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Extended Kalman-Yakubovich-Popov Lemma in a Hilbert Space and Fenchel Duality.

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Abstract— **The Kalman-Yakubovich-Popov (KYP) lemma is extended with new conditions that are equivalent to solvability of the Lur'e equation or the corresponding linear operator inequality. The relation established between the KYP lemma and an extremum problem on the set of positive semi-definite solutions of the generalized Lyapunov inclusion. It is proved that the statements of the KYP lemma are necessary and sufficient conditions for value to be bounded in this problem. The approach is based on the special Fenchel duality theorem and presents the new proof of the KYP lemma as well. The linear-quadratic optimization problem for a behavioral system in a Hilbert space is considered to illustrate the application of the new statements that are added to the KYP lemma.**

I. INTRODUCTION

The Kalman-Yakubovich-Popov (KYP) lemma is a cornerstone of systems theory. It has numerous applications in stability theory of nonlinear systems, optimal and robust control and filtration, stochastic realization theory, adaptive control and other areas. The KYP lemma was formulated by V.M.Popov [1] as an open problem. The first proofs were obtained by V.A.Yakubovich [2] and R.Kalman [3] for the single input systems. The comprehensive overview of the results concerned with the finite dimensional KYP lemma can be found in [4], [5].

The infinite dimensional versions of the KYP lemma are presented in [6], [7] for bounded operators and in [8], [9], [10] for some classes of unbounded operators. We deal with bounded operators in a Hilbert space.

In view of fundamental importance of the KYP lemma the search continues for new proofs [11], [12] and generalizations [13] of this result.

The original KYP lemma claims the equivalence of the following three statements:

1°. There exists a solution of the Lur'e equation.
2°. There exists a solution of the linear matrix (KYP) inequality.
3°. The so-called frequency domain condition is fulfilled. (This condition is expressed in terms of some quadratic form that depends on a complex parameter. The condition holds if this form is positive semi-definite on the imaginary axis.)

In many papers devoted to the KYP lemma [9], [10], [11], [12], [13] the problem of the Lur'e equation solvability is not considered, the attention is focused on equivalence of the linear matrix inequality and frequency domain condition or their generalizations. It should be noted that under some conditions of regularity the Lur'e equation is equivalent to the algebraic Riccati equation. The solvability of one or another equation is important in a number of applications. Our purpose is to supplement lemma with new statements that are equivalent to generalized versions of the statements 1° – 3°.

We consider an auxiliary extremum problem in the space of self-adjoint trace-class operators. The problem is to minimize a linear functional on the set of positive semi-definite solutions of the generalized Lyapunov equation (or inclusion). The main tool for solution of the problem is a specialized version of the Fenchel duality theorem. Using duality we demonstrate that each of statements 1° – 3° is a necessary and sufficient condition for the value to be finite in this extremum problem.

In our formulation of the KYP lemma the frequency condition is fulfilled on an arbitrary straight line or circle on the complex plane. The statements 1° and 2° are modified accordingly. This allows to consider both continuous-time and discrete-time versions of KYP lemma. The idea of the modification was proposed by Yakubovich in the editorial comment to Russian translation of [14] and it was realized in [15]. In addition, the statements are formulated in a form that is convenient for application to behavioral systems. In finite dimensions our generalized statements 2° and 3° of the KYP lemma are special cases of corresponding statements in [13]. The role of KYP inequality in dissipativity analysis of finite dimensional behavioral systems was considered in [16].

In section III we apply the extended KYP lemma to infinite horizon optimization for the linear behavioral system. The cost function is exponentially weighted integral of quadratic form of system behavior. The example demonstrates a new approach to linear-quadratic optimization.

Some preliminary results on the extended KYP lemma in a Hilbert space were presented on Fourth European Congress of Mathematics (Stockholm, 2004) and are published in [17]. The finite dimensional version of the result is to appear in [18].

II. EXTENDED KALMAN-YAKUBOVICH-POPOV LEMMA

Let us introduce some notation. Let $\mathcal{X}$, $\mathcal{U}$ be separable Hilbert spaces over field of complex numbers $\mathbf{C}$, $\mathcal{W} = \mathcal{X} \oplus \mathcal{U}$. The space of linear bounded operators from $\mathcal{W}$ to $\mathcal{X}$, is denoted $\mathcal{B}(\mathcal{W},\mathcal{X})$, $\mathcal{B}(\mathcal{W}) = \mathcal{B}(\mathcal{W},\mathcal{W})$, $\hat{\mathcal{B}}(\mathcal{W}) \subset \mathcal{B}(\mathcal{W})$ is a space of self-adjoint operators, $\mathcal{C}_1(\mathcal{W}) \subset \mathcal{B}(\mathcal{W})$ is a space of trace-class operators, $\hat{\mathcal{C}}_1(\mathcal{W}) \subset \mathcal{C}_1(\mathcal{W})$ is a space

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of self-adjoint trace-class operators, $\hat{\mathcal{C}}_1^+(\mathcal{W}) \subset \hat{\mathcal{C}}_1(\mathcal{W})$ is a cone of positive semi-definite trace-class operators. The space $\mathcal{C}_1(\mathcal{W})$ is an ideal of the algebra $\mathcal{B}(\mathcal{W})$, i.e., for any $S \in \mathcal{C}_1(\mathcal{W})$, $A \in \mathcal{B}(\mathcal{W})$, $SA \in \mathcal{C}_1(\mathcal{W}), AS \in \mathcal{C}_1(\mathcal{W})$. The trace is a linear functional on $\mathcal{C}_1(\mathcal{W})$ that for any $S \in \mathcal{C}_1(\mathcal{W})$ is given by $\mathrm{tr}(S) = \sum \lambda_i(S)$, where $\lambda_i(S),\ i = 1, 2, \dots$ are eigenvalues of S. Any linear functional on $\hat{\mathcal{C}}_1(\mathcal{W})$ can be represented as $\mathrm{tr}(GS)$, where $G \in \hat{\mathcal{B}}(\mathcal{W})$, $S \in \hat{\mathcal{C}}_1(\mathcal{W})$.

