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Mixed $\mathrm{H}_2/\mathrm{H}_\infty$ and robust control of differential linear repetitive processes

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***Abstract*— Repetitive processes are a distinct class of 2D systems (i.e. information propagation in two independent directions) of both systems theoretic and applications interest. They cannot be controlled by direct extension of existing techniques from either standard (termed 1D here) or 2D systems theory. Here we give new results on the design of physically based control laws and, in particular, the first results on a mixed H_2/H_∞ approach and on H_2 control in the presence of uncertainty in the process model.**

I. INTRODUCTION

Repetitive processes are a distinct class of 2D systems of both system theoretic and applications interest. The essential unique characteristic of such a process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(t)$, $0 \leq t \leq \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(t)$, $0 \leq t \leq \alpha$, $k \geq 0$.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (see, for example, [9]). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. For example, they can be used to analyze include classes of iterative learning control (ILC) schemes [5]. More recently another application has arisen in the context of self-servo writing in disk drives [4]).

Attempts to control these processes using standard (or 1D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass and along a given pass. In seeking a rigorous foundation on which to develop a control theory for these processes, it is natural to attempt to exploit structural links which exist between these processes and other classes of 2D linear systems.

The case of 2D discrete linear systems recursive in the positive quadrant $(i, j) : i \geq 0, j \geq 0$ (where i and j denote the directions of information propagation) has been the subject of much research effort over the years using, in the main, the well known Roesser and Fornasini Marchesini state space models. More recently, productive research has been reported on H_∞ and H_2 approaches to analysis and controller design – see, for example, [1] and [11].

A key distinguishing feature of repetitive processes is that information propagation in one of the independent directions, along the pass, only occurs over a finite duration — the pass length. Moreover, in this paper the subject is so-called differential linear repetitive processes where the dynamics along the pass are governed by a linear matrix differential equation. This means that results for 2D discrete linear systems are not applicable.

The structure of linear repetitive processes means that there is a natural way to write down control laws for them which can be based on current pass state or output (pass profile) feedback control and feedforward control from the previous pass profile. For example, in the ILC application, one such family of control laws is composed of output feedback control action on the current pass combined with information 'feedforward' from the previous pass (or trial in the ILC context) which, of course, has already been generated and is therefore available for use.

Previous work has established the basic feasibility of this general approach and provided some algorithms for the design of these (and other) control laws (see, for example, [2]). This paper considers control law design based on the use mixed H_2/H_∞ and robust H_2 approaches to augment existing results using, for example, H_∞ and H_2 settings.

Throughout this paper, the null matrix and the identity matrix with appropriate dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ (respectively, ≥ 0) denotes a real symmetric positive definite (respectively, semi-definite) matrix. Similarly, $M < 0$ (respectively, ≥ 0) denotes a real symmetric negative definite (respectively, semi-definite) matrix.

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II. BACKGROUND

The state space model of the differential linear repetitive processes considered in this paper has the following form over $0 \le t \le \alpha$, $k \ge 0$

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + B_0 y_k(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + Du_{k+1}(t) + D_0 y_k(t) \end{aligned} \quad (1)$$

Here on pass k, $x_k(p)$ is the $n \times 1$ state vector, $y_k(p)$ is the $m \times 1$ pass profile vector and $u_k(p)$ is the $l \times 1$ vector of control inputs. To complete the process description, it is necessary to specify the boundary conditions i.e. the state initial vector on each pass and the initial pass profile (i.e. on pass 0). For the purposes of this paper, no loss of generality occurs from assuming these to be zero.

The stability theory [9] for linear repetitive processes is based on an abstract model in a Banach space setting which includes all such processes as special cases. In this model it is the pass-to-pass coupling (noting again the unique control problem) which is critical. This is of the form $y_{k+1} = L_\alpha y_k$, where $y_k \in E_\alpha$ (E_α a Banach space with norm $||\cdot||$) and L_α is a bounded linear operator mapping E_α into itself. Two concepts of stability can be defined but it is the stronger of these, so-called stability along the pass which is usually required. This holds if, and only if there exists numbers $M_\infty > 0$ and $\lambda_\infty \in (0,1)$ independent of α such that $||L_\alpha^k|| \le M_\infty \lambda_\infty^k$, $k \ge 0$ (where $||\cdot||$ also denotes the induced operator norm) and can be interpreted as bounded-input bounded-output stability independent of the pass length.

