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# Robust stabilisation of nonlinear systems using output measurements via finite data-rate communication channels

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*Abstract*— **This paper addresses a robust stabilisation problem of nonlinear systems with decaying external disturbances using output measurements via finite data-rate communication channels. We assume that there exist an observer and a control law for the systems in the absence of any finite data-rate communication channel. Based on the observer and the control law, we construct an encoder/decoder pair and provide conditions that will guarantee the stability of the closed-loop systems when finite data-rate communication channels are introduced.**

## I. Introduction

It is typical that in a large-scale control system, its sensors and actuators are physically far apart. Under this circumstance, the most convenient or economical way to transmit the feedback signals is through a finite data-rate communication channel. In order to meet the finite data-rate constraint in the channel, the signals need to be sampled and quantised, and converted into packets which have finite number of bits. Both the sampling and quantisation processes reduce the quality and the precision of the feedback signals. As a result, using feedback signals transmitted through a finite data-rate communication channel can have an adverse effect on the performance of the closed-loop system.

In terms of stability performance, a number of control schemes and algorithms have been developed, especially for linear systems, to recover the stability performance with feedback signals transmitted via finite data-rate communication channels (see, e.g., [1] and references therein). Recently, there is a growing attention on stabilising nonlinear systems with feedbacks via finite data-rate communication channels.

For discrete-time nonlinear systems, the concept of feedback topological entropy was introduced and a necessary and sufficient data rate for the local stabilisation was given in [2]. In [3], the concept of topological entropy was extended to uncertain dynamical systems and adopted to study the robust observability of uncertain nonlinear systems and solvability of the optimal control problem via limited capacity communication channels. The paper [4] studied a stabilisation problem under the stabilizability assumption.

As for continuous-time nonlinear systems, a nonlinear stabilisation problem was studied in [5] with the assumption that there exists a feedback law renders the closed-loop systems input-to-state stable (ISS) with respect to the estimation errors. The work [6] replaced the ISS condition with the weaker integral input-to-state stable (iISS) condition. The ISS and iISS assumptions are restrictive. In response to this, a nonlinear stabilisation problem was studied under the stabilizability assumption in [7], similar to [4].

To solve the above-mentioned problems, the full states of the systems are required to be measurable. One the other hand, nonlinear stabilisation using output measurements has drawn a much less attention. Recently, the works [6], [8] solved a semi-global output feedback stabilisation problem for a class of nonlinear systems by the using the observer design introduced by [9]. In order to apply such an observer, the systems were assumed to be transformable into a particular form. For a special class of nonlinear systems, the paper [10] solved an output feedback stabilisation problem and provided algorithms to explicitly design the observer and the control law, and hence *a priori* assumptions on the existence of an observer and a control law are not required.

In terms of the robust stabilisation of nonlinear systems with disturbances using output measurements via finite data-rate communication channels, it is a new area. Therefore, the objective of this paper is to explore this area and propose an algorithm to solve such a stabilisation problem. We assume that there already exist a nonlinear observer and a control law for the systems, and also impose some conditions on the nonlinear observer. To solve the problem, we introduce an encoder/decoder pair for transmitting the observer estimate through the communication channels. We also determine the sampling period and the number of bits required to encode the observer estimate. The approach we take is based on the works of [7], [8]. In [7], they considered state feedback problem, whereas in [8], the output feedback problem for a special class of systems without any disturbances was studied. In this paper, we considered an output feedback problem of a general class of nonlinear systems with an external disturbance input.

The rest of the paper is organised as follows. In section II, we present the problem in this paper, including the class of nonlinear systems we considered and the standing assumptions. After stating the problem, in Section III, we introduce an encoder/decoder pair. In Section IV, the main result is stated. Finally, Section V gives the conclusion and some future directions.

## II. Problem Statement

Consider the following nonlinear uncertain systems:

$$\dot{x} = f(x, w, u), \qquad y = h(x) \tag{1}$$

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where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^p$ is the external disturbance input, and $y \in \mathbb{R}^m$ is the measured output. We assume that the maps $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^q$ are smooth, and satisfy $f(0,0,0) = 0$ and $h(0) = 0$, respectively.

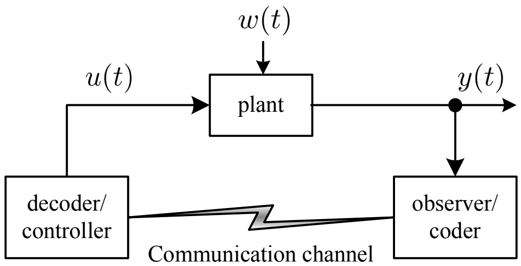


Fig. 1. Control system with a communication channel.

We consider a stabilisation problem of the nonlinear system (1) using output measurements $y$ via a communication channel. The communication channel is assumed to have a finite data-rate constraint, but is free from noise and time delay. We assume that there exist an observer $\dot{\hat{x}} = g(\hat{x}, y, u)$ and a control law $u = k(\hat{x})$ for system (1).

