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$\mathcal{H}_2$ Performance Limitation of Congestion Controller for TCP/AQM Network Systems

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Abstract— **In this paper, we deal with congestion control of one bottleneck type for TCP/AQM network systems and investigate the $\mathcal{H}_2$ control performance limitation. To this end, we first represent the dynamics of the congestion window size at source nodes and the queue length at the bottleneck link by a discrete time model. We then prove the stability of the plant and provide the Youla parametrization of all stabilizing controllers. With this parametrization, we consider an $\mathcal{H}_2$ optimization problem of the queue length at the bottleneck link. We derive an explicit form of the best achievable $\mathcal{H}_2$ norm and discuss the control performance limitation with respect to the ratio of the transmission time of the packets from source nodes to the link and the round trip time.**

keywords: TCP network system, congestion control, Youla parametrization, performance limitation

I. Introduction

The internet has been expanded enormously and the data flow is increasing explosively. This fact requires the necessity of highly sophisticated congestion control of the computer networks. In order to control packet flows and to avoid congestion with high performance, it is necessary to understand the mechanism and dynamics of the congestion of data flows.

The dynamical models of computer network systems with TCP protocols, e.g. Tahoe, Reno, Vegas, RED and their efficiency of those protocols has been examined based on control theory in the last few years [5], [6], [7], [8], [10], [1], [2], [3], [4], [11]. In particular, TCP/AQM (Active Queue Management) protocol, which manages "packet loss rate" and "packet mark rate" according to the queue length at relay nodes, has attracted much attention of researchers.

Typical TCP/AQM protocol is composed of Reno at end nodes and RED at relay nodes. However, the queue length of this system has been shown to be unstable around an equilibrium state when the transmission delay becomes large by using simple one bottleneck link model [5], [6]. This fact clearly reveals the necessity of the improvement of such protocols in order to manage the congestion.

In the above mentioned researches, the recovery of stability or control performance has been discussed and a natural extension of our interests is on the optimization and performance limitation of the congestion control. This is the motivation of this paper and the research focus is on to find the class of all stabilizing controllers for the optimization. From the above observations, we formulate a discrete time model of the network system composed of end nodes with Reno and a bottleneck node with a control protocol which we intend to design. In the previous researches [5], [6], such network system is modeled by a continuous time system and the time delay makes the synthesis problem difficult because of its infinite dimensional operator.

In this paper, we avoid such difficulty by treating the discretized system, and then provide a Youla parametrization [14] of the stabilizing controllers for the closed loop system. Related work is [11], in which an ARX model of discrete time systems is derived from system identification of a network simulator, and performance limitation was qualitatively discussed by using Bode integral relation. On the other hand, our approach is explicitly to provide the optimal performance of closed loop system including the network model given in [5], [6] and discuss the performance limitation analytically.

With the Youla parametrization, we can deal with control synthesis of the network system. The perturbation of the queue length at the bottleneck node should be controlled for stable congestion control. We formulate this problem as an $\mathcal{H}_2$ optimization of the queue length. The optimal controller for the problem and an explicit form of the best achievable $\mathcal{H}_2$ norm can be obtained and discuss the control performance limitation with respect to the ratio of the transmission time of the packets from source nodes to the link and the round trip time. This result exactly provides a performance limitation of the congestion control.

This paper is organized as follows. We first propose a discretized dynamic model of TCP/AQM network congestion system in Section 2. Section 3 is devoted to the stability analysis of the system. The control performance and the related $\mathcal{H}_2$ optimal control problem are discussed in Section 4. Finally in Section 5, the $\mathcal{H}_2$ control performance is examined, and Section 6 is the conclusion.

Notation : $\mathcal{B}(r)$: $\{z|z \in \mathcal{C}, |z| \leq r\}$, $\mathcal{H}_\infty$: analytic functions of z in $\overline{\mathcal{B}}(1)$, $\mathcal{RH}_\infty$: rational functions of $\mathcal{H}_\infty$, $\mathcal{N}$: set of integers.