Several equivalent sets of conditions for stability along the pass are known but here it is one expressed in terms of the 2D transfer function matrix description of the process dynamics, and hence the 2D characteristic polynomial, which is the basic starting point. Since the state on pass 0 plays no role, it is convenient to re-label the state vector as $x_{k+1}(t) \mapsto x_k(t)$ (keeping of course the same interpretation). Also define the pass-to-pass shift operator as z_2 applied e.g. to $y_k(t)$ as $y_k(t) := z_2 y_{k+1}(t)$, and for the along the pass dynamics we use the Laplace transform variable s, where it is routine to argue that finite pass length does not cause a problem provided the variables considered are suitably extended from $[0,\alpha]$ to $[0,\infty]$, and here we assume that this has been done.

Let $Y(s,z_2)$ and $U(s,z_2)$ denote the results of applying these transforms to the sequences $\{y_k\}_k$ and $\{u_k\}_k$ respectively. Then the process dynamics can be written as

$$Y(s,z_2) = G(s,z_2)U(s,z_2)$$

where the 2D transfer function matrix $G(s,z_2)$ is given by

$$G(s,z_2) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} sI - A & -B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix}^{-1} \begin{bmatrix} B \\ D \end{bmatrix}$$

The 2D characteristic polynomial is given by

$$\mathcal{C}(s,z_2) := \det\left(\begin{bmatrix} sI - A & -B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix} \right)$$

and it has been shown elsewhere [9] that stability along the pass holds if, and only if,

$$\mathcal{C}(s,z_2) \ne 0 \quad (2)$$

in $\mathcal{U}(s,z_2) := \{(s,z_2) : \mathrm{Re(s)} \ge 0,\ |z_2| \le 1\}$.

It also possible to use the Laplace transform to conclude that stability along the pass requires each frequency component of the previous pass profile is attenuated from pass-to-pass. In 1D control systems theory and design, the H_2 norm of the system, i.e. the average gain over all frequencies, is a very powerful analysis and controller design tool and the new results here include ones which address the question of what H_2 control means for differential linear repetitive processes.

Here we will use the sufficient condition for stability along the pass of Lemma 1 below which is based on the following Lyapunov function

$$V(k,t) = x_{k+1}^T(t)P_1 x_{k+1}(t) + y_k^T(t)P_2 y_k(t)$$

with associated increment

$$\begin{aligned} \Delta V(k,t) =& \dot{x}_{k+1}^T(t)P_1 x_{k+1}(t) + x_{k+1}^T(t)P_1 \dot{x}_{k+1}(t) \\ &+ y_{k+1}^T(t)P_2 y_{k+1}(t) - y_k^T(t)P_2 y_k(t) \end{aligned} \quad (3)$$

where $P_1 > 0$ and $P_2 > 0$. The proof of this result is omitted here as it follows from a routine extension of results in, for example, [2].

Lemma 1: A differential linear repetitive process described by (1) is stable along the pass if

$$\Delta V(k,t) < 0 \quad (4)$$

We will also use the following signal space.

Definition 1: Consider a $q \times 1$ vector sequence $\{w_k(t)\}$ defined over the real interval $0 \le t \le \infty$ and the nonnegative integers $0 \le k \le \infty$, which is written as $\{[0,\infty],[0,\infty]\}$ Then the L_2 norm of this vector sequence is given by

$$\|w\|_2 = \sqrt{\sum_{k=0}^{\infty} \int_0^{\infty} w_k^T(t) w_k(t)\, dt}$$

and this sequence is said to be a member of $L_2^q\{[0,\infty],[0,\infty]\}$, or L_2^q for short, if $||w||_2 < \infty$.

III. H_2 NORM AND STABILITY

Using the 1D case as motivation, consider a single input stable along the pass process (note that this can be analyzed mathematically by letting the pass length $\alpha \to \infty$) and let the $m \times 1$ vector $g(k,t)$ denote the response to an impulse, denoted by $\delta(k,t)$, applied at $t = 0$ on pass k. Then, by invoking Parseval's theorem in the along pass direction on each pass and summing over the pass index, the H_2 norm is given by

$$\|G\|_2 = \sqrt{\|g(k,t)\|_2^2} = \sqrt{\sum_{k=0}^{\infty} \int_0^{\infty} g^T(k,t) g(k,t) dt} \quad (5)$$

This last result is easily extended to the multiple input case and leads to the following result — see [7] for the details.