By $M^* \in \mathcal{B}(\mathcal{X}, \mathcal{W})$ denote adjoint of an operator $M \in \mathcal{B}(\mathcal{W}, \mathcal{X})$. Let $\langle .,. \rangle$ be inner product in $\mathcal{W}$. We use shorthand ww^* for the one-dimensional operator $\langle ., w\rangle w \in \hat{\mathcal{B}}(\mathcal{W})$, $w \in \mathcal{W}$. This notation is well defined provided we identify any vector $w \in \mathcal{W}$ with the operator from $\mathbf{C}$ to $\mathcal{W}$ that takes λ to λw.

Let operator $\Lambda : \hat{\mathcal{C}}_1(\mathcal{W}) \to \hat{\mathcal{C}}_1(\mathcal{X})$ be given by

$$\Lambda(S) = (M, N)(\Theta \otimes S)(M, N)^*, \tag{1}$$

where $M, N \in \mathcal{B}(\mathcal{W}, \mathcal{X})$, $\Theta = (\Theta_{i,j})_{i,j=1,2}$ is self-adjoint matrix, $\det \Theta < 0$, $\Theta \otimes S = \begin{pmatrix} \Theta_{11}S & \Theta_{12}S \\ \Theta_{21}S & \Theta_{22}S \end{pmatrix}$. If $\Theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathcal{W} = \mathcal{X}$, $N = I_{\mathcal{X}}$ ($I_{\mathcal{X}}$ is identity operator in $\mathcal{X}$), then $\Lambda(S) = MS + SM^*$ is the Lyapunov operator. On this account the defined by (1) operator Λ can be called the generalized Lyapunov operator. The conjugate to Λ is given by

$$\Lambda'(H) = (M^*, N^*)(\Theta^\top \otimes H)(M^*, N^*)^*.$$

Take $G \in \hat{\mathcal{B}}(\mathcal{W})$, $\mathcal{Q} \subset \hat{\mathcal{C}}_1(\mathcal{X})$. Consider the extremum problem:

(A) *minimize* $\mathrm{tr}(GS)$ *over the set of operators* $S \in \hat{\mathcal{C}}_1^+(\mathcal{W})$ *that satisfy the generalized Lyapunov inclusion*

$$\Lambda(S) \in \mathcal{Q}. \tag{2}$$

We shall demonstrate that the KYP lemma defines necessary and sufficient conditions for the value in problem **(A)** be finite (singular case) or be achieved (regular case).

Define sets $\Gamma = \{\lambda \in \mathbf{C} \mid (\lambda, 1)\Theta(\lambda, 1)^* = 0\}$, $\Omega^\pm = \{\lambda \in \mathbf{C} \mid \pm(\lambda, 1)\Theta(\lambda, 1)^* > 0\}$. The curve Γ is a straight line or circle, $\Omega^\pm$ are open domains that are separated by Γ. Varying the matrix Θ we can define any straight line or circle on the complex plane.

Let the functional $\mathrm{P}_{\mathcal{Q}} : \hat{\mathcal{B}}(\mathcal{X}) \to \bar{\mathbf{R}}$ be given by

$$\mathrm{P}_{\mathcal{Q}}(H) = \inf_{Q \in \mathcal{Q}} \mathrm{tr}(QH),$$

where $\mathcal{Q} \subset \hat{\mathcal{C}}_1(\mathcal{X})$. If $\mathcal{Q} = \{Q\}$ ($Q \in \hat{\mathcal{C}}_1(\mathcal{X})$), then $\mathrm{P}_{\mathcal{Q}}(H) = \mathrm{tr}(QH)$.

Consider the following condition:

(Y) *there exist* $f^\pm \in \mathcal{B}(\mathcal{W},\mathcal{U})$ *such that* $\mathrm{Sp}\begin{pmatrix} N \\ f^\pm \end{pmatrix} \not\ni 0$ *and* $\mathrm{Sp}\left(M \begin{pmatrix} N \\ f^\pm \end{pmatrix}^{-1} \begin{pmatrix} I_{\mathcal{X}} \\ 0 \end{pmatrix} \right) \subset \Omega^\pm$.

This condition is analogous to the condition that was introduced by Yakubovich in [7].

First we consider the singular case dealing with non-strict KYP inequality.

Theorem 1: If condition **(Y)** is fulfilled, then for any $G \in \hat{\mathcal{B}}(\mathcal{W})$ the following statements are equivalent:

1°. There exist $H \in \hat{\mathcal{B}}(\mathcal{X})$, $h \in \mathcal{B}(\mathcal{W},\mathcal{U})$ that satisfy the generalized Lur'e equation

$$\Lambda'(H) - G = -h^*h. \tag{3}$$

2°. There exists $H \in \hat{\mathcal{B}}(\mathcal{X})$ that satisfies the generalized KYP inequality

$$\Lambda'(H) - G \le 0. \tag{4}$$

3°. The frequency condition is fulfilled, i. e., the inequality

$$w^*Gw \ge 0 \tag{5}$$

holds for all $w \in \mathcal{W}$ that satisfy the equation

$$(\lambda N - M)w = 0 \tag{6}$$

with some $\lambda \in \Gamma$.