II. Discretized Dynamics of TCP/AQM Network Congestion

We deal with one bottle neck link model of TCP/AQM network congestion with Reno introduced by S. Low et al. [5] (Fig. 1). The system is composed of source nodes, which send out packets, receiving nodes, and a bottleneck link. We assume homogeneous system managed by TCP, i.e. the all source nodes behave equivalent and continue sending packets with the TCP connection. The all transmission times of the packets from the source nodes to the bottleneck or that from

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the bottleneck to the receiving nodes are assumed to be same. At the source nodes, Reno controls the congestion window sizes for packets, on the other hand, at the bottleneck link, ECN (Explicit Congestion Notification) marking and a congestion controller, which we intend to design, transmit the information of the congestion state.

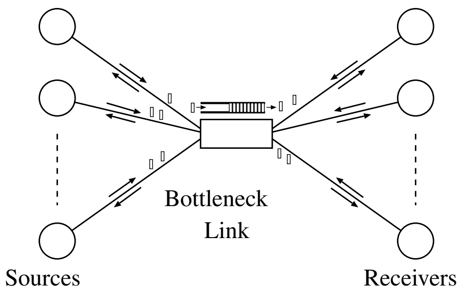

Fig.1: Network topology

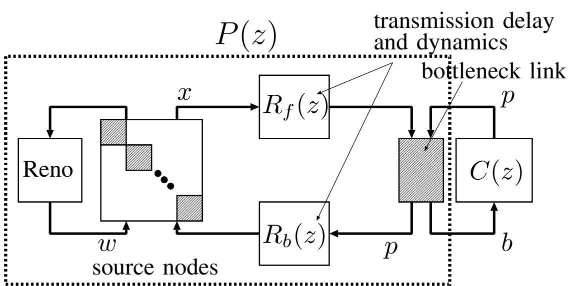

Fig. 2: The closed loop controlled system for one bottleneck link congestion

The corresponding block diagram of the closed loop system is shown in Fig. 2. The system behavior is governed by the flow of packet mark rate p generated at the bottleneck link, window size w of packets at source nodes, source rate x, and queue length b at the bottleneck link. We denote the dynamics from the packet mark rates p to the queue length b by P and the controller at the link by C, and they compose the feedback loop as depicted in Fig. 2.

S. Low et al. [5] introduced a model of continuous time dynamical system for this one bottle neck link. We transform their original model to a corresponding discrete time model in order to derive the parametrization of stabilizing controllers, which is given in Section III.

A. TCP Reno

The dynamics of the congestion avoidance mode of Reno algorithm is modeled as [5] :

$$\begin{aligned}\dot{w}(t) &= (1-p(t-\tau_r^*))\frac{w(t-\tau^*)}{\tau(t-\tau^*)}\frac{1}{w(t)} \\ &\quad -\frac{1}{2}p(t-\tau_r^*)\frac{w(t-\tau^*)w(t)}{\tau(t-\tau^*)},\end{aligned}$$

where $\tau(t)$: round trip time of packets, $\tau_r(t)$: transmission time of marked packets from the link to a source, τ^* and τ_r^*: τ and τ_r at equilibrium state. By applying linearization at the equilibrium state (w^*, p^*) and rewrite $w(t)$ as the perturbation from that, we get the following linear approximated perturbation model.

$$\dot{w}(t) = -\frac{1}{\tau^* p^*}p(t-\tau_r^*) - \frac{p^* w^*}{\tau^*}w(t) \qquad (1)$$

In order to avoid the difficulties of infinite dimensional dynamics of the above delay system, we assume that τ_r^* and t can be represented by

$$\tau_r^* = b^* \cdot \delta, \quad t = T \cdot \delta$$

where r^* and T are integers and δ is a constant time clock, and consider to transform the original differential equation to an approximated difference equation. We apply substitution:

$$\dot{w}(t) \longleftarrow \frac{w(T\cdot\delta) - w(T\cdot\delta-\delta)}{\delta},$$

for the derivative $\dot{w}(t)$ in (1), and get

$$\begin{aligned}&\frac{w(T\cdot\delta)-w(T\cdot\delta-\delta)}{\delta} \\ &= -\frac{1}{\tau^* p^*}p(T\cdot\delta - r^*\cdot\delta) - \frac{p^* w^*}{\tau^*}w(T\cdot\delta).\end{aligned}$$