Theorem 1: A differential linear repetitive process described by (1) is stable along the pass and has H_2 disturbance attenuation $\gamma_2 > 0$, i.e. $\|G\|_2 < \gamma_2$, if there exist matrices $P_1 > 0$ and $P_2 > 0$ such that the following LMIs hold

$$\begin{bmatrix} -P_2 & P_2C & P_2D_0 \\ C^TP_2 & A^TP_1+P_1A+C^TC & P_1B_0+C^TD_0 \\ D_0^TP_2 & B_0^TP_1+D_0^TC & -P_2+D_0^TD_0 \end{bmatrix} < 0$$

and

$$\mathrm{trace}(D^TD + B^TP_1B + D^TP_2D) - \gamma_2^2 < 0 \qquad (6)$$

Some applications areas will clearly require the design of control laws which guarantee stability along the pass and also have the maximum possible disturbance attenuation (here as measured by the H_2 norm). How to address this question in an H_2 setting for processes described by the following state space model over $0 \le t \le \alpha,\ k \ge 0$, is now considered

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + B_0y_k(t) + B_{11}w_{k+1}(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + Du_{k+1}(t) + D_0y_k(t) + B_{12}w_{k+1}(t) \end{aligned} \qquad (7)$$

In this model, $w_{k+1}(t)$ is an $r \times 1$ disturbance vector which belongs to L_2^r, i.e. the model of the previous section with disturbance terms added to the state and pass profile vector updating equations and without loss of generality zero boundary conditions are assumed. Also it is easy to see that stability along the pass for such a process is also governed by the 2D characteristic polynomial condition given by (2).

The control law employed is given by

$$u_{k+1}(t) = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix} \qquad (8)$$

where K_1 and K_2 are appropriately dimensioned matrices to be designed.

Applying Theorem 1 to the state space model resulting from application of the control law now yields the following sufficient condition for stability along the pass

$$\begin{bmatrix} -P_2 & P_2C+P_2DK_1 & P_2D_0+P_2DK_2 \\ K_1^TD^TP_2+C^TP_2 & \Lambda_1 & P_1B_0+P_1BK_1+C^TD_0 \\ K_2^TD^TP_2+D_0^TP_2 & B_0^TP_1+K_2^TB^TP_1+D_0^TC & -P_2+D_0^TD_0 \end{bmatrix} < 0 \qquad (9)$$

where $\Lambda_1 = A^TP_1+P_1A+K_1^TB^TP_1+P_1BK_1+C^TC$. Pre- and post-multiplying (9) by $\mathrm{diag}(P_2^{-1}, P_1^{-1}, P_2^{-1})$ and then setting $W_1 = P_1$, $W_2 = P_2^{-1}$, $N_1 = K_1W_1$ and $N_2 = K_2W_2$ yields the following result which gives a solution to this problem with an algorithm for designing the control law.

Theorem 2: Suppose that a control law of the form (8) is applied to a differential linear repetitive process described by (7). Then the resulting process is stable along the pass and has the prescribed H_2 disturbance rejection bound $\gamma_2 > 0$ if there exist matrices $W_1 > 0$, $W_2 > 0$, N_1 and N_2 such that the following LMIs hold

$$\begin{bmatrix} -W_2 & CW_1+DN_1 & D_0W_2+DN_2 & 0 \\ N_1^TD^T+W_1C^T & W_1A^T+AW_1+N_1^TB^T+BN_1 & B_0W_2+BN_2 & W_1C^T \\ N_2^TD^T+W_2D_0^T & W_2B_0^T+N_2^TB^T & -W_2 & W_2D_0^T \\ 0 & CW_1 & D_0W_2 & -I \end{bmatrix} < 0 \qquad (10)$$

and

$$\begin{aligned} &\mathrm{trace}(X) < \gamma_2^2 - \mathrm{trace}(D^TD) \\ &\begin{bmatrix} X & B_{11}^T & B_{12}^T \\ B_{11} & W_1 & 0 \\ B_{12} & 0 & W_2 \end{bmatrix} > 0 \end{aligned} \qquad (11)$$

where X is additional symmetric matrix variable of compatible dimensions. If these conditions hold, the control law matrices K_1 and K_2 are given by

$$K_1 = N_1W_1^{-1}, \ \ K_2 = N_2W_2^{-1} \qquad (12)$$

IV. THE MIXED H_2/H_∞ CONTROL PROBLEM

Consider a differential linear repetitive process represented by the following state space model over $0 \le t \le \alpha,\ k \ge 0$,

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + B_0y_k(t) + Bu_{k+1}(t) \\ &+ B_{11}w_{k+1}(t) + B_{21}\nu_{k+1}(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + D_0y_k(t) + Du_{k+1}(t) \\ &+ B_{12}w_{k+1}(t) + B_{22}\nu_{k+1}(t) \end{aligned} \qquad (13)$$

where the vectors $x_{k+1}(t)$, $y_k(t)$ and $u_{k+1}(t)$ are defined as in (1), and $w_{k+1}(t)$ and $\nu_{k+1}(t)$ are disturbance vectors.