In order to solve the problem, we need to introduce an encoder/decoder pair, as shown in Figure 1. The encoder takes the observer estimate $\hat{x}(\cdot)$ and convert it to the finite-length codeword $s(kT)$, selected from a coding alphabet of size $l$, at time $kT$, $k = 0, 1, 2, \cdots$ and $T$ is the sampling period. The codeword $s(kT)$ is then transmitted through the communication channel. At the other end of the channel, a decoder decodes the codeword $s(kT)$ and generates the signal $\bar{x}(t)$ $t \in [kT, (k+1)T)$ which is then used to produce the control input $u(t)$ for system (1) at the actuator. The encoder and the decoder are in the following form:

*Encoder:*

$$s(kT) = \mathcal{F}(\hat{x}(\cdot)|_0^{kT}); \tag{2}$$

*Decoder:*

$$u(t)|_{kT}^{(k+1)T} = \mathcal{G}(s(0), s(T), s(2T), \cdots, s(kT)), \tag{3}$$

where $k = 0, 1, 2, \cdots$.

*Definition 2.1:* System (1) is said to be robustly stabilisable via a finite data-rate communication channel if for a given $X > 0$, there exist a constant $W > 0$, a sampling period $T > 0$, and an encoder/decoder pair with a coding alphabet of size $l$ such that from any initial condition[1] $\|x(0)\|_\infty \le X$, the solutions of the closed-loop system (1) with the control (3) satisfy:

$$\|x(t)\|_\infty \le W, \ \ \forall t \ge 0, \ \text{and} \ \lim_{t\to\infty} \|x(t)\|_\infty = 0. \tag{4}$$

Our aim is to design such an encoder/decoder pair with a coding alphabet of size $l$ and to determine a sampling period $T$ which robustly stabilise system (1) via a finite data-rate communication channel with data rate= $\lceil \log_2 l \rceil / T$ bits per unit time ($\lceil \cdot \rceil :=$ ceil$(\cdot)$).

[1]Let $x = [x_1 \cdots x_n]^T$ be a vector from $\mathbb{R}^n$. Then $\|x\|_\infty := \max_{i=1,\cdots,n} |x_i|$.

### A. Assumptions

In order to solve the stabilisation problem, the following assumptions are imposed.

*Assumption 2.1:* For some known constant $X > 0$, the initial state $x(0)$ of system (1) satisfies $\|x(0)\|_\infty \le X$.

For the external disturbance input $w$, we assume that:

*Assumption 2.2:* The external disturbance $w(t)$ decays exponentially, namely there exist *known* strictly positive constants $\lambda_1$ and $\lambda_2$ such that for all $t \ge 0$,

$$\|w(t)\|_\infty \le \lambda_1 \exp(-\lambda_2 t). \tag{5}$$

Since the state $x$ in system (1) cannot be measured, we assume that there exists a nonlinear observer to construct the observer state estimate $\hat{x}$.

*Assumption 2.3:* System (1) admits a nonlinear observer

$$\dot{\hat{x}} = g(\hat{x}, h(x), u), \quad \|\hat{x}(0)\|_\infty \le X, \tag{6}$$

and there exists a positive definite function $V_e$ such that with the observer estimation error $e := x - \hat{x}$ the following inequalities hold:

$$\begin{aligned} &\underline{\alpha}_e(\|e\|_\infty) \le V_e(e) \le \bar{\alpha}_e(\|e\|_\infty) \\ &\dot{V}_e \le -c_e V_e + \gamma_e(\|w\|_\infty) \end{aligned} \tag{7}$$

where scalar $c_e > 0$, $\underline{\alpha}_e(\cdot)$, $\bar{\alpha}_e(\cdot)$ and $\gamma_e(\cdot)$ are some class $K_\infty$ functions[2].

We also impose some properties on the observer.

*Assumption 2.4:* For all $\xi_1$, $\xi_2$ in $\mathbb{R}^n$ and $u$ in $\mathbb{R}^m$, there exist scalars $L_1$, $L_2 > 0$ such that the functions $h(\cdot)$ of the system (1) and $g(\cdot)$ of the observer (6) satisfy

$$\begin{aligned} \|g(\xi_1, h(\xi_1), u) - g(\xi_2, h(\xi_2), u)\|_\infty &\le L_1 \|\xi_1 - \xi_2\|_\infty, \\ \|g(\xi_1, h(\xi_1 + \xi_2), u) - g(\xi_1, h(\xi_1), u)\|_\infty &\le L_2 \|\xi_2\|_\infty. \end{aligned} \tag{8}$$

Using the observer state estimate $\hat{x}$, we consider the following stabilisability property for the system (1).

*Assumption 2.5:* There exists a smooth control law

$$u = k(\hat{x}), \quad k(0) = 0, \tag{9}$$

such that the system $\dot{x} = f(x, w, k(\hat{x})) = f(x, w, k(x - e))$ is input-to-state stable with respect to the observer error $e$ and the external disturbance $w$.

Essentially, there are two kinds of disturbances in the system $\dot{x} = f(x, w, k(x - e))$, namely the estimation error $e$ and the external disturbances $w$, indicating that the sampling period $T$ cannot be too large. Initially, we assume $T$ is less than unity, but the exact value of the sampling period will be determined later.