With the definition

$$w_T := w(T\cdot\delta),$$

we get

$$(1+\frac{\delta p^* w^*}{\tau^*})w_T - w_{T-1} = -\frac{\delta}{\tau^* p^*}p_{T-r^*},$$

and its Z-transform is represented by

$$w(z) = G_R(z)p(z)\ ,$$

where

$$G_R(z) := \frac{-\delta z^{-r^*+1}}{p^*\{(\tau^* + \delta p^* w^*)z - \tau^*\}}\ . \qquad (2)$$

B. Queue Length

A model of the dynamics of the queue length $b(t)$ at the bottleneck link is given as [5]:

$$\dot{b}(t) = N\frac{w(t-\tau_f^*)}{\tau(t-\tau_f^*)} - c = N\frac{w(t-\tau_f^*)}{d+\frac{b(t-\tau_f^*)}{c}} - c,$$

where $\tau_f(t)$: the transmission time of the packets from source nodes to the link, τ_f^*: τ_f at equilibrium state, c: transmission capacity of the link, N: the number of the source nodes, d: time delay only depended on the network given by $\tau(t) - \frac{b(t-\tau(t))}{c}$.

The linearized perturbation model from the equilibrium state (w^*, b^*) is given by

$$\dot{b}(t) = N\frac{w(t-\tau_f^*)}{\tau^*} - N\frac{w^*}{(\tau^*)^2 c}b(t-\tau_f^*).$$

The corresponding discrete time model is expressed as

$$\begin{aligned}&\frac{b(T\cdot\delta)-b(T\cdot\delta-\delta)}{\delta} \\ &= N\frac{w(T\cdot\delta - f^*\cdot\delta)}{\tau^*} - N\frac{w^*}{(\tau^*)^2 c}b(T\cdot\delta - f^*\cdot\delta),\end{aligned}$$

and the Z-transformation is given by

$$b(z) \;=\; \frac{\delta\tau^* cN}{(\tau^*)^2 cz^{f^*} - (\tau^*)^2 cz^{f^*-1} + \delta N w^*} w(z).$$

From $w^* = c\tau^*/N$, we have

$$b(z) = G_L(z)w(z) \;,$$

where

$$G_L(z) := \frac{\delta N}{\tau^* z^{f^*} - \tau^* z^{f^*-1} + \delta} \;. \tag{3}$$

III. Parametrization of Stabilizing Congestion Controllers

By unifying the elements of TCP/AQM network system introduced in the previous section, the transfer function $P(z)$ from p to b is represented by

$$P(z) = G_L(z) \cdot G_R(z) \tag{4}$$

where $G_R(z)$ and $G_L(z)$ are given by (2) and (3), respectively. In this subsection, we first investigate the stability of $P(z)$, and then a parametrization of its stabilizing controllers at the bottleneck link is provided.

A. Stability of Plant and Youla Parametrization of Controllers

Here we show the stability of $P(z)$ whose the characteristic equation is given by

$$\left(\left(\frac{\tau^*}{\delta} + p^* w^*\right) z - \frac{\tau^*}{\delta}\right)\left(\frac{\tau^*}{\delta} z^{f^*} - \frac{\tau^*}{\delta} z^{f^*-1} + 1\right) = 0.$$

Since τ^*, p^*, $w^* > 0$, $\delta > 0$, it is clear that the first factor is stable, but the stability of the second factor is not trivial. We can show the stability by the following theorem under a reasonable assumption.

Theorem 3.1: The following characteristic polynomial is stable for any $\alpha, \beta \in \mathcal{N}$ satisfying $\alpha > \beta \geq 2$.

$$\alpha x^\beta - \alpha x^{\beta-1} + 1 = 0 \;. \tag{5}$$

See the next subsection for the proof.

Let us define two positive integers α and β as

$$\alpha := \frac{\tau^*}{\delta} \;,\; \beta := f^* \;. \tag{6}$$

Since δ is a clock time step and we have

$$\tau^* = \tau_r^* + \tau_f^* = (r^* + f^*)\delta \;,$$

the assumption $\alpha > \beta \geq 2$ is quite reasonable. Therefore, we can see from Theorem 3.1 that the plant is stable. Consequently, the Youla parametrization [14] of the stabilizing controllers in $\mathcal{RH}_\infty$ is immediately given by

$$C(z) = \frac{Q(z)}{1 - P(z)Q(z)},\; Q(z) \in \mathcal{RH}_\infty. \tag{7}$$

B. Proof of Theorem 3.1

We can show that the statement is correct for the cases $\beta = 2$, 3, 4 by direct calculation. It is also satisfied for the general case by using induction as follows. At first assume the statement is true for the case $\beta = n$ $(n \geq 4)$, then the following is the Jury table.