Now we address the question of when does there exist a control law of the form (8) which minimizes the H_2 norm from w to y, denoted here by $||G_{d2}||_2$, and keeps the H_∞ norm from ν to y, denoted here by $||G_{d1}||_2$, below some prescribed level. Note also that if only w is present then this problem reduces to the H_2 control problem already solved in [7]. Similarly, if only ν is present then we obtain the H_∞ control problem which is treated next.

First, we have the following so-called bounded real lemma for differential linear repetitive processes.

Theorem 3: [6] A differential linear repetitive process described by (13) is stable along the pass and has H_∞ disturbance attenuation $\gamma_\infty > 0$ if there exist matrices $R_1 > 0$, $R_2 > 0$ and $R_3 > 0$ such that

$$\begin{bmatrix} -S & S\widehat{A}_2 & S\widehat{D}_1 & 0 \\ \widehat{A}_2^TS & \widehat{A}_1^TP + P\widehat{A}_1 - R & P\widehat{B}_1 & L^T \\ \widehat{D}_1^TS & \widehat{B}_1^TP & -\gamma_\infty^2 I & 0 \\ 0 & L & 0 & -I \end{bmatrix} < 0 \qquad (14)$$

where

$$P = \mathrm{diag}\,(R_1, 0)\,,\ S = \mathrm{diag}\,(R_3, R_2)\,, R = \mathrm{diag}\,(0, R_2)$$
$$\widehat{B}_1 = \begin{bmatrix} B_{21} \\ 0 \end{bmatrix},\ \widehat{D}_1 = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix},\ L = \begin{bmatrix} 0 & I \end{bmatrix}$$

$$\widehat{A}_1=\begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{A}_2=\begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix} \tag{15}$$

Equivalently, since the matrix R_3 has no influence on the result and can hence be deleted, we have the following result.

Theorem 4: A differential linear repetitive process described by (13) is stable along the pass and $\|G_{d1}\|_\infty < \gamma_\infty$, for given $\gamma_\infty > 0$ if there exist matrices $R_1 > 0$, and $R_2 > 0$ such that the following LMI holds

$$\begin{bmatrix} -R_2 & R_2C & R_2D_0 & R_2B_{22} \\ C^TR_2 & A^TR_1+R_1A & R_1B_0 & R_1B_{21} \\ D_0^TR_2 & B_0^TR_1 & -R_2+I & 0 \\ B_{22}^TR_2 & B_{21}^TR_1 & 0 & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{16}$$

Now, based on the results listed in the previous section and [6], it is clear that there exists a control law of the form (8) which minimizes the H_2 norm from w to y and keeps the H_∞ norm of the closed loop from ν to y below γ_∞ if the inequalities (6), (9) and

$$\begin{bmatrix} -R_2 & CR_1+DK_1R_1 & D_0R_2+DK_2R_2 & B_{22} & 0 \\ R_1C^T+R_1K_1^TD^T & R_1A^T+R_1K_1^TB^T+AR_1+BK_1P_1 & B_0R_2+BK_2R_2 & B_{21} & 0 \\ R_2D_0^T+R_2K_2^TD^T & R_2B_0^T+R_2K_2^TB^T & -R_2 & 0 & R_2 \\ B_{22}^T & B_{21}^T & 0 & -\gamma_\infty^2 I & 0 \\ 0 & 0 & R_2 & 0 & -I \end{bmatrix} < 0$$

are satisfied for some $P_1 > 0$, $P_2 > 0$, $R_1 > 0$ and $R_2 > 0$.

The main problem now is that we cannot linearize simultaneously the terms K_1R_1, K_1P_1 and K_2R_2, K_2P_2. This can be overcome (at the possible cost of increased conservativeness) by enforcing $P_1 = R_1$ and $P_2 = R_2$.

