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Reciprocal realization and modeling of textured images

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***Abstract*— This paper attempts to view texture analysis and synthesis in image processing as a problem of realization and subspace identification in terms of reciprocal processes defined on a finite interval. We discuss modeling of textures in the framework of acausal stochastic systems and stochastic realization theory.**

***Index Terms*— Reciprocal processes, image analysis, texture synthesis and recognition, stochastic realization, identification.**

I. Introduction

We shall consider a similar setup for texture analysis as that recently described in [11]. Let $\mathcal{I}(k,l)$, $k = 1,\dots,N$, $l = 0,1,\dots,m-1$ be the pixel intensity values of an image. We shall group all the intensity values of the $k-$th row of the image into an m-dimensional vector

$$\mathbf{y}(k) := \begin{bmatrix}\mathcal{I}(k,0) & \dots & \mathcal{I}(k,m-1)\end{bmatrix}^\top$$

Our goal will be to discuss a class of stochastic models of the signal $\{\mathbf{y}(k),\, k = 1,\dots,N\}$ which seems to be particularly adapted to describe the spatial dynamics (or spatial correlation structure) of textured images. These models are the simplest class of linear models able to describe "nearest neighborhood" interactions with a natural acausal structure.

II. Reciprocal processes on a finite interval

The process $\{\mathbf{y}(k)\}$ is only defined for $1 \le k \le N$ but it can be extended periodically on the whole line of integers by setting $\mathbf{y}(k+\nu N) = \mathbf{y}(k)$ for arbitrary $\nu \in \mathbb{Z}$. In this way it becomes periodic, or equivalently, a process on the discrete unit circle $\mathbb{T}$. In this paper all processes will be vector valued zero-mean second order processes defined on the *discrete unit circle* $\mathbb{T} := \{1,\dots,N\}$ where the arithmetics is to be interpreted modulo N. Subintervals of $\mathbb{T}$ are defined in the obvious way, say $[k_0,k_1] = \{k_0, k_0+1,\dots,k_1\}$, $(k_0,k_1) = \{k_0+1,\dots,k_1-1\}$, etc. Let $\mathbb{H}$ be some ambient Hilbert space of second-order zero-mean random variables with the usual inner product $\langle \xi, \eta\rangle := \mathbb{E}\{\xi\eta\}$, containing all random quantities which we shall discuss later on. The following definition introduces a natural generalization of the Markov property to processes defined on a finite set (and in fact to processes defined on rather general partially ordered sets).

Definition 2.1: Let $\mathbf{x} := \{\mathbf{x}(k), k \in \mathbb{T}\}$ be a n-dimensional process and $[k_0,\, k_1] \subseteq \mathbb{T}$ an arbitrary interval. The process $\mathbf{x}$ is *reciprocal* if for all $k \in [k_0,\, k_1]$ and $h \in (k_0,\, k_1)^c$ (the complementary interval of $(k_0,\, k_1)$, namely $[k_1, k_0]$) it holds that $\mathbf{x}(k)$ and $\mathbf{x}(h)$ are conditionally uncorrelated given the boundary values $\mathbf{x}(k_0)$ and $\mathbf{x}(k_1)$.

There is a vast literature on reciprocal processes [5], [6], [9], [10]. Throughout this paper symbols like $\mathbf{x}$ $\mathbf{y}$ etc. will denote the random vector obtained by stacking all (vector) variables of the process sequentially into a unique big $N*n$ dimensional column vector. A reciprocal process $\mathbf{x}$ is said to be of *full rank* (or *non-singular*) if the Toeplitz matrix $\Sigma := \mathbb{E}\mathbf{x}\mathbf{x}^\top$ is positive definite (i.e. nonsingular).