4°. The inequality (5) holds for all $w \in \mathcal{W}$ that satisfy the equation

$$\Lambda(ww^*) = 0. \tag{7}$$

5°. The inequality

$$\mathrm{tr}(GS) \ge 0$$

holds for all $S \in \hat{\mathcal{C}}_1^+(\mathcal{W})$ that satisfy the equation

$$\Lambda(S) = 0. \tag{8}$$

6°. There exists $Q \in \hat{\mathcal{C}}_1(\mathcal{X})$ such that

$$\inf_{S \in \hat{\mathcal{C}}_1^+(\mathcal{W}) : \Lambda(S) = Q} \mathrm{tr}(GS) > -\infty.$$

7°. The duality relation

$$\inf_{\substack{S \in \hat{\mathcal{C}}_1^+(\mathcal{W}) \\ \Lambda(S) \in \mathcal{Q}}} \mathrm{tr}(GS) = \max_{\substack{H \in \hat{\mathcal{C}}_1(\mathcal{X}) \\ \Lambda'(H) - G \le 0}} \mathrm{P}_{\mathcal{Q}}(H) \tag{9}$$

is fulfilled for any convex bounded nonempty set $\mathcal{Q} \subset \hat{\mathcal{C}}_1(\mathcal{X})$.

There are uniquely defined solutions of (4) $H^\pm \in \hat{\mathcal{B}}(\mathcal{X})$ such that the inequalities

$$H^- \le H \le H^+$$

hold for any $H \in \hat{\mathcal{B}}(\mathcal{X})$ that satisfies (4). There exist $h^\pm \in \mathcal{B}(\mathcal{W},\mathcal{U})$ such that the pairs H^+, h^+ and H^-, h^- satisfy (3).

Let us demonstrate how standard formulation of the KYP lemma [4], [7] can be obtained from Theorem 1. Take

$$\begin{gathered} \Theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, N = (I_{\mathcal{X}}, 0), \\ M = (A, B), A \in \mathcal{B}(\mathcal{X}), B \in \mathcal{B}(\mathcal{U}, \mathcal{X}). \end{gathered} \tag{10}$$

In this case $\Gamma = \mathbf{iR}$ is imaginary axis, $\Omega^\pm = \mathbf{C}^\pm$ are right and left open half-planes, and

$$\Lambda'(H) = \begin{pmatrix} HA + A^*H & HB \\ B^*H & 0 \end{pmatrix}.$$

Thus, we see that (3) and (4) coincide with the standard Lur'e equation and the KYP inequality respectively. Equation (6) takes the form

$$\mathbf{i}\omega x = Ax + Bu, \qquad (11)$$

where $\omega \in \mathbf{R}$. If we suppose that $\mathrm{Sp}(A) \cap \mathbf{i}\mathbf{R} = \emptyset$, then (11) is equivalent to $x = (\mathbf{i}\omega I_{\mathcal{X}} - A)^{-1}Bu$. Consider the Popov operator

$$\Pi(\mathbf{i}\omega) = \begin{pmatrix} (\mathbf{i}\omega I_{\mathcal{X}} - A)^{-1}B \\ I_{\mathcal{U}} \end{pmatrix}^* G \begin{pmatrix} (\mathbf{i}\omega I_{\mathcal{X}} - A)^{-1}B \\ I_{\mathcal{U}} \end{pmatrix},$$

then statement 3° of Theorem 1 takes the form of standard frequency domain condition

$$\Pi(\mathbf{i}\omega) \geq 0, \ \forall \omega \in \mathbf{R}.$$

Let $f^{\pm} \in \mathcal{B}(\mathcal{W},\mathcal{U})$ be operators from condition **(Y)**. In considered case $\begin{pmatrix} N \\ f^{\pm} \end{pmatrix}^{-1} \begin{pmatrix} I_{\mathcal{X}} \\ 0 \end{pmatrix} = \begin{pmatrix} I_{\mathcal{X}} \\ K^{\pm} \end{pmatrix}$, where $K^{\pm} \in \mathcal{B}(\mathcal{X},\mathcal{U})$. Thus, condition **(Y)** can be formulated as follows: there exist $K^{\pm}$ such that $\mathrm{Sp}(A + BK^{\pm}) \subset \mathbf{C}^{\pm}$, i. e., operator $A + BK^{-}$ is Hurwitz and $A + BK^{+}$ is anti-Hurwitz. This coincides with formulation in [7]. In the finite dimensional case ($\mathcal{W},\mathcal{X},\mathcal{U}$ are finite dimensional) **(Y)** is fulfilled iff the pair A, B is controllable. In the infinite dimensional case **(Y)** does not imply controllability of A, B.

If in (10) we put $\Theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then Γ is the unit circle, $\Lambda'(H) = \begin{pmatrix} A^*HA - H & A^*HB \\ B^*HA & B^*HB \end{pmatrix}$. In this case we obtain the discrete-time version of the KYP lemma or the Kalman-Szego lemma.

In the regular case inequality (4) is replaced with the strict one.

Theorem 2: If condition **(Y)** is fulfilled, then for any $G \in \hat{\mathcal{B}}(\mathcal{W})$ the following statements are equivalent:

1°. There exist $H \in \hat{\mathcal{B}}(\mathcal{X})$, $h \in \mathcal{B}(\mathcal{W},\mathcal{U})$ that satisfy (3) and

$$\mathrm{Sp}\begin{pmatrix} N \\ h \end{pmatrix} \not\ni 0, \ \mathrm{Sp}\left(M\begin{pmatrix} N \\ h \end{pmatrix}^{-1}\begin{pmatrix} I_{\mathcal{X}} \\ 0 \end{pmatrix}\right) \cap \Gamma = \emptyset. \qquad (12)$$

2°. There exist $H \in \hat{\mathcal{B}}(\mathcal{X})$, $\delta > 0$ such that

$$\Lambda'(H) - G \leq -\delta I_{\mathcal{X}}. \qquad (13)$$

3°. There exists $\delta > 0$ such that

$$w^*Gw \geq \delta |w|^2 \qquad (14)$$

for all $w \in \mathcal{W}$ that satisfy (6) with some $\lambda \in \Gamma$.