1	α	$-\alpha$	0	$\cdots$	0	0	0	1
2	1	0	0	$\cdots$	0	0	$-\alpha$	α
3	α	0	0	$\cdots$	0	$-\alpha^2$	α^2-1	
4	α^2-1	$-\alpha^2$	0	$\cdots$	0	0	α	
5	c_{n-2}	0	0	$\cdots$	c_1	c_0		
6	c_0	c_1	0	$\cdots$	0	c_{n-2}		
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$				
$2n-5$	p_3	0	p_1	p_0				
$2n-4$	p_0	p_1	0	p_3				
$2n-3$	q_2	q_1	q_0					
$2n-2$	q_0	q_1	q_2					
$2n-1$	r_1	r_0						
$2n$	r_0	r_1						
$2n+1$	s_0							
$2n+2$	s_0							

The conditions $\alpha^2 - 1 > 0$, $c_0 < 0$, $\ldots$, $p_0 < 0$, $r_0 < 0$, $s_0 < 0$ are derived from the Jury criterion with the assumption. Note that the table has a repeated structure of the positions of 0 from 3rd and 4th rows to the $2n-5$th and $2n-4$th rows. We can also show the following conditions from the direct calculations.

$$c_1, \ldots,\; p_1,\; q_1 > 0,\;\; c_{n-2}, \ldots,\; p_3,\; q_2 < 0$$

Then, we give the following lemma which corresponds to the lower part of the table.

Lemma 3.1: For the lower part of the Jury table:

$\cdots$	$\cdots$	$\cdots$	$\cdots$
$2n-3$	q_2	q_1	q_0
$2n-2$	q_0	q_1	q_2
$2n-1$	r_1	r_0	
$2n$	r_0	r_1	
$2n+1$	s_0		
$2n+2$	s_0		

the following two facts hold:

(i) If $q_2 < 0$, $q_1 > 0$, $q_0 < 0$, $q_2 + q_1 + q_0 < 0$, $|q_2| < |q_0|$, then $r_0 < 0$, $s_0 < 0$.

(ii) If $q_2 < 0$, $q_1 > 0$, $q_0 < 0$, $s_0 < 0$, then $q_2 + q_1 + q_0 < 0$.

Next make the Jury table at $b = n + 1$:

1	α	$-\alpha$	0	$\cdots$	0	0	0	1
2	1	0	0	$\cdots$	0	0	$-\alpha$	α
3	α	0	0	$\cdots$	0	$-\alpha^2$	α^2-1	
4	α^2-1	$-\alpha^2$	0	$\cdots$	0	0	α	
5	c_{n-2}	0	0	$\cdots$	c_1	c_0		
6	c_0	c_1	0	$\cdots$	0	c_{n-2}		
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$				
$2n-5$	p_3	0	0	p_1	p_0			
$2n-4$	p_0	p_1	0	0	p_3			
$2n-3$	q_2	0	q_1	q_0				
$2n-2$	q_0	q_1	0	q_2				
$2n-1$	x_2	x_1	r_0					
$2n$	r_0	x_1	x_2					
$2n+1$	y_1	y_0						
$2n+2$	y_0	y_1						
$2n+3$	z_0							
$2n+4$	z_0							

Note that the new elements are x_1, x_2, y_0, y_1 and z_0, and $x_1 > 0$, $x_2 < 0$ are shown by the direct calculation as the case of $b = n$. Therefore, from the first statement of Lemma 3.1, by showing the following two conditions:

$$x_2 + x_1 + r_0 < 0, \;\; |x_2| < |r_0|$$

the statement of this theorem is shown to be correct for $b = n + 1$. Here, the condition

$$x_2 + x_1 + r_0 = (q_2 + q_1 + q_0)(q_2 - q_0) < 0$$

is derived by using the second statement of Lemma 3.1 and the calculation of the tables for $\beta = n + 1$ and $\beta = n$. The second condition is shown as follows.