In order to capture the notion of spatial homogeneity (i.e. stationarity in the space variable) which we consider as a defining characteristic of textures, we shall restrict to stationary processes. Abstractly, one says that $\mathbf{y}$ is a *stationary process* on $\mathbb{T}$ if there is a unitary operator U on $\mathbb{H}$ such that

$$\mathbf{y}(k) = U^k\mathbf{y}(0) \qquad k \le N \tag{II.1}$$

which obviously implies that the covariance of the process depends only on the difference of the arguments, so that we can set

$$R(k-h) := \mathbb{E}\mathbf{y}(t+k)\mathbf{y}(t+h)^\top. \tag{II.2}$$

Since $\mathbf{y}(t+k+\nu N) = \mathbf{y}(t+k)$, $R(k)$ is periodic of period N. From [9], [10] we quote the following fundamental representation theorem

Theorem 2.2: Every stationary reciprocal process on $\mathbb{T}$ can be represented by a three terms recursion of the following form

$$M\mathbf{x}(k) = F^\top\mathbf{x}(k-1) + F\mathbf{x}(k+1) + \mathbf{e}(k) \tag{II.3}$$

where M, F are constant matrices, M is symmetric and positive definite, and $\mathbf{e}$ is a locally correlated process, i.e

$$\mathbb{E}\mathbf{e}(k)\mathbf{e}(h)^\top = 0 \qquad |k| > 1 \tag{II.4}$$

such that

$$\mathbb{E}\mathbf{x}(k)\mathbf{e}(k)^\top = I \qquad \mathbb{E}\mathbf{x}(k)\mathbf{e}(h)^\top = 0 \qquad k \ne h \tag{II.5}$$

A technical condition which is needed in the analysis of reciprocal processes is that the determinant of the matrix $M - F^\top z^{-1} - Fz$ should not be identically zero. This is guaranteed if the time horizon is large enough [8].

Note that (II.4) is just saying that $\{\mathbf{e}(k)\}$ is a stationary vector MA process of order one. As we shall see in a moment this process will be of full rank n if and only if $\mathbf{x}$ is full rank. It follows that there must exist an n-dimensional stationary

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white noise $\{\mathbf{w}(k)\}$ and $n \times n$ matrices $Q = Q^\top > 0 = \mathrm{Var}[\mathbf{w}(k)]$, and B such that

$$\mathbf{e}(k) = \mathbf{w}(k) + B\mathbf{w}(k-1) \tag{II.6}$$

Introducing the block-circulant matrix (for circulant matrices see [2]),

$$\Lambda := \begin{bmatrix} M, & -F & 0 & \dots & 0 & -F^\top \\ -F^\top & M & -F & \dots & 0 & 0 \\ 0 & -F^\top & M & -F & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & -F^\top & M & -F \\ -F & \dots & \dots & \dots & -F^\top & M \end{bmatrix} := \mathrm{Circ}\begin{bmatrix} M, & -F, & \dots & -F^\top \end{bmatrix} \tag{II.7}$$

it is immediate to see that equation (II.3) can be rewritten in compact form as

$$\Lambda \mathbf{x} = \mathbf{e}.$$

Right-multiplying by $\mathbf{e}^\top$, this equation provides immediately an expression for the variance of the process $\mathbf{e}$ namely

$$\mathbb{E}\mathbf{e}\mathbf{e}^\top = \Lambda$$

which, in accordance with (II.4) is block-tridiagonal circulant. It then follows that $\mathbf{x}$ is full rank if and only if Λ is invertible. In this case the variance matrix of $\mathbf{x}$ is

$$\Sigma := \mathbb{E}\mathbf{x}\mathbf{x}^\top = \Lambda^{-1} \tag{II.8}$$

which, as anticipated, clearly implies that M should also be positive definite. In other words, *the covariance of a reciprocal stationary process on the circle must be the inverse of a block-tridiagonal circulant matrix*. In fact if we extract the boundary values, the *internal process* $\mathbf{e}_0 = \{\mathbf{e}(k) \mid k = 2, \dots, N-1\}$ subordinated by $\mathbf{e}$ on $(1, N)$, has exactly a block-tridiagonal covariance

$$\Lambda_0 := \begin{bmatrix} M, & -F & 0 & \dots & 0 \\ -F^\top & M & -F & \dots & 0 \\ 0 & -F^\top & M & -F & \dots \\ \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & -F^\top & M & -F \\ 0 & \dots & \dots & -F^\top & M \end{bmatrix}. \tag{II.9}$$

