed-potential* function which usually takes the form

$$\mathcal{P}^e(u,i) = i^\top \gamma u + \mathcal{G}^e(i) - \mathcal{J}^e(u). \tag{5}$$

Here γ is a unit-less matrix derivable from Kirchhoff's laws, whereas the functions $\mathcal{G}^e : \mathbb{I} \to \mathbb{R}$ and $\mathcal{J}^e : \mathbb{U} \to \mathbb{R}$ represent the *content* and *co-content*, respectively. It will appear later on that the (co-)content function plays a role similar to the Rayleigh (co-)dissipation function in a mechanical system. Note that if the network does not contain resistors and/or sources, the mixed-potential reduces to

$$\mathcal{P}^e(u,i) = i^\top \gamma u. \tag{6}$$

For ease of reference we introduce the definition:

Definition 1: A set of BM equations (4), defined on the voltage and current space $\mathbb{U}$ and $\mathbb{I}$, respectively, together with a mixed-potential of the form (5), is called a *canonical* BM description. Any other set of equations that admit structurally the *same* mixed-potential (5), though not necessarily defined on $\mathbb{U}$ and $\mathbb{I}$, is called a *homonymous* BM description.

Adopting the 'spring-capacitor' and 'mass-inductor' analogy, the construction of the mechanical analogue of (4) will basically be concerned with the construction of a mixed-potential of the form (5) in terms of mechanical forces and velocities, either directly in terms of $\mathbb{F}$ and $\mathbb{V}$ (i.e., canonical), or indirectly in terms of e.g., $\mathbb{Q}$ and $\mathbb{P}$ (i.e., homonymous). Additionally, the corresponding metric is desired to coincide with a form comparable to:

$$Q(\cdot) = \begin{pmatrix} \text{'springs'} & * \\ * & -\text{'mass'} \end{pmatrix}. \tag{7}$$

[2] Note that the units of $\nabla_f\mathcal{H}^*(v,f)$ and $\nabla_v\mathcal{H}^*(v,f)$ are displacement (or position) and momenta, while the units of $\dot{v}$ and $\dot{f}$ are the time-derivatives of velocity (i.e., acceleration) and force, respectively.

[3] Passivity-Based Control (PBC) is a control method that has its roots in the fi eld robots and the closely related Lagrangian framework. For a detailed elaboration on this subject the interested reader is referred to [8], and the references cited therein.

III. Preliminaries

As discussed in Section I, a standard mechanical system can be represented by the Hamiltonian equation set (2). For ease of presentation, we set the external forces $\tau = 0$ and rewrite (2) in a more compact form as

$$\dot{z} = J\nabla\mathcal{H}(z), \tag{8}$$

where $z = \mathrm{col}(q, p)$, and J is a skew-symmetric matrix of the form

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = -J^\top, \tag{9}$$

with I_n the $n \times n$ identity matrix. The Hamiltonian function $\mathcal{H}(z) = \mathcal{H}(q, p)$ is assumed to be

$$\mathcal{H}(q, p) = \tfrac{1}{2} p^\top M^{-1}(q)p + \mathcal{V}(q), \tag{10}$$

where $M(q) = M^\top(q) > 0$ is the inertia matrix, and $\mathcal{V} : \mathbb{Q} \to \mathbb{R}$ is a twice-differentiable potential energy function.

Clearly, for standard mechanical systems $J^{-1} = J^\top$ is well-defined. Hence, the Hamiltonian equations (8) can be rewritten as

$$J^{-1}\dot{z} = \nabla\mathcal{H}(z), \tag{11}$$

which directly gives rise to the suggestion of a BM type of gradient system (compare with (4)). However, apart from the fact that the system is described in terms of displacement and momenta instead of some force and velocity variables, the matrix J^{-1} is skew-symmetric and dimensionless, while the 'potential' function $\mathcal{H}(z)$ still represents the total energy. On the other hand, since we now formally have two different pairs, $\{J, \mathcal{H}\}$ and $\{J^{-1}, \mathcal{H}\}$, both describing the same dynamics, the next question is whether there exists other pairs, say $\{\tilde{J}, \tilde{\mathcal{H}}\}$ or $\{\tilde{J}^{-1}, \tilde{\mathcal{H}}\}$, that equivalently describe the system's dynamics. Borrowing inspiration from [2], such pairs can be generated as illustrated in the following lemma.