Under these assumptions, the following result provides the LMI condition for mixed H_2/H_∞ control law design.

Theorem 5: Suppose that a control law of the form (8) is applied to a differential linear repetitive process described by (13). Then the resulting process is stable along the pass and has prescribed H_2 and H_∞ norms bounds $\gamma_2 > 0$ and $\gamma_\infty > 0$ respectively if there exist matrices $W_1 > 0$, $W_2 > 0$, N_1 and N_2 such that the LMIs (10), (11) and

$$\begin{bmatrix} -W_2 & CW_1+DN_1 & D_0W_2+DN_2 & B_{22} & 0 \\ W_1C^T+N_1^TD^T & W_1A^T+N_1^TB^T+AW_1+BN_1 & B_0W_2+BN_2 & B_{21} & 0 \\ W_2D_0^T+N_2^TD^T & W_2B_0^T+N_2^TB^T & -W_2 & 0 & W_2 \\ B_{22}^T & B_{21}^T & 0 & -\gamma_\infty^2 I & 0 \\ 0 & 0 & W_2 & 0 & -I \end{bmatrix} < 0 \tag{17}$$

hold. If this is the case, then the H_2/H_∞ control law matrices K_1 and K_2 are given by (12).

Proof: This follows immediately from the results given or referenced above and hence the details are omitted. ■

Remark 1: This result is based on choosing a single Lyapunov function for both the H_2 and H_∞ criteria. In the 1D systems case this is a well known procedure termed the "Lyapunov shaping paradigm" in the literature [10].

Recall that in 1D system theory the $\mathcal{H}_\infty$ norm is used as a measure of system robustness. In the same spirt, this last result can be interpreted as follows; keeping the $\mathcal{H}_\infty$ norm of the 2D transfer function matrix from ν to y less than γ_∞ guarantees that the example considered is robust to unstructured perturbations of the form

$$\nu = \mathbf{\Delta} y, \quad \|\mathbf{\Delta}\|_\infty \le \gamma_\infty$$

and, simultaneously, the performance cost (in the $\mathcal{H}_2$ norm sense) is minimized. This means that choosing a lower value of γ_∞ reduces the process robustness and vice versa.

Remark 2: Note that by adjusting γ_∞ we can trade-off between $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance. Hence, a trade-off curve can be constructed for a given example which allows the designer to choose the control law which satisfies the compromise between robustness (measured with the $\mathcal{H}_\infty$ norm) and performance (measured with the $\mathcal{H}_2$ norm).

V. H_2 CONTROL OF UNCERTAIN PROCESSES

In this section we extend the results of previous two sections to the control of differential linear repetitive processes in the case when there is uncertainty in the matrices A, B_0, C and D_0. The analysis will make use of the following well known result.

Lemma 2: [3] Let Σ_1, Σ_2 be real matrices of appropriate dimensions. Then for any matrix $\mathcal{F}$ satisfying $\mathcal{F}^T\mathcal{F} \le I$ and a scalar $\epsilon > 0$ the following inequality holds

$$\Sigma_1\mathcal{F}\Sigma_2 + \Sigma_2^T\mathcal{F}^T\Sigma_1^T \le \epsilon^{-1}\Sigma_1\Sigma_1^T + \epsilon\Sigma_2^T\Sigma_2. \tag{18}$$

Consider now a differential linear repetitive process with the uncertainty modelled as additive perturbations to the nominal model matrices, resulting in the state space model over $0 \le t \le \alpha$, $k \ge 0$

$$\begin{aligned} \dot{x}_{k+1}(t) &= (A+\Delta A)x_{k+1}(t) + Bu_{k+1}(t) \\ &+ (B_0+\Delta B_0)y_k(t) + B_{11}w_{k+1}(t) \\ y_{k+1}(t) &= (C+\Delta C)x_{k+1}(t) + Du_{k+1}(t) \\ &+ (D_0+\Delta D_0)y_k(t) + B_{12}w_{k+1}(t) \end{aligned} \tag{19}$$

where ΔA, ΔB_0, ΔC, ΔD_0 represent admissible uncertainties. We also assume that these uncertainty matrices can be expressed in the form

$$\begin{bmatrix} \Delta A & \Delta B_0 \\ \Delta C & \Delta D_0 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \sigma^{-1}\mathcal{F}\begin{bmatrix} E_1 & E_2 \end{bmatrix} \tag{20}$$

where H_1, H_2, E_1, E_2 are known constant matrices and $\mathcal{F}$ is an unknown matrix with constant entries which satisfies

$$\mathcal{F}^T\mathcal{F} \le I \tag{21}$$

The design parameter σ here can be considered as a term available for use to attenuate the effects of the uncertainty.