*Assumption 2.6:* The sampling period $T$ lies in the interval $(0, 1]$.

Instead of using $\hat{x}$, the final control law uses the state $\bar{x}$ which is generated from the system $\dot{\bar{x}} = g(\bar{x}, h(\bar{x}), u)$ in the decoder. We need to guarantee the existence of solution $\bar{x}(\cdot)$

[2]A continuous function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to class $K_\infty$ if it is strictly increasing, $\alpha(0) = 0$, and $\alpha(r) \to \infty$ as $r \to \infty$. It is worth mentioning that for any class $K_\infty$ function $\gamma$ and any $a$, $b \ge 0$, the inequality $\gamma(a + b) \le \gamma(2a) + \gamma(2b)$ holds.

with the feedback control law $u = k(\bar{x})$ between sampling times. Hence, we impose the following assumption:

*Assumption 2.7:* For every initial state $\bar{x}(t_0) \in \mathbb{R}^n$, the solution of the system

$$\dot{\bar{x}} = g(\bar{x}, h(\bar{x}), k(\bar{x})), \tag{10}$$

is defined for all $t \in [t_0, t_0 + T)$.

We introduce the following property for a special class of class $K_\infty$ functions which will be useful in determining the decay rate of $e$.

*Definition 2.2 (Exponentially invariant):* A class $K_\infty$ function $\sigma$ is said to be *exponentially invariant*[3] if for any positive constants $a$ and $b$, there exist positive constants $a_0$ and $b_0$ such that $\sigma(a\exp(-bt)) \leq a_0 \exp(-b_0 t)$ holds for all $t \geq 0$.

*Assumption 2.8:* The class $K_\infty$ functions $\gamma_e$ and $\underline{\alpha}_e^{-1}$, i.e., the inverse of $\underline{\alpha}_e$ in (7)[4], are exponentially invariant.

Assumption 2.5 guarantees the existence of an ISS Lyapunov functions $V_x$ for the system $\dot{x} = f(x, w, k(\hat{x})) = f(x, w, k(x - e))$. Then, Assumptions 2.3 and 2.5 together imply that the augmented system $\dot{x}_e := [\dot{e}\ \dot{x}]^T$ is ISS with respect to the external disturbance $w$. Therefore, there exists an ISS Lyapunov function $V$ for the augmented system $\dot{x}_e$ and have the following properties:

$$\begin{aligned} &\underline{\alpha}(\|x_e\|_\infty) \leq V(x_e) \leq \bar{\alpha}(\|x_e\|_\infty) \\ &\dot{V} = \dot{V}_e + \frac{\partial V_x}{\partial x} f(x, w, k(\hat{x})) \leq -\alpha(\|x_e\|_\infty) + \gamma_w(\|w\|_\infty), \end{aligned} \tag{11}$$

where $\underline{\alpha}$, $\bar{\alpha}$, $\alpha$, $\gamma_w$ are some class $K_\infty$ functions (see [11]). In other words, the feedback law $u = k(\hat{x})$ in Assumption 2.5 renders the origin $(x, e) = 0$ of the augmented system semi-globally asymptotically stable, since the external disturbance $w$ decays exponentially.

## III. Encoding the Observer Estimate

In this section, we propose an encoder/decoder pair and show that the difference between the observer estimate $\hat{x}$ and the state $\bar{x}$ generated by the encoder/decoder pair satisfies certain conditions which will be useful at the later stage.

First of all, we present some preliminary results which are useful in constructing the encoder/decoder pair.

*Lemma 3.1:* Let Assumptions 2.3, 2.2 and 2.8 hold. Then, there exist strictly positive constants $c_1$, $c_2$, $\eta_1$ and $\eta_2$ such that the observer estimation error $e(t)$ satisfies

$$\|e(t)\|_\infty \leq c_1 \exp(-\eta_1 t) + c_2 \exp(-\eta_2 t), \tag{12}$$

for all $t \geq 0$ that the solution $x(\cdot)$ is defined.

*Proof:* See Appendix VII-A. ∎

*Lemma 3.2:* Let Assumption 2.4 hold. Let $\mathcal{I} = [t_0, t_1]$ be the interval of time that the solutions of systems

$$\dot{\hat{x}} = g(\hat{x}, h(\hat{x} + e), u), \quad \dot{\bar{x}} = g(\bar{x}, h(\bar{x}), u) \tag{13}$$

and $x(\cdot)$ are defined. Then, the inequality

$$\begin{aligned} &\|\hat{x}(t) - \bar{x}(t)\|_\infty \leq \|\hat{x}(t_0) - \bar{x}(t_0)\|_\infty \exp(L_1(t - t_0)) \\ &\quad + L_2\left(\frac{c_1}{\eta_1}\exp(-\eta_1 t_0) + \frac{c_2}{\eta_2}\exp(-\eta_2 t_0)\right)\exp(L_1(t - t_0)) \end{aligned} \tag{14}$$

holds for all $t \in \mathcal{I}$.