Lemma 1: Consider a standard mechanical system represented by (11). If $\nabla^2\mathcal{H}(z)$ is full-rank, then for any constant symmetric matrix K the dynamics of (11) can be equivalently expressed by

$$\tilde{J}^{-1}(z)\dot{z} = \nabla\tilde{\mathcal{H}}(z), \tag{12}$$

where

$$\tilde{\mathcal{H}}(z) = \tfrac{1}{2}(\nabla\mathcal{H}(z))^\top K\nabla\mathcal{H}(z), \tag{13}$$

and

$$\tilde{J}^{-1}(z) = \nabla^2\mathcal{H}(z)KJ^{-1}. \tag{14}$$

Proof: The result follows directly by computing the gradient of $\tilde{\mathcal{H}}(z)$ and substitution of (14) and (8). $\triangleleft$

Having made these observations, our next task is to select a constant and symmetric matrix K such that (13) coincides with the mechanical equivalent of (6), while (14) represents a metric similar to (7).

IV. Main Result

Theorem 1: Consider a standard mechanical system described by the Hamiltonian equations (11). The dynamics of (11) can be equivalently expressed as

$$Q(z)\dot{z} = \nabla\mathcal{P}(z), \tag{15}$$

where

$$Q(z) = \begin{pmatrix} \nabla_q^2\mathcal{V}(q) + \frac{1}{2}\nabla_q^2\big(p^\top M^{-1}(q)p\big) & -\nabla_q\big(p^\top M^{-1}(q)\big) \\ \nabla_q\big(M^{-1}(q)p\big) & -M^{-1}(q) \end{pmatrix}, \tag{16}$$

and

$$\begin{aligned}\mathcal{P}(z) = {} & (\nabla\mathcal{V}(q))^\top M^{-1}(q)p \\ & + \tfrac{1}{2}\left(\nabla_q\big(p^\top M^{-1}(q)p\big)\right)^\top M^{-1}(q)p.\end{aligned} \tag{17}$$

The pair (16) and (17) defines a homonymous BM description of mechanical type.

Proof: The key is to select in (13) and (14) of Lemma 1 a constant symmetric K-matrix such that

$$KJ^{-1} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \tag{18}$$

which means that K should be chosen as

$$K = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} = K^\top.$$

Hence, if we define $\mathcal{P}(z) \triangleq \tilde{\mathcal{H}}(z)$ and $Q(z) \triangleq \tilde{J}^{-1}(z)$, then by substitution of K into (13) and (14), we obtain (16) and (17) from $\mathcal{P}(q, p) \triangleq (\nabla_q\mathcal{H}(q, p))^\top\nabla_p\mathcal{H}(q, p)$ and

$$Q(q, p) \triangleq \begin{pmatrix} \nabla_q^2 & -\nabla_q\nabla_p \\ \nabla_p\nabla_q & -\nabla_p^2 \end{pmatrix}\mathcal{H}(q, p),$$

respectively. The claim that the pair (16) and (17) defines a homonymous BM description (in the light of Definition 1) follows from the fact that, although expressed in terms of q and p, the units and form of $\mathcal{P}(q, p)$ coincide with power, and correspond to the mechanical analogue of a mixed-potential for a lossless electrical network as given in (6). Indeed, this is most easily seen from (11) since the generalized velocities are defined by

$$\dot{q} = \nabla_p\mathcal{H}(q, p) = M^{-1}(q)p,$$

and the generalized forces[4] by

$$\dot{p} = -\nabla_q\mathcal{H}(q, p) = -\tfrac{1}{2}\nabla_q\big(p^\top M^{-1}(q)p\big) - \nabla_q\mathcal{V}(q).$$

Thus, according to the latter equations, $\mathcal{P}(q, p)$ as defined in (17), can be written in terms of $\dot{q} \in \mathbb{V}$ and $\dot{p} \in \mathbb{F}$ as

$$\mathcal{P}(\cdot) = -\dot{q}^\top\dot{p}, \tag{19}$$

i.e., $\mathcal{P}(\cdot)$ = (minus) velocity × force = power. $\triangleleft$

Regarding Theorem 1, we observe that the diagonal terms

[4]Note that the term $\frac{1}{2}\nabla_q(p^\top M^{-1}(q)p)$ is part of the centrifugal and Coriolis forces [10].