The process model (19) can be rewritten in a form similar to (13) as

$$\begin{aligned} \dot{x}_{k+1}(t) =& Ax_{k+1}(t) + B_0 y_k(t) + Bu_{k+1}(t) \\ &+ B_{11} w_{k+1}(t) + H_1 \nu_{k+1}(t) \\ y_{k+1}(t) =& Cx_{k+1}(t) + D_0 y_k(t) + Du_{k+1}(t) \\ &+ B_{12} w_{k+1}(t) + H_2 \nu_{k+1}(t) \\ z_{k+1}(t) =& E_1 x_{k+1}(t) + E_2 y_k(t) \\ \nu_{k+1}(t) =& \sigma^{-1}\mathcal{F} z_{k+1}(t) \end{aligned} \tag{22}$$

where the parametric uncertainty is viewed as a fictitious feedback between the exogenous output $z_{k+1}(t)$ and the exogenous input $\nu_{k+1}(t)$ (see e.g. [8] for further details on such an analysis for uncertain 1D systems). Hence on applying the control law (8), the resulting process state space model can be written in the form

$$\begin{bmatrix} \dot{x}_{k+1}(t) \\ y_{k+1}(t) \end{bmatrix} = \overline{A} \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix} \tag{23}$$

where

$$\overline{A} = \begin{bmatrix} A+BK_1 & B_0+BK_2 \\ C+DK_1 & D_0+DK_2 \end{bmatrix} + \begin{bmatrix} \Delta A & \Delta B_0 \\ \Delta C & \Delta D_0 \end{bmatrix}$$

and we have the following result.

Theorem 6: Suppose that a control law of the form (8) is applied to a differential linear repetitive process described by (19) with uncertainty structure satisfying (20) and (21). Then the resulting process is stable along the pass and has prescribed H_2 disturbance attenuation $\gamma_2 > 0$ if there exist matrices $W_1 > 0$, $W_2 > 0$, N_1 and N_2 such that the LMIs (10) and (11) of Theorem 2 and

$$\begin{bmatrix} -W_2 & (\star) & (\star) & (\star) & (\star) \\ W_1 C^T + N_1^T D^T & \Theta & (\star) & (\star) & (\star) \\ W_2 D_0^T + N_2^T D^T & W_2 B_0^T + N_2^T B^T & -W_2 & (\star) & (\star) \\ H_2^T & H_1^T & 0 & -\sigma^2 I & (\star) \\ 0 & E_1 W_1 & E_2 W_2 & 0 & -I \end{bmatrix} < 0 \tag{24}$$

hold (where $\Theta = W_1 A^T + AW_1 + BN_1 + N_1^T B^T$). Also when these hold, the stabilizing control law matrices K_1 and K_2 are given by (12).

Proof: Introduce the matrices

$$\overline{A}_1 = \begin{bmatrix} A+BK_1 & B_0+BK_2 \\ 0 & 0 \end{bmatrix}, \overline{A}_2 = \begin{bmatrix} 0 & 0 \\ C+DK_1 & D_0+DK_2 \end{bmatrix},$$
$$\overline{H}_1 = \begin{bmatrix} H_1 \\ 0 \end{bmatrix},\ \overline{H}_2 = \begin{bmatrix} 0 \\ H_2 \end{bmatrix},\ E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}$$

Then it follows immediately that (22) is stable along the pass if there exist matrices $P_1 > 0$, $P_2 > 0$ such that the following LMI holds

$$\begin{aligned} &\left(\overline{A}_1 + \overline{H}_1 \sigma^{-1}\mathcal{F}E\right)^T P + P\left(\overline{A}_1 + \overline{H}_1 \gamma^{-1}\mathcal{F}E\right) \\ &+ \left(\overline{A}_2 + \overline{H}_2 \sigma^{-1}\mathcal{F}E\right)^T S \left(\overline{A}_2 + \overline{H}_2 \gamma^{-1}\mathcal{F}E\right) - R < 0 \end{aligned}$$