*Proof:* By Assumption 2.4, we have

$$\begin{aligned} \|\dot{\hat{x}} - \dot{\bar{x}}\|_\infty &\leq \|g(\hat{x}, h(\hat{x}), u) - g(\bar{x}, h(\bar{x}), u)\|_\infty \\ &\quad + \|g(\hat{x}, h(\hat{x} + e), u) - g(\hat{x}, h(\hat{x}), u)\|_\infty \\ &\leq L_1\|\hat{x} - \bar{x}\|_\infty + L_2\|e\|_\infty \end{aligned} \tag{15}$$

Therefore, by Lemma 3.1 and using Gronwall-Bellman inequality, we have the inequality (14). ∎

Before moving on, we define the following strictly positive scalars which will be frequently used:

$$\begin{aligned} &\zeta := \bar{\alpha} \circ 2\alpha^{-1} \circ 2\gamma_w(\lambda_1), \quad \Lambda := \exp(L_1 T) \\ &c := \bar{\alpha}(3X), \quad W := \underline{\alpha}^{-1}(c + \zeta + 1), \quad \hat{W} := c_1 + c_2 + W \end{aligned} \tag{16}$$

where the constants: $X$ is from Assumption 2.1; $\lambda_1$ is from (2.2); $c_1$ , $c_2$ are from (12); and the class $K_\infty$ functions are from the conditions of the ISS Lyapunov function (11) of the augmented system $\dot{x}_e = [\dot{e}\ \dot{x}]^T$.

*Encoder:* The observer estimate $\hat{x}(\cdot)$ is sampled at $t = kT$, $k = 0, 1, 2, \cdots$. For each $k \geq 0$, we define a hypercube $\Omega(kT) \subset \mathbb{R}^n$ which has the centroid $\lim_{\epsilon>0,\epsilon\to 0} \bar{x}(kT - \epsilon) =: \bar{x}(kT^-)$ and has length $L(kT)$ at each side of its edge. We call $\Omega(kT)$ as the quantisation region and $L(kT)$ as the range of $\Omega(kT)$ at time $kT$. The quantisation region $\Omega(kT)$ is then uniformly partitioned into $N^n$ smaller hypercubes, where $N > 1$ is a design parameter. We call $N$ as the number of quantisation levels. If $\hat{x}(kT)$ falls into one of these smaller hypercubes, the difference between $\hat{x}(kT)$ and the centroid of the smaller hypercube, $z(kT)$, satisfies $\|\hat{x}(kT) - z(kT)\|_\infty \leq L(kT)/2N$. The location of $z(kT)$ for this particular smaller hypercube can be determined by

$$z(kT) = \bar{x}(kT^-) + b(kT)L(kT)/2N \tag{17}$$

where $b(kT) := [b_1(kT), \cdots, b_n(kT)]^T$ and $b_i(kT)$, $i = 1, \cdots, n$, is a suitable integer taking values in the set

$$\{-(N-1), \cdots, -5, -3, -1, 1, 3, 5, \cdots, (N-1)\} \tag{18}$$

if $N$ is even, or in the set

$$\{-(N-1), \cdots, -6, -4, -2, 0, 2, 4, 6, \cdots, (N-1)\} \tag{19}$$

if $N$ is odd. We therefore define a suitable map $\Phi(\cdot, \cdot, \cdot)$ such that

$$b(kT) = \Phi(z(kT), \bar{x}(kT^-), L(kT)). \tag{20}$$

The codeword $s(kT)$, which will be transmitted, is chosen from a coding alphabet of size $l = N^n$ corresponding to the binary representation of the integer vector $b(kT)$. Again, we define a suitable map $\phi(\cdot)$ to represent such an operation as follows:

$$s(kT) = \phi(b(kT)). \tag{21}$$

[3]For example: $r(s) = s^n$, $n > 0$, is exponentially invariant.

[4]The inverse of a class $K_\infty$ function is also of class $K_\infty$.

The time evolution of the centroid $\bar{x}(\cdot)$ is governed by the centroid update law represented by the differential equation

$$\begin{aligned}\dot{\bar{x}} &= g(\bar{x}, h(\bar{x}), u), \quad t \in [kT, (k+1)T),\ k \geq 0,\\ \bar{x}(kT) &= z(kT)\end{aligned} \tag{22}$$

with initial condition $\bar{x}(0^-) = 0$, $z(kT)$ is the centroid of the smaller hypercube in the quantisation region $\Omega(kT)$ where $\hat{x}(kT)$ lies.