of the Q-matrix (16) correspond to the inverse of the diagonal terms of (7). Hence, $Q(q,p)$ in its present form is not (yet) interpretable as the precise mechanical analogue of $Q^{\mathrm{e}}(u,i)$ in (4). This is due to the fact the dynamics are still expressed in terms of the generalized displacements and momenta, instead of the generalized forces and velocities, respectively (i.e., the mechanical analogues to voltages and currents as adopted at the end of Section II). Furthermore, the presence of the skew-symmetric off-diagonal terms stem from the fact that the non-symmetrical part of the system's drift vector field $J\nabla\mathcal{H}(z)$ is shifted to the left-hand side of the equations in order to guarantee the integrability needed for the construction of the mixed-potential function.

As is highlighted in [4], and according to Definition 1, a precise mechanical analog (or, in a different parlance: a canonical BM equation set of mechanical type) of the BM equations (4) can only be obtained if the Legendre transformations from $\mathbb{P} \to \mathbb{V}$ and $\mathbb{Q} \to \mathbb{F}$, and preferably vice-versa, are well-defined relations. Unfortunately, in general this is not always the case. For example, if a system operates under the influence of a (constant) gravitational force, the mapping $q \mapsto f$ (recall that f represents the generalized forces) simply does not exist.[5] On the other hand, suppose for simplicity that $M(q)$ is constant, i.e., $M(q) = M$, then (16) reduces to

$$Q(q) = \begin{pmatrix} \nabla_q^2\mathcal{V}(q) & 0 \\ 0 & -M^{-1} \end{pmatrix}.$$

(Compare with Q^{e} in (4).) Hence, Eq. (15), together with the pair defined in (16) and (17), can be written as

$$\begin{pmatrix} \nabla_q^2\mathcal{V}(q) & 0 \\ 0 & -M^{-1} \end{pmatrix}\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \nabla_q\mathcal{P}(q,p) \\ \nabla_p\mathcal{P}(q,p) \end{pmatrix} = \begin{pmatrix} \nabla_q^2\mathcal{V}(q)M^{-1}p \\ M^{-1}\nabla_q\mathcal{V}(q) \end{pmatrix}. \tag{20}$$

Clearly, since M^{-1} is invertible by assumption (even in the non-constant case!), the Legendre transformation of $p \mapsto \dot{q}$ is well-defined, i.e., $p = M\dot{q}$, or equivalently, $\dot{q} = M^{-1}p$. Let again $v = \dot{q} \in \mathbb{V}$, denote the generalized velocities, then the second equation in (20) can be written as

$$-M\dot{v} = \nabla_q\mathcal{V}(q). \tag{21}$$

Moreover, if $\nabla_q^2\mathcal{V}(q)$ is full-rank and there exists a mapping $q \mapsto f$, we have with

$$\mathcal{V}^*(f) \triangleq q^{\mathsf{T}}f - \mathcal{V}(q), \quad f = \nabla_q\mathcal{V}(q),$$

that the first equation in (20) can be rewritten on the $(\mathbb{V},\mathbb{F})$-space as $\nabla_f^2\mathcal{V}^*(f)\dot{f} = v$. Furthermore, elimination of the q-dependency of $\nabla_q\mathcal{V}(q)$ in (21) by $q = \nabla_f\mathcal{V}^*(f)$, gives

$$-M\dot{v} = \nabla_q\mathcal{V}(q)\Big|_{q=\nabla_f\mathcal{V}^*(f)} = f.$$

Hence, the precise mechanical analogue of (4) is given by

$$\underbrace{\begin{pmatrix} \nabla_f^2\mathcal{V}^*(f) & 0 \\ 0 & -M \end{pmatrix}}_{\triangleq \tilde{Q}(f)}\begin{pmatrix} \dot{f} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \nabla_f\tilde{\mathcal{P}}(f,v) \\ \nabla_v\tilde{\mathcal{P}}(f,v) \end{pmatrix}, \tag{22}$$

[5]Of course, we could treat gravity as an external force, input or disturbance. However, in Hamiltonian mechanics gravity is usually included using the potential energy function.