4°. There exists $\delta > 0$ such that (14) holds for all $w \in \mathcal{W}$ that satisfy (7).

5°. There exists $\delta > 0$ such that

$$\mathrm{tr}((G - \delta I_k)S) \geq 0$$

for all $S \in \hat{\mathcal{C}}_1^+(\mathcal{W})$ that satisfy (8).

6°. There exists $Q \in \hat{\mathcal{C}}_1(\mathcal{X})$, $\delta > 0$ such that

$$\inf_{S \in \hat{\mathcal{C}}_1^+(\mathcal{W}):\Lambda(S)=Q} \mathrm{tr}((G - \delta I_{\mathcal{X}})S) > -\infty.$$

7°. For any convex weakly compact nonempty set $\mathcal{Q} \subset \hat{\mathcal{C}}_1(\mathcal{X})$ the following duality relation is fulfilled:

$$\min_{\substack{S \in \hat{\mathcal{C}}_1^+(\mathcal{W}) \\ \Lambda(S) \in \mathcal{Q}}} \mathrm{tr}(GS) = \max_{\substack{H \in \hat{\mathcal{B}}(\mathcal{X}) \\ \Lambda'(H) - G \leq 0}} \mathrm{P}_{\mathcal{Q}}(H). \qquad (15)$$

There are uniquely defined solutions of (4) $H^{\pm} \in \hat{\mathcal{B}}(\mathcal{X})$ such that for any $H \in \hat{\mathcal{B}}(\mathcal{X})$ the inequality $\Lambda'(H) - G < 0$ implies

$$H^- < H < H^+. \qquad (16)$$

There exist $h^{\pm} \in \mathcal{B}(\mathcal{W},\mathcal{U})$ such that the pairs H^+, h^+ and H^-, h^- satisfy (3). If $\pm\Theta_{11} \leq 0$, then

$$\mathrm{Sp}\begin{pmatrix} N \\ h^{\pm} \end{pmatrix} \not\ni 0, \ \mathrm{Sp}\left(M\begin{pmatrix} N \\ h^{\pm} \end{pmatrix}^{-1}\begin{pmatrix} I_{\mathcal{X}} \\ 0 \end{pmatrix}\right) \subset \Omega^{\pm}. \qquad (17)$$

If there is $S \in \mathrm{Argmin}\{\mathrm{tr}(GS) \mid S \in \hat{\mathcal{C}}_1^+(\mathcal{W}),\ \Lambda(S) \in \mathcal{Q}\}$ such that $\pm\Lambda(S) \geq 0$, then $H^{\pm} \in \mathrm{Argmax}\{\mathrm{P}_{\mathcal{Q}}(H) \mid H \in \hat{\mathcal{B}}(\mathcal{X}),\ \Lambda'(H) - G \leq 0\}$.

Remark 1: If $\Theta_{11} = 0$, then Γ is a straight line and (17) is fulfilled for both operators h^+ and h^-. If $\Theta_{11} \neq 0$, then Γ is a circle and (17) can be fulfilled only for one of two operators $h^{\pm}$.

Remark 2: From duality relation (15) it follows that the pair of operators S, H ($S \in \hat{\mathcal{C}}_1^+(\mathcal{W})$, $\Lambda(S) \in \mathcal{Q}$, $H \in \hat{\mathcal{B}}(\mathcal{X})$, $\Lambda'(H) - G \leq 0$) satisfies inclusions $S \in \mathrm{Argmin}\{\mathrm{tr}(GS) \mid S \in \hat{\mathcal{C}}_1^+(\mathcal{W}),\ \Lambda(S) \in \mathcal{Q}\}$, $H \in \mathrm{Argmax}\{\mathrm{P}_{\mathcal{Q}}(H) \mid H \in \hat{\mathcal{B}}(\mathcal{X}),\ \Lambda'(H) - G \leq 0\}$ iff

$$\mathrm{tr}(GS) = \mathrm{P}_{\mathcal{Q}}(H). \qquad (18)$$

Let us consider the special case (10). Putting $G = \begin{pmatrix} G_{xx} & G_{xu} \\ G_{ux} & G_{uu} \end{pmatrix}$, where $G_{xx} \in \hat{\mathcal{B}}(\mathcal{X})$, $G_{xu} \in \mathcal{B}(\mathcal{U},\mathcal{X})$, $G_{ux} \in \mathcal{B}(\mathcal{X},\mathcal{U})$, $G_{uu} \in \hat{\mathcal{B}}(\mathcal{U})$, we can see from (13) that $\mathrm{Sp}(G_{uu}) \subset (0, +\infty)$. In the considered case the pair H, h satisfies the Lur'e equation (3) iff H satisfies the algebraic Riccati equation

$$HA^* + A^*H + (HB - G_{xu})G_{uu}^{-1}(B^*H - G_{ux}) - G_{xx} = 0. \qquad (19)$$

For finite dimensions the ordering (16) of equation (19) solutions was first obtained in [19]. This result is also known in the infinite dimensional case.

The proof of Theorems 1 and 2 is outlined in Section IV.

III. Linear-Quadratic Optimization

As an illustrative example of the extended KYP lemma application we consider the linear-quadratic optimization problem for the behavioral system

$$N\dot{w}(t) = Mw(t), \qquad (20)$$

where $w(.) : [0, +\infty) \to \mathcal{W}$ has derivative at every $t \in [0, +\infty)$, $N, M \in \mathcal{B}(\mathcal{W}, \mathcal{X})$.

In special case $N = (I_{\mathcal{X}}, 0)$ the system (20) has standard state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad (21)$$

where $w = x \oplus u$, $x \in \mathcal{X}$, $u \in \mathcal{U}$, $M = (A, B)$, $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{U}, \mathcal{X})$.