The range update law is defined by the difference equation

$$\begin{aligned}L(0) &= 2X\\ L((k+1)T) &= RL(kT) + 2c_3 \exp(-\eta_3(kT)),\end{aligned} \tag{23}$$

for all $k \geq 0$, where $R := \Lambda/N > 0$ ,

$$c_3 := L_2(\frac{c_1}{\eta_1} + \frac{c_2}{\eta_2}) \exp(L_1), \quad \eta_3 := \min\{\eta_1, \eta_2\}. \tag{24}$$

For any $k \geq 1$, equation (23) can also be written as

$$L(kT) = R^k L(0) + 2c_3 \sum_{j=0}^{k-1} R^{k-1-j} \exp(-\eta_3 jT). \tag{25}$$

*Decoder:* Once the symbol $s(kT)$ arrives at the decoder at time $kT$, it will be converted into the corresponding integer vector $b(kT) = \phi^{-1}(s(kT))$. In the decoder, it consists of both the centroid update law (22) and the range update law (23) with the same initial values as in the encoder. In other words, $\bar{x}(\cdot)$ generated by the encoder will be identical to that of the decoder, similar argument is applied for $L(\cdot)$. Therefore, $\bar{x}(kT^-)$ and $L(kT)$ are also available to the decoder, and $z(kT)$ can then be determined from (17). With this value of $z(kT)$, the state $\bar{x}(t)$ can be calculated for all $t \in [kT, (k+1)T)$, and hence $\bar{x}((k+1)T^-)$ is available to the decoder when it receives $s((k+1)T)$ at time $(k+1)T$.

The construction of our encoder/decoder pair is based on the works [7], [8] with modifications in the centroid update and range update laws to suit our purpose. When choosing the range update law (23), we have considered that the sampling time $T$ belongs to the interval $(0, 1]$.

Next, for convenience, we define the following strictly positive constants:

$$Y := X + 2c_3, \quad \bar{W} := Y + \hat{W}. \tag{26}$$

The following lemmas show some properties of the range update law (23).

*Lemma 3.3:* Consider the difference equation (23) and the definition of $Y$ (26). For any $T \in (0, 1]$, any $R < 1/2$, any $c_3$ and $\eta_3 > 0$, then $L(kT) < 2Y$ for all $k \geq 0$. Furthermore, $L(kT) \to 0$ as $k \to \infty$.

*Proof:* See Appendix VII-B. ∎

*Lemma 3.4:* Consider the difference equation (23). For any $X > 0$, any $\epsilon > 0$, any $\delta \in (0, 1/2)$ and any integer $s$ satisfying

$$s \begin{cases} = 2 & \text{when } \epsilon/4c_3 \geq 1, \\ \geq \dfrac{|\ln(\epsilon/4c_3)|}{|\ln(0.5+\delta)|} + 1 & \text{otherwise,} \end{cases} \tag{27}$$

if $T \in (0, 1]$,

$$\eta_3 \geq \frac{|\ln \delta|}{T}, \quad N > \max\left\{2\Lambda, \Lambda(\epsilon/2X)^{-1/s})\right\}, \tag{28}$$

then $L(kT) < \epsilon$, for all $k \geq s$.

*Proof:* See Appendix VII-C. ∎

*Proposition 3.1:* Let Assumptions 2.1–2.8 hold. Let $t^* \in (0, \infty)$ be the time for the solution $x(t)$ of system (1) exists and satisfies

$$\|x(t)\|_\infty \leq W, \quad \forall t \in [0, t^*]. \tag{29}$$

If $R < 1/2$, then

$$\|\hat{x}(t) - \bar{x}(t)\|_\infty \leq L((k+1)T)/2 \tag{30}$$

holds for all $t \in [kT, (k+1)T)$, for each $k \geq 0$, such that $t \leq t^*$.

*Proof:* See the proof of [8, Lemma 2]. ∎

The following corollary follows immediately from Proposition 3.1 and Lemma 3.3.

*Corollary 3.1:* Let Assumptions 2.1–2.8 hold. Suppose that the solution $x(t)$ of system (1) exists and satisfies $\|x(t)\|_\infty \leq W$ for all $t \geq 0$. If $R < 1/2$, then inequality (30) holds for all $t \in [kT, (k+1)T)$, for all $k \geq 0$, and $\lim_{t\to\infty} \|\hat{x}(t) - \bar{x}(t)\|_\infty = 0$.

## IV. Stabilisation by Encoded Observer Estimate

In Assumption 2.5, the control law $u = k(\hat{x})$ renders the system $\dot{x} = f(x, w, u)$ (1) ISS with respect to the observer estimation error $e$ and the external disturbance $w$. However, the observer estimate $\hat{x}$ is not available for the actuator which locates at the other end of the communication channel. On the other hand, the signal $\bar{x}(\cdot)$ is generated by the centroid and the range update laws (22)–(23) through the use of encoded $\hat{x}(kT)$ at each sampling time $kT$, $k \geq 0$. It is then natural to choose $u = k(\bar{x})$ as the candidate control law to solve our stabilisation problem.

By replacing $u = k(\hat{x})$ with $u = k(\bar{x})$, the $\dot{V}$ inequality in (11) becomes

$$\begin{aligned}\dot{V} &\leq -\alpha(\|x_e\|_\infty) + \gamma_w(\|w\|_\infty)\\ &\quad + \frac{\partial V_x}{\partial x}(x)(f(x, w, k(\bar{x})) - f(x, w, k(\hat{x})))\\ &\leq -\alpha(\|x_e\|_\infty) + \gamma_w(\|w\|_\infty) + \frac{\partial V_x}{\partial x}(x) g(x, \hat{x}, \bar{x}, w)(\hat{x} - \bar{x})\end{aligned} \tag{31}$$

where $g$ is smooth and can be determined by following [7, Appendix]. Later, by using the Lyapunov argument, we will determine a suitable sampling period $T$ and number of quantisation levels $N$ that solve our stabilisation problem.