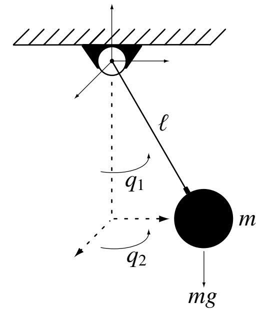

Fig. 2. A frictionless spherical pendulum.

where the associated mixed-potential is defined

$$\tilde{\mathcal{P}}(f,v) \triangleq \mathcal{P}(q,p)\Big|_{\substack{q=\nabla_f\mathcal{V}^*(f) \\ p=M\dot{q}}} = v^{\mathsf{T}}f. \tag{23}$$

Additionally, the previous observations indirectly clarify the role played by $\mathcal{H}^*(f,v)$, i.e., the co-Hamiltonian, as discussed in Section I, since

$$\begin{pmatrix} \nabla_f^2\mathcal{V}^*(f) & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} \dot{f} \\ \dot{v} \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \nabla_f\mathcal{H}^*(f,v) \\ \nabla_v\mathcal{H}^*(f,v) \end{pmatrix}.$$

A similar discussion hold for non-constant inertia matrices $M = M(q) > 0$, but it yields a more complex analysis which is omitted for sake of brevity. In conclusion for this part, we summarize the latter discussion in the following corollary:

Corollary 1: Consider a standard mechanical system described by the Hamiltonian equations (11). If the mapping $q \mapsto f$ is well-defined, then (15), together with the pair defined in (16) and (17), is the canonical mechanical analogue of the BM equations (4).

Let us next illustrate the application of Theorem 1 using an example.

Example 1: Consider the frictionless spherical pendulum shown in Figure 2. The system consists of a massless rigid rod of length ℓ fixed in one end by a spherical joint and having a bulb of mass m at the other end. Let q_1 and q_2 denote angles of the vertical and horizontal movements, and p_1 and p_2 the corresponding momenta. The configuration space of the system is $\mathbb{S}^2$, however we will assume that q_1 and q_2 remain inside the domain $]0,\pi[$ and $]0,2\pi[$, respectively. The Hamiltonian (i.e., the total stored energy) reads

$$\mathcal{H}(q,p) = \tfrac{1}{2}p^{\mathsf{T}}M^{-1}(q)p - mg\ell\cos(q_1),$$

where

$$M^{-1}(q) = \begin{pmatrix} \dfrac{1}{m\ell^2} & 0 \\ 0 & \dfrac{1}{m\ell^2\sin^2(q_1)} \end{pmatrix}.$$

The centrifugal and Coriolis forces are defined by the gradient of the kinetic energy with respect to q_1 and q_2, i.e.,

$$\tfrac{1}{2}\nabla_q\Big(p^\top M^{-1}(q)p\Big) = \begin{pmatrix} -\dfrac{\cos(q_1)}{m\ell^2\sin^3(q_1)}p_2^2 \\ 0 \end{pmatrix},$$

and the potential forces are

$$\nabla\mathcal{V}(q) = \begin{pmatrix} mg\ell\sin(q_1) \\ 0 \end{pmatrix}.$$

Application of Theorem 1 yields that the homonymous mixed-potential for the system is given by

$$\mathcal{P}(q,p) = \frac{g}{\ell}\sin(q_1)p_1 - \frac{\cos(q_1)}{m^2\ell^4\sin^3(q_1)}p_2^2 p_1.$$

Furthermore, we compute the matrix $Q(q,p)$ as

$$Q(q,p) = \begin{pmatrix} \Phi(q,p) & 0 & 0 & \dfrac{2\cos(q_1)}{m\ell^2\sin^3(q_1)}p_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{m\ell^2} & 0 \\ -\dfrac{2\cos(q_1)}{m\ell^2\sin^3(q_1)}p_2 & 0 & 0 & -\dfrac{1}{m\ell^2\sin^2(q_1)} \end{pmatrix},$$

where

$$\Phi(q,p) \triangleq mg\ell\cos(q_1)p_1 - \left(\frac{3\cos^2(q_1)}{m\ell^2\sin^4(q_1)} + \frac{1}{m\ell^2\sin^2(q_1)}\right)p_2^2 p_1.$$

We directly observe that the system is not minimal in the sense that $Q(q,p)$ is rank deficient. However, since q_2 does not explicitly contribute to the dynamics (also not in the original Hamiltonian model), we may delete the second row and column of $Q(q,p)$ as to obtain a minimal homonymous BM description. Also note that the mapping $q \mapsto f$ is not globally defined, and thus we can not obtain a canonical BM equation set for this system.