Lemma 3.2: Let l_k be the ratios of the elements $(2k - 1,\ 1)$ and $(2k,\ 1)$ in the Jury table for $b = n + 1$, that is,

$$\begin{aligned} &\left[\begin{array}{ccccccc} l_2 & l_3 & \cdots & l_k & \cdots & l_{n-1} & l_n \end{array}\right] \\ &= \left[\begin{array}{ccccccc} \frac{\alpha^2-1}{\alpha} & \frac{c_0}{c_{n-2}} & \cdots & \frac{k_0}{k_{n-k+1}} & \cdots & \frac{q_0}{q_2} & \frac{r_0}{x_2} \end{array}\right]. \end{aligned}$$

Then, $l_k > 1$ for $k = 2, \ldots, n$.

From Lemma 3.2, $|x_2| < |r_0|$ is shown and the statements of the theorem is satisfied for $\beta = n+1$. By using induction, the proof of Theorem 3.1 is completed.

IV. H_2 Control Performance

In this section, we formulate the performance of the congestion control such as H_2 optimization control problem under constraints of the bound for the perturbation of queue length or packet mark rate.

A. *Control Performance*

We assume that the exogenous noise for the closed loop system is added to the amount of packets arriving at the bottleneck link. Then, the closed loop system can be represented as in Fig. 3, where P_1, P_2 are defined by

$$P_1(z) = \frac{N}{\alpha} \cdot G_R(z), \;\; P_2(z) = G_L(z) \cdot \frac{\alpha}{N}, \tag{8}$$

and the output y of P_1 represents the perturbation of the amount of packets from source nodes. Such noise is caused by fails of packets in this system or the perturbation of the behaviour of the homogeneous model which we assume.

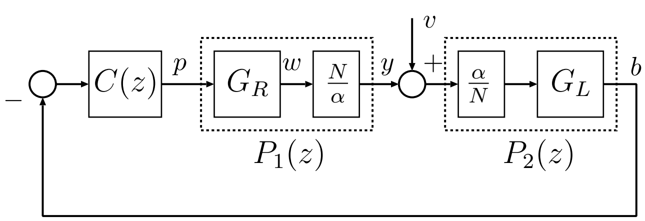

Fig. 3: The block diagram for control synthesis

We here consider to design a stabilizing controller at the bottleneck link which reduces the gain from v to the queue length. The noise v can be considered as white Gaussian, therefore H_2 optimization of the transfer function from v to b is reasonable. However, we should note that there exist two constraints for this system:

- packet mark rate p should satisfy $0 \leq p \leq 1$,
- the queue at the bottleneck link should not be void.

The first constraint comes from the specification of Reno, that is, the bound for the perturbation of the congestion window size. The second constraint is for keeping the maximum throughput of the bottleneck link.

Then, the H_2 optimization problem under the constraints can be formulated as follows:

$$\begin{array}{rcl} \mu(b^*) & = & \inf\limits_{Q \in \mathcal{RH}_\infty} \|G_{bv}(Q)\|_2 \;\; \text{subject to} \\ -p^* & \leq & (G_{pv}(Q) * v)_i \leq 1 - p^* \\ -b^* & \leq & (G_{bv}(Q) * v)_i \leq b_{\text{full}} - b^* \quad (i = 0, 1, \ldots), \end{array} \tag{9}$$

where G_{bv} is the transfer function from v to b, G_{pv} the transfer function from v to p, b_{full} the capacity of the buffer. In other words,

$$G_{bv}(Q) := P_2(1 - P_1P_2Q), \;\; G_{pv}(Q) := -P_2Q. \tag{10}$$

The l_1 norm of the perturbation of the queue length was taken to consider the constraints of (9) and the following optimization problem was investigated in [9]:

$$\begin{array}{rcl} \mu(b^*) & = & \inf\limits_{Q \in \mathcal{RH}_\infty} \|P_2(1 - P_1P_2Q)\|_2 \\ \text{subject to} & & \|-P_2Q * v\|_{l1} \leq p^* \\ & & \|P_2(1 - P_1P_2Q) * v\|_{l1} \leq b^* \end{array} \tag{11}$$

Since this is one of the typical multiobjective mixed H_2/ℓ_1 control problems [12], [13], the optimal controller can be obtained numerically by restricting the class of Q appropriately [9]. However, we can not see which parameters in the network system dominantly affect the control performance. The aim of this paper is to give an answer to the question.