Given $G \in \hat{\mathcal{B}}(\mathcal{W})$, $\alpha \in \mathbf{R}$, we consider the quadratic functional

$$\Phi(w(.)) = \int_0^{+\infty} e^{\alpha t} w^*(t) G w(t)\, \mathrm{d}t,$$

which is defined for all $w(.) \in L_{2,\alpha}(\mathcal{W}) \stackrel{\text{def}}{=} \{w(.) \mid \int_0^{+\infty} e^{\alpha t} |w(t)|^2\, \mathrm{d}t < \infty\}$.

Take $q \in \mathcal{X}$. Consider the following optimization problem:

(B) *minimize $\Phi(w(.))$ over the set of $w(.) \in L_{2,\alpha}(\mathcal{W})$ that satisfy (20) and the initial condition*

$$N w(0) = q. \tag{22}$$

The problem was formulated in [7] for system (21). The linear-quadratic optimization of finite-dimensional behavioral systems was investigated in [20], [21]. We would like to illustrate difference of our approach from others.

Consider operator $\mathcal{I}(w(.)) = \int_0^{+\infty} e^{\alpha t} w(t) w(t)^*\, \mathrm{d}t$. It maps $L_{2,\alpha}(\mathcal{W})$ into $\hat{\mathcal{C}}_1^+(\mathcal{W})$. For any solution of (20) we have

$$\begin{array}{l} N \frac{\mathrm{d}}{\mathrm{d}t}(e^{\alpha t} w w^*) N^* = \alpha N (e^{\alpha t} w w^*) N^* + \\ M (e^{\alpha t} w w^*) N^* + N (e^{\alpha t} w w^*) M^*. \end{array} \tag{23}$$

Integrating (23) on the interval $[0, +\infty)$ and taking into account (22) we obtain the equation

$$\Lambda(\mathcal{I}(w(.))) = -q q^*, \tag{24}$$

where Λ is given by (1) with $\Theta = \begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}$. Using the operator $\mathcal{I}$ we can represent the cost function in the following form

$$\Phi(w(.)) = \mathrm{tr}(G \mathcal{I}(w(.))).$$

In parallel with problem **(B)** let us consider problem **(A)** putting $\mathcal{Q} = \{-q q^*\}$. In this case inclusion (2) takes the form of equation

$$\Lambda(S) = -q q^*. \tag{25}$$

Due to (24) the operator $S = \mathcal{I}(w(.))$ satisfies (25) for any $w(.) \in L_{2,\alpha}(\mathcal{W})$ satisfying (20) and (22). Hence

$$\begin{array}{l} \inf\{\Phi(w(.)) |\, w(.) \in L_{2,\alpha}(\mathcal{W}), w(.) \text{ satisfies } (20), (22)\} \geq \\ \inf\{\mathrm{tr}(GS) \mid S \in \hat{\mathcal{C}}_1^+(\mathcal{W}),\ S \text{ satisfies } (25)\}. \end{array} \tag{26}$$

The idea of our approach is to replace problem **(B)** with problem **(A)**. The relation between the problems is defined by the following corollary of Theorems 1 and 2.

Corollary 1: Let condition **(Y)** be fulfilled.

- If one of statements 1°–7° of Theorem 1 holds, then $\inf\{\Phi(w(.)) \mid w(.) \in L_{2,\alpha}(\mathcal{W}),\ w(.) \text{ satisfies } (20), (22)\} > -\infty$ for any $q \in \mathcal{X}$.
- If one of statements 1°–7° of Theorem 2 holds, then for any $q \in \mathcal{X}$ there exists optimal solution $w^\circ(.)$ of problem **(B)** that is uniquely defined as a solution of interconnected behavioral system (20) and

$$h^- w(t) = 0, \tag{27}$$

satisfying the initial condition (22). The optimal value of cost function is given by

$$\Phi(w^\circ(.)) = -q^* H^- q. \tag{28}$$

Here H^-, h^- are operators defined in statement 7° of Theorem 2. The interconnected system (20), (27) admits the representation in state-space form

$$\dot{x}(t) = M F^- x(t), w(t) = F^- x(t), \tag{29}$$

where $F^- = \begin{pmatrix} N \\ h^- \end{pmatrix}^{-1} \begin{pmatrix} I_{\mathcal{X}} \\ 0 \end{pmatrix}$ and

$$\mathrm{Sp}(M F^-) \subset \{\lambda \in \mathbf{C} \mid \mathrm{Re}\lambda < -\alpha/2\}. \tag{30}$$

- If any one statement of Theorem 1 is not fulfilled, then $\inf\{\Phi(w(.)) \mid w(.) \in L_{2,\alpha}(\mathcal{W}),$ $w(.)$ satisfies $(20), (22)\} = -\infty$ for any $q \in \mathcal{X}$.

Proof: First statement. From statement 7° of Theorem 1 it follows that $\inf\{\mathrm{tr}(GS) \mid S \in \hat{\mathcal{C}}_1^+(\mathcal{W}),\ \Lambda(S) = -q q^*\} > -\infty$. Together with (26) this proves the statement.