V. On the Role of Dissipation

In the previous sections we have concentrated on standard mechanical systems without any external disturbances or dissipative forces. In this section we generalize our developments further by studying the effect of dissipative forces and velocities working on the system. An ideal (translational or rotational) mechanical dissipator is defined as an object which exhibits no kinetic or potential effects. In the analysis hereafter, we assume for simplicity that the dissipators are linear and time-invariant.

A. Mechanical Content and Co-Content

As illustrated in [10], linear dissipation effects are included into a Lagrangian or Hamiltonian equation set by applying a constant negative gain feedback of the associated velocities and forces. For a mechanical system of the form (8) this means that the resulting (closed-loop) system takes the form:

$$\dot{z} = (J - D)\nabla\mathcal{H}(z), \tag{24}$$

where

$$D = \begin{pmatrix} G & 0 \\ 0 & R \end{pmatrix}, \tag{25}$$

with $G = G^\top \geq 0$ and $R = R^\top \geq 0$. We can define in a manner analogues to the definition of electrical resistors

$$\mathcal{G}(v) \triangleq \int^v (Rv')^\top \mathrm{d}v' = \tfrac{1}{2}v^\top Rv \tag{26}$$

as the mechanical content associated to the dissipators contained in mechanical 'resistance' matrix R. Note that (26) coincides with the usual definition of the Rayleigh dissipation function.

Conversely, the quantity

$$\mathcal{J}(f) \triangleq \int^f (Gf')^\top \mathrm{d}f' = \tfrac{1}{2}f^\top Gf \tag{27}$$

is referred to as the mechanical co-content associated to the dissipators contained in the mechanical 'conductance' matrix G. Consequently, the co-content (27) should then be considered as some Rayleigh *co-dissipation* function. This function, although (to our knowledge) only defined conceptually in [6], can be argued to have some physical significance as illustrated in the following example.

Example 2: Consider the linear mass-spring-damper system depicted in Figure 3.

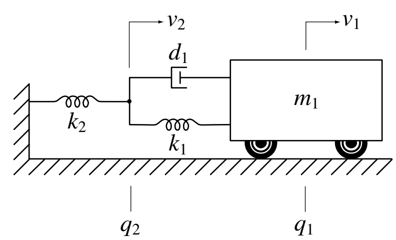

Fig. 3. Example system for mechanical co-content.

Although the equivalent damper velocity v_d can be expressed as $v_d = v_1 - v_2$ $(= \dot{q}_1 - \dot{q}_2)$, the problem, however, is that q_2 (resp., v_2) is not related to a mass element and can therefore not serve as a displacement (resp., velocity) coordinate. As a result, the damper can not be described in terms of a Rayleigh dissipation (or content) function $\mathcal{G}(v)$, but needs to be described by its dual form; the Rayleigh co-dissipation function, or in the terminology used here: the co-content. Let $f_j = k_j q_j$, $j = 1, 2$, denote the forces related to the linear springs with elasticity constants k_j, then

$$\mathcal{J}(f_1, f_2) = \frac{1}{2d}\Big(f_2 - f_1\Big)^2.$$

The Hamiltonian equations (24) can be used to obtain a valid equation set for this system, however the corresponding dissipation matrix D, as introduced in (25), should for this particular example be changed to

$$D = \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}, \text{ with } G = \frac{1}{d}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = G^\top \geq 0,$$

for all $d > 0$.

This enables us to extend Theorem 1 to standard mechanical system with (linear) dissipation.