Using this expression it is easy to see that the parameters of the MA representation can be expressed in terms of the original parameters of the descriptor representation (II.3) (and conversely). For, from the moving average representation (II.6) rewritten in matrix form as

$$\begin{bmatrix} \mathbf{e}(2) \\ \mathbf{e}(3) \\ \vdots \\ \mathbf{e}(N-1) \end{bmatrix} = \begin{bmatrix} B & I & 0 & \dots & \dots \\ 0 & B & I & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & B & I \end{bmatrix} \begin{bmatrix} \mathbf{w}(1) \\ \mathbf{w}(1) \\ \vdots \\ \mathbf{w}(N-1) \end{bmatrix}$$

we compute the covariance matrix of $\mathbf{e}_0$ and impose it to be equal to Λ_0, obtaining

$$M = Q + BQB^\top, \qquad F = -QB^\top \tag{II.10}$$

which can be inverted to get

$$B = -F^\top Q^{-1}, \qquad Q = -F^\top Q^{-1} F + M. \tag{II.11}$$

The last equation is an algebraic Riccati-type equation which, by positive definiteness of Λ_0, can be shown to have a unique positive definite solution Q [7], [4]. Hence there is a one-to-one map $(M, F) \leftrightarrow (Q, B)$ mapping the parameters of the reciprocal descriptor model (II.3) into those of the MA model (II.6). This fact will be useful in solving the identification problem later on.

Incidentally, observe that the process $\mathbf{e}$ on the whole interval $[1, N]$ can be represented as

$$\begin{bmatrix} \mathbf{e}(1) \\ \mathbf{e}(2) \\ \vdots \\ \mathbf{e}(N) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \dots & B \\ B & I & 0 & \dots & \dots \\ 0 & B & I & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & B & I \end{bmatrix} \begin{bmatrix} \mathbf{w}(1) \\ \mathbf{w}(2) \\ \vdots \\ \mathbf{w}(N) \end{bmatrix}.$$

We now come to the fundamental object of interest in this paper.

Definition 2.3: An m-dimensional stationary process $\mathbf{y} := \{\mathbf{y}(k), k \in \mathbb{T}\}$ admits a reciprocal realization, if there is an n-dimensional reciprocal stationary process $\mathbf{x}$ such that

$$\mathbf{y}(k) = C\mathbf{x}(k) \qquad k \in \mathbb{T} \tag{II.12}$$

for a suitable constant matrix C.

Hence reciprocal realizations (the models we are interested in) are of the following form

$$M\mathbf{x}(k) = F^\top \mathbf{x}(k-1) + F\mathbf{x}(k+1) + \mathbf{e}(k) \tag{II.13}$$

$$\mathbf{y}(k) = C\mathbf{x}(k) \tag{II.14}$$

Note that we could easily make $M = I$ by substituting $\mathbf{e}(k)$ with a non normalized input process say $M^{-1/2}\mathbf{e}(k)$ and by rescaling the other parameters of the realization accordingly. There is a natural notion of *minimality* of reciprocal realizations and a simple rank test to check it [10] which however we shall not go through in this short survey. We shall instead study the structure of the covariance of processes which admit reciprocal realizations.