B. *H_2 control problem*

In this paper we focus on the H_2 control performance limitations related to the above situation, where we want to minimize the H_2 norm of $P_2(1 - P_1P_2Q)$ under constraints of H_2 norms of P_2Q and $P_2(1-P_1P_2Q)$. Instead considering the constrained problem, which is hard to get the analytical expression for the limitation, we will investigate the best possible performance of the following H_2 optimal control problem:

$$J_\epsilon^* = \inf_{Q \in \mathcal{RH}_\infty} J_\epsilon \,, \tag{12}$$

where

$$\begin{array}{rcl} J_\epsilon & := & \delta(\|P_2(1 - P_1P_2Q)\|_2^2 + \epsilon^2 \|P_2Q\|_2^2) \\ & = & \left\| \sqrt{\delta} \left[\begin{array}{c} 1 - P_1P_2Q \\ \epsilon Q \end{array}\right] P_2 \right\|_2^2 \end{array} \tag{13}$$

with $\epsilon > 0$. Note that the factor δ is introduced to make a fare comparison when we change the sampling rate.

We should again emphasize that our aim here is to derive an analytical expression for the optimal value rather than to provide a design procedure for the optimal control problem. We divide the problem into two cases, $\epsilon = 0$ and $\epsilon > 0$. The purpose of the first case is to derive an analytical expression for

$$J_0^* = \inf_{Q \in \mathcal{RH}_\infty} \left\| \sqrt{\delta} P_2(1 - P_1P_2Q) \right\|_2^2 . \tag{14}$$

This can be rewritten as

$$J_0^* = \inf_{Q \in \mathcal{RH}_\infty} \|W(1 - P_i Q)\|_2^2 \ , \tag{15}$$

where P_i and P_o are the inner and outer part of $P_1 P_2$, i.e., $P := P_1 P_2 = P_i P_o$, and W denotes the outer part of $\sqrt{\delta} P_2$.

We first note that

$$\|W(1 - P_i Q)\|_2^2 = \left\|W(P_i^{\#} - Q)\right\|_2^2 = \left\|W P_i^{\#} - \hat{Q}\right\|_2^2$$

holds, where $\hat{Q} = WQ$. Since W is an outer function, we have

$$J_0^* = \inf_{\hat{Q} \in \mathcal{RH}_\infty} \left\|W P_i^{\#} - \hat{Q}\right\|_2^2 = \left\|(W P_i^{\#})_+\right\|_2^2 \ , \tag{16}$$

where

$$(W P_i^{\#})_+ = W P_i^{\#} - (W P_i^{\#})_-$$

and $(W P_i^{\#})_-$ is the proper and stable part of $W P_i^{\#}$, i.e., $(W P_i^{\#})_+$ is the improper or unstable part of $W P_i^{\#}$.

The second case can be treated using the result of the first case. Actually the followings hold.

$$\begin{aligned} J_\epsilon^* &= \inf_{Q \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} W(1 - P_i P_o Q) \\ \epsilon W Q \end{bmatrix} \right\|_2 \\ &= \inf_{\hat{Q} \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} W(1 - P_i \hat{Q}) \\ \epsilon W P_o^{-1} \hat{Q} \end{bmatrix} \right\|_2 \\ &= \inf_{\hat{Q} \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} (W P_i^{\#})_+ + (W P_i^{\#})_- - \hat{Q}) \\ \epsilon W P_o^{-1} \hat{Q} \end{bmatrix} \right\|_2 \\ &= J_0^* + \inf_{\hat{Q} \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} R - \hat{Q} \\ \epsilon V \hat{Q} \end{bmatrix} \right\|_2 \ , \end{aligned} \tag{17}$$

where

$$R := (W P_i^{\#})_- \in \mathcal{RH}_\infty \ , \ V := \epsilon W P_o^{-1} : \text{outer}. \tag{18}$$

V. H_2 Performance Limits: J_0^*

This section is devoted to derive an analytical expression of the H_2 performance limits, J_0^*, for the TCP/AQM network congestion. We first note that

$$\begin{aligned} P(z) &= G_R(z) G_L(z) \\ &= \frac{-\delta z^{-r^*+1}}{p^*\{(\tau^* + \delta p^* w^*) z - \tau^*\}} \cdot \frac{\delta N}{\tau^* z^{f^*} - \tau^* z^{f^*-1} + \delta}. \end{aligned}$$