Second statement. If $x = N w$, then for any solution $w(.)$ of (20), (27) the pair $w(.), x(.)$ satisfies (29). Converse is also true. Thus, (29) is the state-space representation of (20). If $x^\circ(.), w^\circ(.)$ is a solution of (29) that satisfies the initial condition $x^\circ(0) = q$, then (22) is fulfilled. From (30) it follows that $x^\circ(.) \in L_{2,\alpha}(\mathcal{X})$ and $w^\circ(.) \in L_{2,\alpha}(\mathcal{W})$. Multiplying both sides of (3) by $w^\circ(t) w^\circ(t)^*$ and taking into account equality $h^- F^- = 0$ we obtain $\Lambda'(H^-) w^\circ(t) w^\circ(t)^* = G w^\circ(t) w^\circ(t)^*$ for all $t \in [0, +\infty)$. Integrating this equation on $[0, +\infty)$ with the weight $e^{\alpha t}$ we get equation $\Lambda'(H^-) \mathcal{I}(w^\circ(.)) = G \mathcal{I}(w^\circ(.))$. Calculating the trace of both sides of this equation we obtain

$$\mathrm{tr}(\Lambda'(H^-) \mathcal{I}(w^\circ(.))) = \mathrm{tr}(G \mathcal{I}(w^\circ(.))). \tag{31}$$

By definition of conjugate operator

$$\mathrm{tr}(\Lambda(S) H) = \mathrm{tr}(\Lambda'(H) S), \tag{32}$$

for all $S \in \hat{\mathcal{C}}_1(\mathcal{W})$ and $H \in \hat{\mathcal{B}}(\mathcal{X})$. Equations (31), (32) and (24) imply

$$\mathrm{P}_{\mathcal{Q}}(H^-) = \mathrm{tr}(-q q^* H^-) = \mathrm{tr}(G \mathcal{I}(w^\circ(.))). \tag{33}$$

From the statement 7° of Theorem 2 and Remark 2 it follows that $\mathcal{I}(w^\circ(.)) \in \mathrm{Argmin}\{\mathrm{tr}(GS) \,|\, S \in \hat{\mathcal{C}}_1^+(\mathcal{W}),$ $S \text{ satisfies } (25)\}$. Taking into account (26) we get $w^\circ(.) \in \mathrm{Argmin}\{\Phi(w(.)) \mid w(.) \in L_{2,\alpha}(\mathcal{W}),\ w(.)$ satisfies $(20), (22)\}$, i. e., $w^\circ(.)$ is the optimal solution of problem **(B)**. Equation (28) follows from (33).

The proof of the third statement is analogous to the proof of corresponding result in [7].

IV. Key Elements Of Theorems 1 And 2 Proof

Taking into account limited volume of the paper we cannot present complete proofs. In this section we formulate the most important auxiliary statements and outline the proofs.

A. Fenchel duality in extremum problem (A)

Denote $\mathrm{domP}_{\mathcal{Q}} = \{H \in \hat{\mathcal{B}}(\mathcal{X}) \mid \mathrm{P}_{\mathcal{Q}}(H) > -\infty\}$. It is not hard to prove that

$$\inf_{\substack{S \in \hat{\mathcal{C}}_1^+(\mathcal{W}) \\ \Lambda(S) \in \mathcal{Q}}} \mathrm{tr}(GS) \geq \sup_{\substack{H \in \mathrm{domP}_{\mathcal{Q}} \\ \Lambda'(H) - G \leq 0}} \mathrm{P}_{\mathcal{Q}}(H). \tag{34}$$

Hereafter we assume that $\sup\emptyset = -\infty$.

Theorems 1 and 2 are based on the following duality result.

Theorem 3: Let $\mathcal{Q} \subset \hat{\mathcal{C}}_1(\mathcal{X})$ be convex nonempty set. If condition **(Y)** is fulfilled, then:

- The duality relation

$$\inf_{\substack{S \in \hat{\mathcal{C}}_1^+(\mathcal{W}) \\ \Lambda(S) \in \mathcal{Q}}} \mathrm{tr}(GS) = \sup_{\substack{H \in \mathrm{domP}_{\mathcal{Q}} \\ \Lambda'(H) - G \leq 0}} \mathrm{P}_{\mathcal{Q}}(H)$$

holds.

- If there exists $H \in \mathrm{domP}_{\mathcal{Q}}$ that satisfies (4), then $\mathrm{Argmax}\{\mathrm{P}_{\mathcal{Q}}(H) \mid H \in \mathrm{domP}_{\mathcal{Q}},\ \Lambda'(H) - G \leq 0\} \neq \emptyset$.
- If there exist $H \in \mathrm{domP}_{\mathcal{Q}}$, $\delta > 0$ such that (13) holds and

$$\mathrm{Argmin}\{\mathrm{tr}(QH) \mid Q \in \mathcal{Q}\} \neq \emptyset \ \ \forall H \in \mathrm{domP}_{\mathcal{Q}}, \tag{35}$$

then $\mathrm{Argmin}\{\mathrm{tr}(GS) \mid S \in \hat{\mathcal{C}}_1^+(\mathcal{W}),\ \Lambda(S) \in \mathcal{Q}\} \neq \emptyset$ and for any $S^\circ \in \mathrm{Argmin}\{\mathrm{tr}(GS) \mid S \in HM_k^+,\ \Lambda(S) \in \mathcal{Q}\}$, $H^\circ \in \mathrm{Argmax}\{\mathrm{P}_{\mathcal{Q}}(H) \mid H \in \mathrm{domP}_{\mathcal{Q}},\ \Lambda'(H) - G \leq 0\}$

$$\mathrm{tr}(GS^\circ) = \mathrm{P}_{\mathcal{Q}}(H^\circ) = \mathrm{tr}(\Lambda(S^\circ)H^\circ). \tag{36}$$

For reasons of space we only can comment briefly the proof of Theorem 3. Theorem 3 is a version of Fenchel duality results [22], [23]. It is shown in [18] that for finite dimensions Theorem 3 is a corollary of the corresponding results in [22]. In the infinite dimensional case the known results [23] cannot be applied, because the operator Λ is defined on $\hat{\mathcal{C}}_1(\mathcal{W})$, which is not a reflexive space.