First note that the state covariance matrix $\Sigma(\tau) := \mathbb{E}\mathbf{x}(k+\tau)\mathbf{x}(k)^\top$ also satisfies a three terms recursion

$$M\Sigma(k) = F^\top \Sigma(k-1) + F\Sigma(k+1) + I\delta(k) \qquad k \in \mathbb{T} \tag{II.15}$$

where $\delta(k)$ is the usual Kronecker function equal to one if $k = 0$ and zero otherwise. Introducing $\Sigma_1(k) := \Sigma(k-1)$ we can formally rewrite the recursion as

$$\begin{bmatrix} F & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma(k+1) \\ \Sigma_1(k+1) \end{bmatrix} = \begin{bmatrix} M & -F^\top \\ I & 0 \end{bmatrix} \begin{bmatrix} \Sigma(k) \\ \Sigma_1(k) \end{bmatrix} \tag{II.16}$$

which, assuming F nonsingular can be solved to get

$$\begin{bmatrix} \Sigma(k) \\ \Sigma_1(k) \end{bmatrix} = \Phi^k \begin{bmatrix} \Sigma_0 \\ \Sigma_1 \end{bmatrix} \qquad \begin{matrix} \Sigma_0 = \Sigma(0) \\ \Sigma_1 = \Sigma_1(0) \end{matrix} \tag{II.17}$$

where

$$\Phi = \begin{bmatrix} F^{-1}M & -F^{-1}F^\top \\ I & 0 \end{bmatrix} \tag{II.18}$$

In conclusion we see that the covariance function of a process realizable by a reciprocal realization has the following structure

$$R(k) = \mathbb{E}\mathbf{y}(t+k)\mathbf{y}(t)^\top = \begin{bmatrix} C & 0 \end{bmatrix} \Phi^k \begin{bmatrix} \Sigma_0 \\ \Sigma_1 \end{bmatrix} C^\top \qquad \forall k \in \mathbb{T} \tag{II.19}$$

where the expression holds for all k' s. Assuming that the realization we are using is minimal, and recalling standard results from linear system realization theory, the expression above shows that *the block-Hankel matrix constructed with the covariances lags* $\{R(k);\, k = 1, \ldots, N\}$ *must have rank exactly equal to twice the dimension of the state process, i.e.* $2n$. A similar conclusion holds in case F is singular, in which case however we need to bring in the fine structure of the matrix pencil appearing in (II.16), see [8] .

Our main interest in this paper is to solve the inverse problem of computing the parameters of a (minimal) realization, (C, M, F) from the output covariance data $\{R(k)\}$.

III. Identification of Reciprocal Processes

In the applications we have in mind, the state vector will generally have smaller dimension than the output and hence the matrix C (which without loss of generality we assume with linearly independent columns) will have $n < m$ columns. In this case it is easy to show that the process $\mathbf{y}$ *is itself reciprocal* but it is in general a *singular* reciprocal process. There must hence exist a $n \times m$ matrix $C^\sharp$ (any left inverse of C would do) such that $\mathbf{x}(k) := C^\sharp \mathbf{y}(k)$ is reciprocal of full rank and the Toeplitz matrix $R := \mathbb{E}\mathbf{y}\mathbf{y}^\top$ admits a factorization ($\otimes$ denoting Kronecker product)

$$R = (I_N \otimes C)\Sigma(I_N \otimes C)^\top \tag{III.1}$$

where $\Sigma > 0$ is the covariance of an n-dimensional full-rank reciprocal process. A possible way of determining such a C matrix is to compute the numerical range space of the first block row of the matrix R, say by Singular Value Decomposition

$$\begin{bmatrix} R(0) & R(1) & R(2) & \ldots \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}$$

Here the diagonal matrix D_2 contains the "small" singular values which describe the "noise" in the data and are to be discarded. The "denoised" range space is then approximated by that of the matrix U_1. In this way we choose a C matrix with orthonormal columns and a very convenient left inverse to choose is just $C^\sharp = U_1^\top$.

For the rest of the section we shall concentrate on the problem (which we have been reducing our original problem to) of computing (i.e. estimating) the parameters (M, F) of a second-order descriptor model (II.3) of an observed full-rank reciprocal process $\mathbf{x}$.

Under the assumption of a Gaussian distribution for $\mathbf{x}$,

$$p_\Lambda(x) = \frac{1}{\sqrt{(2\pi)^{nN} \det\left(\Lambda^{-1}\right)}} \exp\left(-\frac{1}{2} x^\top \Lambda x\right),$$

the estimation of the model parameters (M, F) can be cast in the framework of maximum likelihood estimation. Observe that the covariance Λ is parameterized through M and L as given by formula (II.7).