An obvious application of the Schur's complement formula to this last expression now yields

$$\begin{bmatrix} -S^{-1} & \left(\overline{A}_2 + \overline{H}_2 \sigma^{-1}\mathcal{F}E\right) \\ \left(\overline{A}_2 + \overline{H}_2 \sigma^{-1}\mathcal{F}E\right)^T & \Omega^T P + P\Omega - R \end{bmatrix} < 0$$

where $\Omega = \overline{A}_1 + \overline{H}_1 \sigma^{-1}\mathcal{F}E$, or

$$\begin{aligned} &\begin{bmatrix} -P_2 & (\star) & (\star) \\ C^T P_2 + K_1^T D^T P_2 & \Theta_1 & (\star) \\ D_0^T P_2 + K_2^T D^T P_2 & B_0^T P_1 + K_2^T B^T P_1 & -P_2 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ E_1^T \\ E_2^T \end{bmatrix} \mathcal{F}^T \begin{bmatrix} \sigma^{-1} H_2^T P_2 & \sigma^{-1} H_1^T P_1 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \sigma^{-1} P_2 H_2 \\ \sigma^{-1} P_1 H_1 \\ 0 \end{bmatrix} \mathcal{F} \begin{bmatrix} 0 & E_1 & E_2 \end{bmatrix} < 0 \end{aligned}$$

where $\Theta_1 = A^T P_1 + P_1 A + P_1 BK_1 + K_1^T B^T P_1$. Now make an obvious application of the result of Lemma 2, and pre- and post-multiply the result by $\mathrm{diag}(\epsilon^{-\frac{1}{2}} I, \epsilon^{-\frac{1}{2}} I, \epsilon^{-\frac{1}{2}} I)$. Also introduce the notation $\overline{P}_1 = \epsilon^{-1} P_1$, $\overline{P}_2 = \epsilon^{-1} P_2$ and $\Theta_2 = A^T \overline{P}_1 + \overline{P}_1 A + \overline{P}_1 BK_1 + K_1^T B^T \overline{P}_1$, and then an obvious application of the Schur's complement formula gives

$$\begin{bmatrix} -\overline{P}_2 & (\star) & (\star) & (\star) & (\star) \\ C^T \overline{P}_2 + K_1^T D^T \overline{P}_2 & \Theta_2 & (\star) & (\star) & (\star) \\ D_0^T \overline{P}_2 + K_2^T D^T \overline{P}_2 & B_0^T \overline{P}_1 + K_2^T B^T \overline{P}_1 & -\overline{P}_2 & (\star) & (\star) \\ H_2^T \overline{P}_2 & H_1^T \overline{P}_1 & 0 & -\sigma^2 I & (\star) \\ 0 & E_1 & E_2 & 0 & -I \end{bmatrix} < 0$$

Note that this last condition is not linear in $\overline{P}_1$, $\overline{P}_2$, $\overline{P}_3$, K_1 and K_1. However, this difficulty can be avoided by employing the following transformations. First, pre- and post-multiply the last expression by $\mathrm{diag}(\overline{P}_2^{-1}, \overline{P}_1^{-1}, \overline{P}_2^{-1}, I, I)$ and finally set $W_1 = P_1^{-1}$, $W_2 = \overline{P}_2^{-1}$, $N_1 = K_1 \overline{P}_1^{-1}$, $N_2 = K_2 \overline{P}_2^{-1}$ to obtain (24). ■

Remark 3: The term σ in the LMI of (24) can be minimized by using linear objective minimization procedure

$$\min_{W_1>0, W_2>0, X, N_1, N_2} \mu$$
$$\text{subject to (10), (11) and (24) with } \mu = \sigma^2$$

which, due to the presence of the term σ^{-1} in the uncertainty model of (20), provides an essential advantage of allowing us to extend the uncertainty boundaries, i.e. increase the robustness.

The last result (i.e. the LMI (24)) shows that there exists a link between mixed H_2/H_∞ and robust H_2 control problems. This means that the same control law solves the mixed H_2/H_∞ and robust H_2 control problems.