Using (31), we first define a time $\theta$ such that, for all $t \in [0, \theta]$, $x_e(t) = [e(t)\ x(t)]^T$ belongs to the level set

$$\Gamma_{c+\zeta+1} := \{x_e \in \mathbb{R}^{2n} : V(x_e) \leq c + \zeta + 1\}. \tag{32}$$

Later on, we will choose $T$ and $N$ to render the level set $\Gamma_{c+\zeta+1}$ invariant, and hence the solution $x_e(t) = [e(t)\ x(t)]^T$ does not leave this set for all time $t \geq 0$.

For the sake of convenience, we define a region $\Delta$ and a constant $M$ as follows:

$$\begin{aligned}\Delta &:= \{(x,\hat{x},\bar{x},w) : \|x\|_\infty \le W, \|\hat{x}\|_\infty \le \hat{W},\\ &\qquad \|\bar{x}\|_\infty \le \bar{W}, \|w\|_\infty \le \lambda_1\},\\ M &:= \max_{(x,\hat{x},\bar{x},w)\in\Delta} \|\frac{\partial V_x}{\partial x}(x) g(x,\hat{x},\bar{x},w)\|_\infty.\end{aligned} \tag{33}$$

*Lemma 4.1:* Let $R < 1/2$. There exists a finite time

$$\theta := 1/2(\gamma_e(\lambda_1) + MY + 1) \tag{34}$$

such that, for all $t \in [0,\theta]$, $x_e(t) \in \mathrm{int}(\Gamma_{c+\zeta+1})$ and

$$\dot{V}(t) \le -\alpha(\|x_e(t)\|_\infty) + \gamma_w(\lambda_1) + M\|\hat{x}(t) - \bar{x}(t)\|_\infty. \tag{35}$$

*Proof:* See Appendix VII-D. ■

Before stating the main result, we introduce our proposed encoder/decoder pair: for each $k \ge 0$,

*Encoder:*

$$\begin{aligned}&\bar{x}(0^-) = 0, \quad L(0) = 2X\\ &L(kT) = \Lambda L((k-1)T)/N + 2c_3 e^{-\eta_3((k-1)T)},\ k \ne 0\\ &b(kT) = \Phi(\hat{x}(kT), \bar{x}(kT^-), L(kT))\\ &\bar{x}(kT) = \bar{x}(kT^-) + b(kT)L(kT)/2N\\ &\quad\dot{\bar{x}}(t) = g(\bar{x}(t), h(\bar{x}(t)), k(\bar{x}(t))), \forall t \in [kT,(k+1)T)\\ &s(kT) = \phi(b(kT))\end{aligned} \tag{36}$$

*Decoder:*

$$\begin{aligned}&\bar{x}(0^-) = 0, \quad L(0) = 2X\\ &b(kT) = \phi^{-1}(s(kT))\\ &L(kT) = \Lambda L((k-1)T)/N + 2c_3 e^{-\eta_3((k-1)T)},\ k \ne 0\\ &\bar{x}(kT) = \bar{x}(kT^-) + b(kT)L(kT)/2N\\ &\quad\dot{\bar{x}}(t) = g(\bar{x}(t), h(\bar{x}(t)), k(\bar{x}(t))), \forall t \in [kT,(k+1)T).\end{aligned} \tag{37}$$

*Theorem 4.1 (Main result):* Consider system (1) and let Assumptions 2.1–2.8 hold. For any $X > 0$, any $\rho \in (0, c+1)$, any $\delta \in (0,1/2)$ and any integer $s$ satisfying

$$s\begin{cases} = 2 & \text{when } \dfrac{\alpha\circ\frac{1}{2}\bar{\alpha}^{-1}(\rho)}{4Mc_3} \ge 1,\\ \ge \dfrac{\left|\ln\left(\dfrac{\alpha\circ\frac{1}{2}\bar{\alpha}^{-1}(\rho)}{4Mc_3}\right)\right|}{|\ln(0.5+\delta)|} + 1 & \text{otherwise,}\end{cases} \tag{38}$$

if there exists a sampling period $T$ satisfying both $T \in (0, \theta/s]$ and

$$T \ge \frac{|\ln\delta|}{\eta_3}, \tag{39}$$

and the number of bits $B = \lceil \log_2 N^n \rceil$ for encoding is chosen to satisfy

$$N > \max\left\{2\Lambda, \Lambda\left(\frac{1}{2MX}\alpha\circ\frac{1}{2}\circ\bar{\alpha}^{-1}(\rho)\right)^{-1/s}\right\}, \tag{40}$$

then system (1) is robustly stabilisable via a finite data-rate communication channel by the encoder/decoder pair (36)–(37) with coding alphabet of size $l = N^n$, the observer (6) and the control $u = k(\bar{x})$.