Assume that T independent samples $\underline{x} := \left(x^{(1)}, .., x^{(T)}\right)$ of the process $\mathbf{x}$ are available (ideally, assume that T independent sample images of the same texture are available). Then it is possible to write the exact log-likelihood in the form

$$L(M, F) = \log\det(\Lambda) - \frac{1}{T}\mathrm{Trace}\left\{\sum_{i=1}^{T}\left[x^{(i)}\right]^\top \Lambda x^{(i)}\right\} \tag{III.2}$$

where we have neglected constant terms not depending on the parameters.

After some manipulations and grouping of terms in the expression (III.2) and using the structure of the matrix Λ one obtains

$$\begin{aligned} L(M, F) \;=\; & \log\det(\Lambda) + \\ & -\mathrm{Trace}\left\{M T_0(\underline{x})\right\} + \\ & +\mathrm{Trace}\left\{F T_1(\underline{x})\right\} \end{aligned} \tag{III.3}$$

where T_0 and T_1 are given by:

$$T_0(\underline{x}) = \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{N} x^{(i)}(j)\left[x^{(i)}(j)\right]^\top \tag{III.4}$$

and

$$\begin{aligned} T_1(\underline{x}) \;=\; & \tfrac{2}{T}\textstyle\sum_{i=1}^{T}\sum_{j=2}^{N} x^{(i)}(j)\left[x^{(i)}(j-1)\right]^\top \\ & +\tfrac{2}{T}\textstyle\sum_{i=1}^{T} x^{(i)}(1)\left[x^{(i)}(N)\right]^\top \end{aligned} \tag{III.5}$$

from (III.3) we can easily see that hat T_0 and T_1 are sufficient statistics for the parameters (M, F)[1]. It is a well-known fact in statistics that the maximum likelihood estimator is a function of a minimal sufficient statistics, which in our case are T_0 (its upper triangular part) and T_1. From the theory of exponential families, it also follows (see [1]) that T_0 and T_1 are Maximum Likelihood (and therefore efficient) estimators for their expected values, which are given by

$$ET_0 = N\Sigma(0) \qquad ET_1 = N\Sigma(1)$$

Hence, the maximum likelihood estimators of M ad L must be expressible as functions of the maximum likelihood estimators of $\Sigma(0)$ and $\Sigma(1)$

$$\hat{\Sigma}(0) := \frac{T_0(\underline{x})}{N} \qquad \hat{\Sigma}(1) := \frac{T_1(\underline{x})}{N}. \tag{III.6}$$

In fact the problem of estimating (M, F) from $\hat{\Sigma}(0)$ and $\hat{\Sigma}(1)$ is a classical *Covariance Selection Problem*. For details we shall refer the reader to the seminal paper [3]. From Dempster's covariance selection theory it follows that our problem can be reformulated as that of completing the covariances $\hat{\Sigma}(0), \hat{\Sigma}(1)$ with a sequence of estimates $\hat{\Sigma}(2), ...\hat{\Sigma}(N-1)$ in such a way that the (block-circulant) estimate $\hat{\Sigma} = \mathrm{Circ}\left[\hat{\Sigma}(0), \hat{\Sigma}(1)^\top, \ldots, \hat{\Sigma}(N-1)^\top\right]$ admits

[1]Note that the matrix M is symmetric. Therefore, strictly speaking, only the upper triangular part of $T_0(\underline{x})$ in (III.4) is a sufficient statistics, T_0 being symmetric.

an inverse $\hat{\Lambda} = \hat{\Sigma}^{-1}$ which has zero elements exactly in the positions where Λ in (II.7) is zero.