Our proof uses the fact that $\hat{\mathcal{C}}_1(\mathcal{W})$ is a vector lattice. The proof is based on usual technique of convex sets separation. Condition **(Y)** guarantees that the so-called Slater condition is fulfilled. This can be seen from the following simple

Lemma 1: If **(Y)** is fulfilled, then $\Lambda(\hat{\mathcal{C}}_1^+(\mathcal{W})) = \hat{\mathcal{C}}_1(\mathcal{X})$.

B. The structure of solutions set of homogeneous generalized Lyapunov equation

Theorem 4: Every solution $S \in \hat{\mathcal{C}}_1^+(\mathcal{W})$ of (8) has the form

$$S = \sum_{j \in J} w_j w_j^*, \tag{37}$$

where vectors $w_j \in \mathcal{W}$, $j \in J$, are solutions of (7), $J \subset \mathbf{N}$ is a finite or infinite set of indexes. In addition

$$\{w \text{ satisfies } (7)\} = \mathrm{cl}\ \{w \text{ satisfies } (6)\}. \tag{38}$$

In finite dimensional case Theorem 4 is a corollary of corresponding results in [11]. However the technique employed in [11] cannot be used in the infinite dimensional case. Our proof [17] is based on a theorem on completeness of the system of root vectors of some unbounded operators [24].

C. Proof of implications $2° \Rightarrow 3° \Rightarrow 4° \Rightarrow 5° \Rightarrow 6° \Rightarrow 2°$

Let us prove the implications sequence of Theorem 1. The corresponding implications of Theorem 2 are proved analogously.

Consider the implication $2° \Rightarrow 3°$. If w satisfies (6), then $w^*\Lambda'(H)w = 0$. Multiplying both sides of (4) by w^* from the left and by w from the right, we obtain (5). The implication $3° \Rightarrow 4°$ follows from (38). The implication $4° \Rightarrow 5°$ follows from (37). The implication $5° \Rightarrow 6°$ is trivial. The implication $6° \Rightarrow 2°$ follows from the first statement of Theorem 3.

D. Proof of implication $2° \Rightarrow 7°$

Consider the implication $2° \Rightarrow 7°$ in Theorem 2. Taking into account that for any weakly compact set $\mathcal{Q}$ (35) holds and $\mathrm{domP}_{\mathcal{Q}} = \hat{\mathcal{B}}(\mathcal{X})$ we obtain (15) as a corollary of the third statement of Theorem 3.

The rest of statement 7° can also be proved using Theorem 3, but this proof is too cumbersome to be presented here. Therefore we give another shorter proof that does not include inequality (16) and is based on Theorem 1 from [7].

Take a square matrix Ψ of order 2, $\det\Psi \neq 0$, and an operator $T \in \mathcal{B}(\mathcal{W})$, $\mathrm{Sp}(T) \not\ni 0$. Consider the transformation $\mathcal{T}$ of variables that were introduced in Section II

$$\begin{aligned} &(M_{\mathcal{T}}, N_{\mathcal{T}}) = (M, N)(\Psi \otimes I_k)(I_2 \otimes T^{-1}), \\ &\Theta_{\mathcal{T}} = \Psi^{-1}\Theta(\Psi^{-1})^*,\ G_{\mathcal{T}} = (T^{-1})^* G T^{-1}, \\ &\Lambda_{\mathcal{T}}(S) = (M_{\mathcal{T}}, N_{\mathcal{T}})(\Theta_{\mathcal{T}} \otimes S)(M_{\mathcal{T}}, N_{\mathcal{T}})^*. \end{aligned} \tag{39}$$

It is easy to see that

$$\begin{aligned} &\mathrm{tr}(G_{\mathcal{T}} S_{\mathcal{T}}) = \mathrm{tr}(GS),\ \Lambda_{\mathcal{T}}(S_{\mathcal{T}}) = \Lambda(S), \\ &\Lambda'_{\mathcal{T}}(H) = (T^{-1})^*\Lambda'(H)T^{-1}, \end{aligned} \tag{40}$$

where $S_{\mathcal{T}} = TST^*$. Additional properties of transformation (39) are given in the following

Lemma 2: If Θ, M, N satisfies **(Y)**, then there exists the transformation $\mathcal{T}$ defined by (39) such that the triple $\Theta_{\mathcal{T}}, N_{\mathcal{T}}, M_{\mathcal{T}}$ has the form (10) and satisfies **(Y)**.

Suppose that $\pm\Theta_{11} \leq 0$. Let an operator $h \in \mathcal{B}(\mathcal{W}, \mathcal{U})$ be such that $h_{\mathcal{T}} = hT^{-1}$ satisfies the conditions

$$\mathrm{Sp}\begin{pmatrix} N_{\mathcal{T}} \\ h_{\mathcal{T}} \end{pmatrix} \not\ni 0,\ \ \mathrm{Sp}\left(M_{\mathcal{T}}\begin{pmatrix} N_{\mathcal{T}} \\ h_{\mathcal{T}} \end{pmatrix}^{-1}\begin{pmatrix} I_{\mathcal{X}} \\ 0 \end{pmatrix}\right) \subset \mathbf{C}^{\pm}. \tag{41}$$

Then

$$\mathrm{Sp}\begin{pmatrix} N \\ h \end{pmatrix} \not\ni 0,\ \mathrm{Sp}\left(M\begin{pmatrix} N \\ h \end{pmatrix}^{-1}\begin{pmatrix} I_{\mathcal{X}} \\ 0 \end{pmatrix}\right) \subset \Omega^{\pm}.$$

The proof of Lemma 2 is not hard and is omitted.