*Proof:* Given $X > 0$, $\rho \in (0, c+1)$, $\delta \in (0, 1/2)$, an integer $s$ satisfies (38). Define $\epsilon := \alpha\circ\frac{1}{2}\bar{\alpha}^{-1}(\rho)/M$. We pick a sampling period $T \in (0, \theta/s]$. Since $\theta < 1$ (34) and $s > 1$, we have $T \in (0, 1)$. Therefore, if (39) and (40) hold, then $L(sT) < \epsilon$ by applying Lemma 3.4.

Since $\theta \ge Ts$, there exists $s' > s$ such that $\theta \in [(s'-1)T, s'T)$. Using Lemma 3.4, we have

$$\|\hat{x}(\theta) - \bar{x}(\theta)\|_\infty \le \frac{L(s'T)}{2} < \frac{L(sT)}{2} < \epsilon/2. \tag{41}$$

It is also true for any time $\bar{\theta} > \theta$ if $x_e(t) \in \mathrm{int}(\Gamma_{c+\zeta+1})$ for all $t \in [\theta, \bar{\theta}]$ (See [7, Corollary 1]).

Next, by using (16), (35) and (41), we obtain at $t = \theta$,

$$\begin{aligned}\dot{V}(x_e(\theta)) < &-\alpha(\|x_e(\theta)\|_\infty) + \frac{1}{2}\alpha\circ\frac{1}{2}\bar{\alpha}^{-1}(k)\\ &+ \frac{1}{2}\alpha\circ\frac{1}{2}\bar{\alpha}^{-1}(\rho).\end{aligned} \tag{42}$$

By Lemma 4.1, $x_e(\theta) \in \mathrm{int}(\Gamma_{c+\zeta+1})$. Therefore, if $x_e(\theta)$ is in the set $\{\rho+\zeta \le V(x_e) \le c+\zeta+1\}$, it is straightforward to show that $\dot{V}_e(x_e(\theta)) < 0$. As a result, the solution $x_e(t)$ will enter the set $\{\rho+\zeta \le V(x_e)\}$ in some finite time $t_1 \ge \theta$, and cannot leave this set since $\dot{V} < 0$ on the boundary $V(x_e) = \rho+\zeta$. Similarly, if $x_e(\theta)$ is already in the set $\{\rho+\zeta \le V(x_e)\}$, it will not leave this set. In other words, the solution $x_e(t)$ stays in the set $\Gamma_{c+\zeta+1}$ for all $t \ge 0$. It then implies that $\|x_e(t)\|_\infty \le W$ and hence, $\|x(t)\|_\infty \le W$ for all $t \ge 0$.

Using Lemma 3.1 and (16), we have $\|\hat{x}(t)\|_\infty \le \hat{W}$ for all $t \ge 0$. In addition, by Lemma 3.3 and Corollary 3.1, $\|\bar{x}(t)\|_\infty \le \bar{W}$ for all $t \ge 0$. Therefore, $x(t)$, $\hat{x}(t)$ and $\bar{x}(t)$ are uniformly bounded for all $t \ge 0$.

Furthermore, by Corollary 3.1, since $\|x(t)\|_\infty \le W$ for all $t \ge 0$ and $R < 1/2$, we have $\lim_{t\to\infty}\|\hat{x}(t) - \bar{x}(t)\| = 0$. Also, by Assumption 2.2, $\lim_{t\to\infty}\|w(t)\|_\infty = 0$. Therefore, by using (35) and (31), $\dot{V}$ satisfies

$$\dot{V} \le -\alpha(\|x_e\|_\infty) + \gamma_w(\|w\|_\infty) + M\|\hat{x} - \bar{x}\|_\infty. \tag{43}$$

By observing (43), we conclude that $\lim_{t\to\infty}\|x_e(t)\|_\infty = 0$. This implies that both $\|x(t)\|_\infty$, $\|e(t)\|_\infty$ converge to zero as $t \to \infty$. In addition, it is clear that $\lim_{t\to\infty}\|\hat{x}(t)\| = 0$ and $\lim_{t\to\infty}\|\bar{x}(t)\| = 0$. This completes the proof of Theorem 4.1. ■

*Remark 4.1:* Condition (39) indicates that the decay rates of the observer estimation error and the external disturbances are required to be sufficiently fast.

## V. Conclusion

This paper studied a robust stabilisation problem of nonlinear systems with decaying external disturbances using output measurements via finite data-rate communication channels. Given an observer and a control law for the systems in the absence of any finite data-rate communication channel, we constructed an encoder/decoder pair and provided conditions that the stability of the closed-loop systems was guaranteed when introducing finite data-rate communication channels in the feedback loop. In the future research, we will follow [10] and remove some of the assumptions by restricting the class of nonlinear systems, and provide algorithms to explicitly design the observer, control law and encoder/decoder pair.