It is shown in [3] that this completion problem has a unique solution. The only viable numerical scheme (to the best of our knowledge) to compute the solution turns out still to be Dempster's original iterative algorithm. This algorithm, starting from any (positive definite) estimate of Σ, converges to the unique matrix $\hat{\Sigma}$ which

- has $\hat{\Sigma}(0)$ and $\hat{\Sigma}(1)$ in positions (i,i) and $(i,i-1)$ respectively
- the inverse $\hat{\Sigma}^{-1} = \hat{\Lambda}$ is of the form (II.7) where the $n \times n$ blocks in position $(1,1)$ and $(1,2)$ are the maximum likelihood estimators of M and $-F$ respectively.

For the purpose of exposition we report a simple (scalar, i.e. $n = 1$) example where $N = 15$, $T = 30$, $M = 1.25$ $F = -0.25$. The process has been generated taking T independent samples from a Gaussian random vector with covariance matrix $\Sigma = \Lambda^{-1}$. In figure 1 we report a view of the likelihood as a function of B and Q and in figure 2 the values of the estimated parameters $\hat{M}_i$ and $\hat{F}_i$ estimated at the i-th iteration of the algorithm. As predicted the solutions converge to the maximum likelihood estimates (dotted line, M_{ML} and F_{ML}).

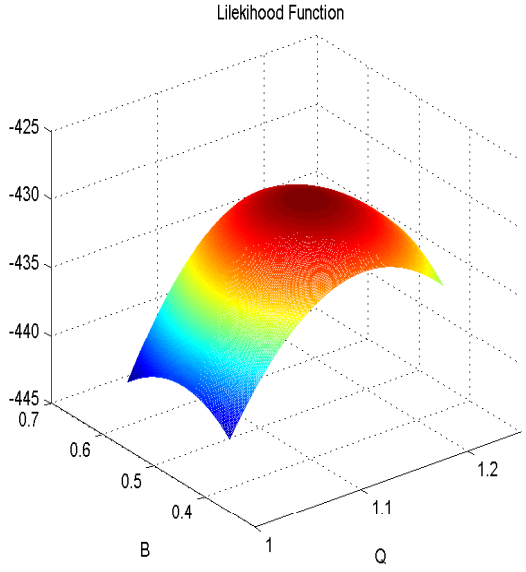

Fig. 1. Likelihood Function

Unfortunately, Dempster's iterative algorithm used above is computationally intensive; in particular it requires repeated inversion of matrices of size $O(N * n)$. It does not seem to be useful for real size images.

We may therefore have to resort to approximations. To this purpose, it should be stressed that the length of the interval N is a parameter playing and essential role in the modeling problem we are addressing. Note in fact, that for fixed M and F, the corresponding $\Sigma(0)$, $\Sigma(1)$ vary with N. It is remarkable (but not surprising) that this dependence generally decreases (and vanishes asymptotically) as N increases. In

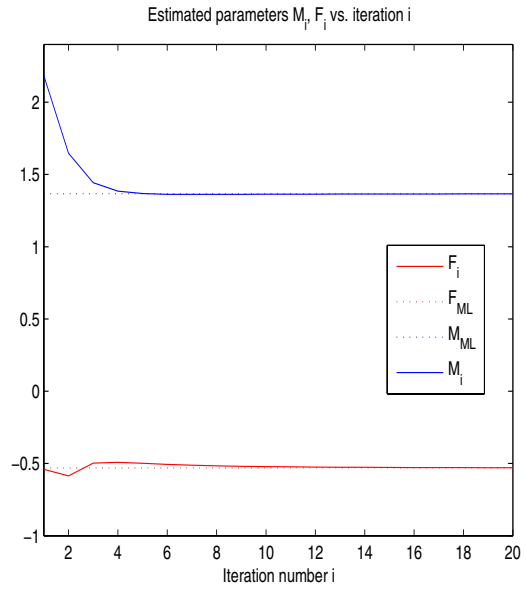

Fig. 2.

figure 3 we show, for the same values of M and F chosen in the previous example, how $\Sigma(0)$ and $\Sigma(1)$ vary as a function of N. It is apparent that as N increases their values converge.