Consider solvability of Lur'e equation (3). By Lemma 2 it follows that the triple Θ, M, N can be transformed to $\Theta_{\mathcal{T}}, M_{\mathcal{T}}, N_{\mathcal{T}}$ satisfying (10). By (40) the operators $\Theta_{\mathcal{T}}, M_{\mathcal{T}}, N_{\mathcal{T}}, G_{\mathcal{T}}$ satisfy (13). Applying Theorem 1 from [7] we can conclude that there exist operators $H^{\pm} \in \hat{\mathcal{B}}(\mathcal{X})$, $K^{\pm} \in \mathcal{B}(\mathcal{X}, \mathcal{U})$, $R \in \hat{\mathcal{B}}(\mathcal{U})$ such that $\Lambda'_{\mathcal{T}}(H^{\pm}) - G_{\mathcal{T}} = (K^{\pm}, I_{\mathcal{U}})^* R (K^{\pm}, I_{\mathcal{U}})$ and $\mathrm{Sp}(A - BK^{\pm}) \subset \mathbf{C}^{\pm}$. Define $h^{\pm} = R^{1/2}(K^{\pm}, I_{\mathcal{U}})T$, then (3) holds for the pairs $H^{\pm}, h^{\pm}$. From the equations $\begin{pmatrix} N_{\mathcal{T}} \\ h_{\mathcal{T}}^{\pm} \end{pmatrix} = \begin{pmatrix} I_{\mathcal{X}} & 0 \\ R^{1/2}K^{\pm} & R^{1/2} \end{pmatrix}$,

$M_{\mathcal{T}} \left(\begin{array}{c} N_{\mathcal{T}} \\ h^{\pm}_{\mathcal{T}} \end{array} \right)^{-1} \left(\begin{array}{c} I_{\mathcal{X}} \\ 0 \end{array} \right) = A - BK^{\pm}$ it follows that (41) holds. Applying Lemma 2 we obtain (17).

The implication $2^\circ \Rightarrow 7^\circ$ in Theorem 1 is proved analogously.

E. Proof of implications $7^\circ \Rightarrow 1^\circ \Rightarrow 2^\circ$

The implications $7^\circ \Rightarrow 1^\circ$ are trivial in both theorems. The implication $1^\circ \Rightarrow 2^\circ$ is trivial in Theorem 1.

It is easier to prove the implication $1^\circ \Rightarrow 3^\circ$ instead of $1^\circ \Rightarrow 2^\circ$ in Theorem 2. Let the transformation (39) be defined by the identity matrix $\Psi = I_{\mathbf{C}^2}$ and the operator $T = \left(\begin{array}{c} N \\ h \end{array} \right)$. Then $N_{\mathcal{T}} = (I_{\mathcal{X}}, 0)$, the operator $M_{\mathcal{T}}$ can be represented as $M_{\mathcal{T}} = (A, B)$, $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{U}, \mathcal{X})$, and $A = M \left(\begin{array}{c} N \\ h \end{array} \right)^{-1} \left(\begin{array}{c} I_{\mathcal{X}} \\ 0 \end{array} \right)$. From (12) it follows that $\mathrm{Sp}(A) \cap \Gamma = \emptyset$ and hence there exists $\delta_1 > 0$ such that

$$\|(\lambda I_{\mathcal{X}} - A)^{-1} B\| \leq \delta_1 \qquad (42)$$

for all $\lambda \in \Gamma$. Taking into account that $h_{\mathcal{T}} = hT^{-1} = (0, I_{\mathcal{U}})$ we have

$$\Lambda'_{\mathcal{T}}(H) - G_{\mathcal{T}} = - \left(\begin{array}{cc} 0 & 0 \\ 0 & I_{\mathcal{U}} \end{array} \right).$$

Let the vector $w_{\mathcal{T}} = Tw$, $w \in \mathcal{W}$, be represented as $w_{\mathcal{T}} = x_{\mathcal{T}} \oplus u_{\mathcal{T}}$, where $x_{\mathcal{T}} = Nw$, $u_{\mathcal{T}} = hw$. Then for any w satisfying (6) with some $\lambda \in \Gamma$ we have

$$\lambda x_{\mathcal{T}} = A x_{\mathcal{T}} + B u_{\mathcal{T}}. \qquad (43)$$

Take $\delta_2 = (1 + \delta_1^2)^{-1}$. Then from (42) it follows that for any $w_{\mathcal{T}}$ satisfying (43) $|u_{\mathcal{T}}|^2 \geq \delta_2 |w_{\mathcal{T}}|^2$. Therefore for any w satisfying (6) we have $w^* G w = w^*(G - \Lambda'(H)) w = w^*_{\mathcal{T}} (G_{\mathcal{T}} - \Lambda'_{\mathcal{T}}(H)) w_{\mathcal{T}} = |u_{\mathcal{T}}|^2 \geq \delta_2 |w_{\mathcal{T}}|^2 \geq \delta_2 \|T^{-1}\|^{-2} |w|^2$. Consequently, (14) is fulfilled with $\delta = \delta_2 \|T^{-1}\|^{-2}$. This completes the proof.

V. Conclusion

The paper is devoted to extension of the KYP lemma with additional statements. Although we focus on infinite dimensional case the result is also new in finite dimensions. Besides, we present the new proof of the generalized version of the KYP lemma. The crucial points of the proof are Theorems 3 and 4, which are of independent interest. These theorems can be used in various linear-quadratic optimization problems.

The illustrative example presented in Section III is a version of the standard linear-quadratic optimization problem for a behavioral system. The consideration of the exponentially weighted functional allows to look for optimal behavior that has the desired decay rate. The example demonstrates the new approach to the linear-quadratic optimization. The optimization problem for the differential equation in a Hilbert space is replaced with the extremum problem on the solutions of the generalized Lyapunov equation, which is an algebraic equation in the space of trace-class operators.

The proposed result can be used in the linear-quadratic optimization with quadratic constraints. The S-procedure based method for solution of such problems was proposed in [25]. Using Theorems 3 and 4 we can tackle a wider range of problems.

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