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## VII. Appendix

### A. Proof of Lemma 3.1

First, we pick any $\bar{c} \in (0, c_e)$ and any $r^0 \in (0, 2X)$. Using the fact that $\gamma_e$ is class $K_\infty$, Assumptions 2.2 and 2.8 imply that there exist strictly positive constants $\bar{\lambda}_1$ and $\bar{\lambda}_2$ such that, for all $t \geq 0$, $\gamma_e(\|w(t)\|_\infty) \leq \gamma_e(\lambda_1 \exp(-\lambda_2 t)) \leq \bar{\lambda}_1 \exp(-\bar{\lambda}_2 t)$. Therefore, we have $V(e(t)) \leq r(t) + \exp(-\bar{c}t)(e(0) - r^0)$ where $r(t)$ is described by $\dot{r} = -\bar{c}r + \bar{\lambda}_1 \exp(-\bar{\lambda}_2 t)$, $r(0) = r^0 > 0$ (see [12, Lemma 3.1]) . Hence, by using Assumption 2.8 and (7), there exist constants $c_1$, $c_2$, $\eta_1$, $\eta_2 > 0$ such that inequality (12) holds.

### B. Proof of Lemma 3.3

By the definition of $Y$ (26), it is obvious that $L(0) = 2X < 2Y$. If $R < 1/2$, equation (23) gives $L(T) = R2X + 2c_3 \exp(-\eta_3 T) < 2Y$. By induction, we have $L(kT) < 2Y$ for all $k \geq 0$. Observing (25), if $R < 1/2$, $\eta_3 > 0$ and $T > 0$, $\lim_{k \to \infty} L(kT) = 0$.

### C. Proof of Lemma 3.4

Given $X > 0$ and $\delta \in (0, 1/2)$. Let $T \in (0, 1]$. By considering (23) and (25) and using the fact that $(a+b)^n \geq a^n + b^n$, for all $n \geq 1$ and $a, b \geq 0$, we have

$$L(kT) \leq R^k X + 2c_3(R + e^{-\eta_3 T})^{k-1}, \quad \forall k > 1. \tag{44}$$

Therefore, if $s > 1$, then

$$L(sT) \leq R^s X + 2c_3(R + e^{-\eta_3 T})^{s-1}. \tag{45}$$

Note that $(1/2 + \delta) < 1$. If $\epsilon/4c_3 < 1$, by using the conditions that $R < 1/2$, $\eta_3 \geq |\ln \delta|/T$ and $s \geq 1 + |\ln(\epsilon/4c_3)|/|\ln(0.5 + \delta)|$, the second term in (45) satisfies

$$2c_3((R + e^{-\eta_3 T})^{s-1}) < 2c_3(\frac{1}{2} + \delta)^{s-1} < \frac{\epsilon}{2}, \tag{46}$$

since $s \geq 2$. On the other hand, if $\epsilon/4c_3 \geq 1$, we have $s = 2$ and (46) also holds. Also, if $R < (\epsilon/2X)^{1/s}$, the first term of (45) satisfies $R^s X < \frac{\epsilon}{2}$. Therefore, $L(sT) < \epsilon$. Since $R < 1$ and $(R + e^{-\eta_3 T}) < 1$, it is obvious that $L(kT) < \epsilon$, for all $k \geq s$.

### D. Proof of Lemma 4.1

We follow the procedure in the proof of [7, Lemma 2]. First, we define time $\bar{t}$ be the largest time which $x(\cdot)$ exists and satisfies $\|x(t)\|_\infty \leq W$, $\|\hat{x}(t)\|_\infty \leq \hat{W}$, $\|\bar{x}(t)\|_\infty \leq \bar{W}$ for all $t \in [0, \bar{t}]$. Such a time $\bar{t}$ exists, since $\|x(0)\|_\infty < W$, $\|\hat{x}(0)\|_\infty < c_1 + c_2 + W = \hat{W}$ and $\|\bar{x}(0)\|_\infty < Y + \hat{W} = \bar{W}$.

Suppose $\bar{t} \leq \theta$. Then, by using (31), (33), Lemma 3.3 and Proposition 3.1, we have, for all $t \in [0, \bar{t}]$,

$$\dot{V}(t) < -\alpha(\|x_e(t)\|_\infty) + \gamma_w(\lambda_1) + MY \tag{47}$$

Next, by using $\bar{t} \leq \theta$, we have

$$V(x_e(t)) < V(x_e(0)) + \frac{1}{2} < \bar{\alpha}(3X) + \frac{1}{2} < c + \zeta + 1. \tag{48}$$

for all $t \leq \bar{t}$, and hence $x_e(t) \in \mathrm{int}(\Gamma_{c+\zeta+1})$ for all $t \in [0, \bar{t}]$. In particular at $\bar{t}$, $\|x(\bar{t})\|_\infty < W$, $\|\hat{x}(\bar{t})\|_\infty < \hat{W}$ and $\|\bar{x}(\bar{t})\|_\infty < \bar{W}$, which then contradicts the definition of $\bar{t}$. It yields $\bar{t} > \theta$. Therefore, by using (47) and (48) with $t \in [0, \theta]$, we have $x_e(t) \in \mathrm{int}(\Gamma_{c+\zeta+1})$ for all $t \in [0, \theta]$.

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