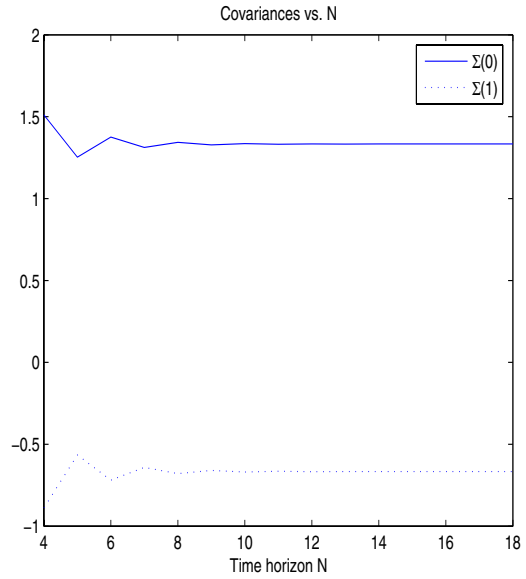

Fig. 3. Covariance matrices $\Sigma(0)$ and $\Sigma(1)$ for fixed values of $M = 1.25$, $F = -0.5$, as N ranges in the interval $[4, 18]$.

We can describe this fact by saying that enlarging the time horizon N, the effect of the boundary condition vanishes. For this reason, provided N is large enough, one may hope that an approximate, but yet accurate estimate of M and F, could be found by just resorting to algebraic techniques which use the asymptotic values $\hat{\Sigma}(0)$ and $\hat{\Sigma}(1)$ corresponding

to "large" N. In fact it will be argued in the following section that in the limit $N \to \infty$ the map relating (M, F) and $(\Sigma(0), \Sigma(1))$ does not depend on N. We shall take advantage of this fact to obtain a simple algorithm which yields (approximate) maximum likelihood estimators $\hat{M}, \hat{F}$.

IV. The stationary infinite-interval approximation

As shown in [7], (stationary) reciprocal processes defined on the whole time axis $\mathbb{Z}$ (i.e. $N = \infty$) are (stationary) Markov. This means that the model parameters of a reciprocal process defined on the whole line are in one-to-one correspondence with the parameters A and $R = Var\{\mathbf{v}\}$ of a standard stationary Markov model

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + \mathbf{v}(t) \tag{IV.1}$$

of the same process. This should not be surprising since, also for Markov processes there is a one to one correspondence between the pair (A, R) and the two covariances $\Sigma(0), \Sigma(1)$ given by:

$$\begin{aligned} A &= \Sigma(1)\Sigma(0)^{-1} \\ R &= \Sigma(0) - A\Sigma(0)A^{\top} = \Sigma(0) - \Sigma(1)\Sigma(0)^{-1}\Sigma(-1) \end{aligned} \tag{IV.2}$$

Hence by using using equation (IV.2) and the one-to-one map attaching (M, F) to (A, R), it is in principle easy to determine the (asymptotic) estimators $(\hat{M}, \hat{F})$ from the covariance estimates $\hat{\Sigma}(0), \hat{\Sigma}(1)$.

For reasons of space we shall not enter into the derivation of these maps; for our purposes suffices it to say that there exist functions Ψ_M and Ψ_F by which, the model parameters M and F of a reciprocal process on the line, can be computed as

$$M = \Psi_M(\Sigma(0), \Sigma(1)) \quad F = \Psi_F(\Sigma(0), \Sigma(1)); \tag{IV.3}$$

explicit expressions for Ψ_M and Ψ_F may be found in the Appendix.

As discussed at the end of the previous Section, when N is"large" [2] one may hope that

$$\hat{M}_a = \Psi_M(\hat{\Sigma}(0), \hat{\Sigma}(1)) \quad \hat{F}_a = \Psi_F(\hat{\Sigma}(0), \hat{\Sigma}(1)); \tag{IV.4}$$

produce approximate estimators $\hat{M}_a$, $\hat{F}_a$ which are close to being maximum likelihood.