Theorem 2: Consider a standard mechanical system with dissipation described by (24). The dynamics of (24) can be equivalently expressed by (15), where $Q(z)$ is of the form (16), while

$$\begin{aligned}\mathcal{P}(q,p) = \mathcal{G}(q,p) + (\nabla_q \mathcal{V}(q))^\top M^{-1}(q)p \\ + \tfrac{1}{2}\Big(\nabla_q\Big(p^\top M^{-1}(q)p\Big)\Big)^\top M^{-1}(q)p - \mathcal{J}(p,q),\end{aligned} \tag{28}$$

where

$$\begin{aligned}\mathcal{G}(q,p) &= \tfrac{1}{2}p^\top M^{-1}(q)RM^{-1}(q)p \\ \mathcal{J}(q,p) &= \tfrac{1}{2}(\nabla_q \mathcal{V}(q))^\top G \nabla_q \mathcal{V}(q) \\ &\quad + \tfrac{1}{2}\Big(\nabla_q\Big(p^\top M^{-1}(q)p\Big)\Big)^\top G\nabla_q\Big(p^\top M^{-1}(q)p\Big).\end{aligned} \tag{29}$$

Proof: In this case, we select K in (13) and (14) such that

$$K(J-D)^{-1} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Hence,

$$K = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}(J-D) = \begin{pmatrix} -G & I_n \\ I_n & R \end{pmatrix},$$

which, since $R = R^\top$ and $G = G^\top$, ensures that $K = K^\top$. The remaining part of the proof follows along the same lines of the proof of Theorem 1 and by noting that $v = \nabla_p \mathcal{H}(q,p)$ and $f = \nabla_q \mathcal{H}(q,p)$. ◁

Corollary 2: Consider a standard mechanical system with dissipation of the form (24). If the mapping $q \mapsto f$ is well-defined, then (15), together with the pair defined in (16) and (28), is precisely the mechanical analogue of the BM equations (4) and, hence, identifies the fourth equation set — including dissipation — suggested by the quadrangle of Figure 1. A description with the latter properties is referred to as a canonical BM equation set of mechanical type.

B. External Signals

During our developments we have assumed that the external signals (e.g., sources and disturbances), as modeled in Section I by the vector τ, are zero. The previous analysis remains unaffected if we include (possibly velocity-dependent) external forces. Indeed, the expressions remain valid if we replace $\mathcal{G}$ in (29) by a new content function of the form

$$\tilde{\mathcal{G}}(q,p,\tau) = \mathcal{G}(q,p) - \int^{\nabla_p \mathcal{H}(q,p)} \tau^\top(v')\mathrm{d}v' \tag{30}$$

A similar construction holds for the inclusion of (possibly force-dependent) external velocity sources.

VI. Discussion

The results reported in this paper are the first steps towards a general power-based modeling and analysis framework for physical systems. The present work first shows that a large class of mechanical systems (referred to as standard mechanical systems) can be described by a homonymous BM equation set. This set appears to be precisely the 'missing link' between the classical Lagrangian, co-Lagrangian and Hamiltonian equation sets on the one-side (defined on the $(\mathbb{Q},\mathbb{V})$, $(\mathbb{P},\mathbb{F})$ and $(\mathbb{Q},\mathbb{P})$ spaces, respectively), and the equation set defined on the $(\mathbb{V},\mathbb{F})$ space — as is illustrated by the quadrangle in Figure 1.

The analysis was carried out for standard mechanical systems with linear dissipation and a constant structure matrix of the form (9). However, since the matrix J is in general state-dependent, i.e., $J(z) = -J^\top(z)$, it is necessary to extend Lemma 1 with a state-modulated K-matrix. Consequently, the new pair $\{\tilde{J}^{-1}, \tilde{\mathcal{H}}\}$ is then obtained as follows:

$$\tilde{J}(z) = \tfrac{1}{2}\Big[\nabla^2\mathcal{H}(z)K(z) + \nabla\Big((\nabla\mathcal{H}(z))^\top K(z)\Big)\Big]J^{-1}(z),$$

and

$$\tilde{\mathcal{H}}(z) = \tfrac{1}{2}(\nabla\mathcal{H}(z))^\top K(z)\nabla\mathcal{H}(z).$$

For mechanical systems having a structure matrix of the form $J(z) = -J^\top(z)$, the corresponding Hamiltonian equation set is usually referred to as a generalized Hamiltonian (or port-Hamiltonian) system [10]. Besides the fact that the state-space is (locally) not restricted to $2n$ (i.e., an even number of) generalized coordinates (q,p), it can be argued to be an excellent tool to describe a very large class of physical models, ranging from standard mechanical systems treated here to electrical, electro-mechanical or even distributed parameter systems in various domains. For that reason, the next step is the search for a general BM equation set, starting from a port-Hamiltonian system description.

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