Then the inner part of $P(z)$ is expressed as

$$P_i(z) = z^{-\alpha} \tag{19}$$

and the outer part of $\sqrt{\delta} P_2(z)$ is given by

$$W(z) = \frac{\alpha \sqrt{\delta} z^\beta}{\alpha z^\beta - \alpha z^{\beta-1} + 1} \ . \tag{20}$$

Hence, we have

$$W P_i^{\#} = \frac{\alpha \sqrt{\delta} z^{\alpha+\beta}}{\alpha z^\beta - \alpha z^{\beta-1} + 1} \ .$$

Since the denominator of $W P_i^{\#}$ is stable, $(W P_i^{\#})_+$ is equal to the improper part of $W P_i^{\#}$. Let

$$\begin{aligned} \phi(z) &:= (W P_i^{\#})_+ \\ &= \phi_0 z^\alpha + \phi_1 z^{\alpha-1} + \cdots + \phi_{\alpha-2} z^2 + \phi_{\alpha-1} z. \end{aligned} \tag{21}$$

Then we can see

$$J_0^* = \left\|(W P_i^{\#})_+\right\|_2^2 = \|\phi(z)\|_2^2 = \sum_{i=0}^{\alpha-1} |\phi_i|^2 \ .$$

Let

$$\lambda := \alpha^{-1} = \delta / \tau^*$$

and suppose $\alpha = q\beta + r$ with q and r are non-negative integers, then the sequence $\phi_i (i = 0 \sim \alpha - 1)$ is the solution of the following linear equation:

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & & 0 \\ 0 & & -1 & 1 & 0 & \cdots & \cdots & 0 \\ \lambda & & 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & -1 & 1 & 0 & 0 \\ \vdots & 0 & \lambda & \cdots & \cdots & -1 & 1 & 0 \\ 0 & \cdots & 0 & \lambda & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \vdots \\ \vdots \\ \phi_{\beta-1} \\ \phi_\beta \\ \vdots \\ \vdots \\ \phi_{2\beta-1} \\ \phi_{2\beta} \\ \vdots \\ \vdots \\ \phi_{\alpha-1} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\delta} & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}^T \ .$$

This implies that

$$\begin{aligned} \phi_0 &= \sqrt{\delta}, \\ \phi_i &= \phi_{i-1} \ \ (i = 1 \sim \beta - 1), \\ \phi_j &= \phi_{j-1} - \lambda \phi_{j-\beta} \ \ (j = \beta \sim 2\beta - 1), \end{aligned}$$

which lead to the following solutions for $\alpha < 3\beta$:

$$\begin{aligned} \phi_i &= \sqrt{\delta}, \\ \phi_{\beta+i} &= \sqrt{\delta}\{1 - (i+1)\lambda\}, \\ \phi_{2\beta+i} &= \sqrt{\delta}\{1 - (\beta + i + 1)\lambda + \frac{(i+1)(i+2)}{2}\lambda^2\} \ , \end{aligned}$$

where $i = 0 \sim \beta - 1$. We can get the result for $\phi_{\ell\beta+i}$ with $\ell \geq 3$ similarly, but we omit the forms due to the space limitation.

Let us now consider the two special cases where $\alpha = \beta$, i.e., $\tau^* = \tau_f^*$ or $\tau_r^* = 0$ and $\beta < \alpha \leq 2\beta$, i.e., $\tau^* \leq 2\tau_f^*$ or $\tau_r^* \leq \tau_f^*$ to obtain closed-form expressions for J_0^* and investigate what we can see from them for the performance limits on the TCP/AQM congestion control.

- $\alpha = \beta$ or $\tau^* = \tau_f^*$:

$$J_0^* = \alpha\delta = \tau^* \tag{22}$$

- $\alpha \leq 2\beta$ or $\tau^* \leq 2\tau_f^*$:

$$\begin{aligned} J_0^* &= \delta[\beta + r\{1 - (r+1)\lambda + \frac{(r+1)(2r+1)}{6}\lambda^2\}] \\ &= \frac{\tau^*}{\alpha}[\beta + (\alpha - \beta)\{1 - (\alpha - \beta + 1)\lambda \\ &\quad + \frac{(\alpha - \beta + 1)(2(\alpha - \beta) + 1)}{6}\lambda^2\}] \end{aligned}$$