To see this, assume that $\sigma = \gamma_\infty$ and apply the same transformation used to obtain (17) from (14) where the matrices $\widehat{B}_1$, $\widehat{D}_1$ and L are now given by

$$\widehat{B}_1 = \begin{bmatrix} H_1 \\ 0 \end{bmatrix},\ \widehat{D}_1 = \begin{bmatrix} 0 \\ H_2 \end{bmatrix},\ L = \begin{bmatrix} E_1 & E_2 \end{bmatrix}$$

This link can also be established using the Lyapunov stability condition of Lemma 1. In particular from (21)

$$\sigma^{-2}\mathcal{F}^T\mathcal{F} \le \sigma^{-2} I$$

and hence, assuming $\sigma = \gamma_\infty$, we have

$$\begin{aligned} \nu_{k+1}^T(t)\nu_{k+1}(t) &= \gamma_\infty^{-2} z_{k+1}^T(t)\mathcal{F}^T\mathcal{F} z_{k+1}(t) \\ &\le \gamma_\infty^{-2} z_{k+1}^T(t) z_{k+1}(t) \end{aligned}$$

which can be rewritten as

$$\gamma_\infty^2 \nu_{k+1}^T(t)\nu_{k+1}(t) - z_{k+1}^T(t) z_{k+1}(t) \leq 0 \tag{25}$$

Hence the inequality

$$\Delta V(k,t) + z_{k+1}^T(t) z_{k+1}(t) - \gamma_\infty^2 \nu_{k+1}^T(t)\nu_{k+1}(t) < 0 \tag{26}$$

can hold only if the term $\Delta V(k,t) < 0$, i.e. stability along the pass holds. Also this inequality can be regarded as arising from the associated Hamiltonian for differential linear repetitive processes — see [6] for further details. Moreover, if (26) holds then the process is stable along the pass and the H_∞ norm from ν to z is kept below the prescribed level $\gamma_\infty > 0$. Finally, routine manipulations establish the link between robust H_2 control and H_2/H_∞ control detailed above.

VI. Numerical Example

As a numerical example, consider the case when

$$A = \begin{bmatrix} -0.0050 & -5.8077 \\ 1 & -0.0050 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 \\ 0.0494 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D_0 = 0.7692$$

and

$$\begin{aligned} B &= \begin{bmatrix} 0.9 \\ 0.2 \end{bmatrix},\ B_{11} = \begin{bmatrix} 0.9 \\ 1.3 \end{bmatrix},\ B_{21} = \begin{bmatrix} 0.6 \\ 1.1 \end{bmatrix} \\ D &= 0.6,\ B_{12} = 1.2,\ B_{22} = 0.9 \end{aligned} \tag{27}$$

Executing the design procedure for Theorem 5 for $\gamma_\infty = 1.9$ gives the solution as $\gamma_2 = 3.3397$ and

$$\begin{aligned} W_1 &= \begin{bmatrix} 2.2752 & -0.1238 \\ -0.1238 & 0.2470 \end{bmatrix},\ W_2 = 0.5070 \\ N_1 &= \begin{bmatrix} -5.8572 & -0.4243 \end{bmatrix},\ N_2 = -0.7337 \end{aligned} \tag{28}$$

The corresponding control law matrices are

$$K_1 = \begin{bmatrix} -2.7427 & -3.0925 \end{bmatrix},\ K_2 = -1.4472 \tag{29}$$

Furthermore, suppose that the H_∞ disturbance rejection bound is set as $\beta\gamma_\infty$, where β is a given positive scalar. Then, by adjusting β we can trade off the H_∞ versus H_2 performance. This leads to a trade-off between $\|G_{d1}\|_\infty \leq \gamma_\infty$ and $\|G_{d2}\|_2 \leq \gamma_2$ as shown in Figure 1.

VII. Conclusions

In this paper we have formulated and solved mixed H_2/H_∞ control problem for linear differential repetitive processes. Furthermore, the solution to the H_2 control problem of uncertain processes has also been developed and a link between them established. Finally, by introducing an additional positive scalar we can trade off the H_∞ versus H_2 performance, as illustrated in the given numerical example. On going work is aimed at a dynamic pass profile controller design with H_2 and H_∞ performance specifications and this will be reported on in due course.

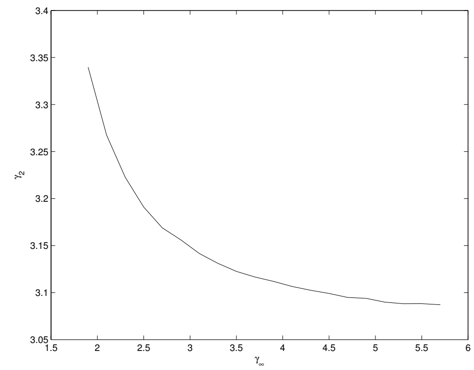

Fig. 1. Trade-off between $\|G_{d1}\|_\infty \leq \gamma_\infty$ and $\|G_{d2}\|_2 \leq \gamma_2$

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