Let define

$$\rho := \frac{\beta}{\alpha} = \frac{\tau_f^*}{\tau^*} \ (\frac{1}{2} \leq \rho < 1),$$

then we have

$$J_0^* = \tau^*[\rho + (1-\rho)\{\rho - \lambda + \frac{(1-\rho+\lambda)(2(1-\rho)+\lambda)}{6}\}] \ , \tag{23}$$

since $(\alpha-\beta)\lambda = 1-\rho$. For example, if $\alpha = 2\beta$ or $\tau_r^* = \tau_f^*$, J_0^* can be rewritten as

$$J_0^* = \frac{\tau^*}{24}(\lambda^2 - 9\lambda + 19) \ .$$

We can see that J_0^* converges to

$$[\rho + (1-\rho)\{\rho + \frac{(1-\rho)^2}{3}\}]\tau^* = \frac{1+3\rho-\rho^3}{3}\tau^* \ ,$$

when λ tends to zero which corresponds to the continuous-time case. We can also easily see that the function is monotone increasing for $1/2 \le \rho < 1$. This implies that the smaller transmission time of the packets from source nodes to the link gives the better control performance in the sense of the H_2 norm. This is also true when $\tau^* > 2\tau_f^*$.

VI. Conclusion

In this paper, we have dealt with congestion control of one bottleneck type for TCP/AQM network systems and investigated the $\mathcal{H}_2$ control performance limitation. We first proved the stability of the discretized model of the dynamics of the congestion window size at source nodes and the queue length at the bottleneck link and provided the Youla parametrization of all stabilizing controllers.

With this parametrization, we have investigated an $\mathcal{H}_2$ optimization problem of the queue length at the bottleneck link. We have obtained an explicit form of the best achievable $\mathcal{H}_2$ norm and discussed the control performance limitation with respect to the ratio of the transmission time of the packets from source nodes to the link and the round trip time. Further examinations based on the analytic expression for the more general H_2 control performance, J_ϵ^*, are on going.

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VII. Proofs of Lemmas

A. Proof of Lemma 3.1

We can easily show the statement (i) by using the following relations:

$$r_0 = q_2^2 - q_0^2 \tag{24}$$
$$s_0 = (q_2-q_0)^2(q_2+q_1+q_0)(-q_2+q_1-q_0), \tag{25}$$

The statement (ii) is clear from (24) and (25).

B. Proof of Lemma 3.2

We can see from the Jury table l_k is shown to satisfy the following recursive equation:

$$l_{k+1} = \frac{1}{l_{k-1}}(l_k^2 - 1), \quad k = 2, \ldots, n-1$$
$$l_2 = \frac{\alpha^2-1}{\alpha}, \quad l_3 = \frac{\alpha^4 - 3\alpha^2 + 1}{\alpha^3}.$$

Its solution is

$$l_k = A\cos\{(k-3)\theta\} \ ,$$

where

$$\theta = \arcsin\left(A^{-1}\right), \ A = \sqrt{l_4^2 - 1},$$
$$l_4 = \frac{\alpha^8 - 7\alpha^6 + 11\alpha^4 - 6\alpha^2 + 1}{\alpha^7 - \alpha^5}$$

for $k = 4, \ldots, n$. Then, l_4 can be written by

$$l_4 = \alpha + \frac{-6\alpha^6 + 11\alpha^4 - 6\alpha^2 + 1}{\alpha^7 - \alpha^5} \ ,$$

where

$$\frac{-6\alpha^6 + 11\alpha^4 - 6\alpha^2 + 1}{\alpha^7 - \alpha^5} > -\frac{6}{5}, \quad \alpha \ge 5.$$

Therefore, with $\alpha > k$, we have

$$A^2 = l_4^2 - 1 > \left(\alpha - \frac{6}{5}\right)^2 - 1 > (\alpha-3)^2 > (k-3)^2.$$

Finally, with $0 < \frac{1}{A} < \frac{1}{2}$, $k - 3 < A$,

$$0 < \frac{k-3}{A} < (k-3)\arcsin\left(A^{-1}\right) < \frac{\pi(k-3)}{3A} < \frac{\pi}{3},$$

holds and hence we have

$$1 < \frac{A}{2} < A\cos\{(k-3)\theta\} < A.$$