Further work is needed to explore the properties of $\hat{M}_a$, $\hat{F}_a$. We shall just illustrate how this approximation behaves for a fixed time horizon N as a function of the roots of $(M - F^{\top}z^{-1} - Fz)$ with a simple example.

We shall consider a scalar reciprocal process defined on the interval $[0, 10]$ and vary the parameters so that the stable root of $(M - F^{\top}z^{-1} - Fz)$ ranges in the interval $[0.1, 0.95]$. We plot in Figure 4 the "true " parameters M and F (solid lines) and the corresponding parameters obtained from $\Sigma(0)$ and $\Sigma(1)$ (which correspond to having $N = \infty$) obtained through equations (IV.3). As expected the approximation becomes worse as the pole becomes closer to the unit circle.

[2]Here "large" should be compared with process dynamics, i.e. the stable roots of $M - F^{\top}z^{-1} - Fz$.

Note that here N is rather small ($N = 10$) as compared to the number of rows of a typical image ($N = 170$ in the example of Figure 5).

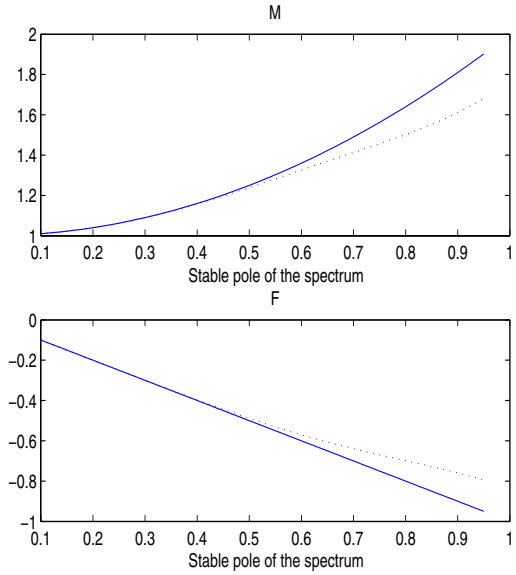

Fig. 4. True parameters M and F (solid lines) vs. their approximations obtained assuming N "large" (dotted lines) as a function of the spectrum pole. The approximation becomes worst as the pole becomes closer to the unit circle.

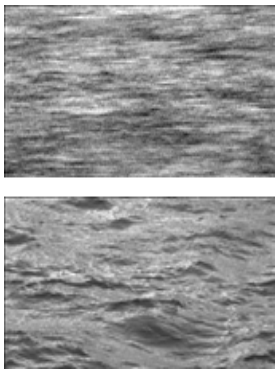

Fig. 5. Comparison between original (bottom) and synthesized (top) texture (River sequence).

V. Experiments

We have tested the approximate technique proposed in this paper on a real texture; the data used for identification is the "river" sequence which can be downloaded from the web site *ftp://whitechapel.media.mit.edu/pub/szummer/temporal-texture/.*

We have first identified the parameters M and F using the algorithm described in Section IV, as if the image sequence was formed by i.i.d. samples. Then we have synthesized a *new* image which matches the second order statistics of a reciprocal process described by the identified parameters $\hat{M}$ and $\hat{F}$. Figure 5 shows a synthesized image (top) compared to an image from the data sequence (bottom).

VI. Texture smoothing

Besides data compression, synthesis and recognition, the reciprocal models (II.13) are extremely useful for data smoothing (i.e. acausal filtering). The acausal filtering problem consists in extracting a minimum error variance estimate of the image signal $C\mathbf{x}(k)$ from observations affected by additive white noise $\{\mathbf{w}(k)\}$,

$$\begin{aligned} M\mathbf{x}(k) &= F^\top \mathbf{x}(k-1) + F\mathbf{x}(k+1) + \mathbf{e}(k) \quad \text{(VI.1)} \\ \mathbf{y}(k) &= C\mathbf{x}(k) + \mathbf{w}(k) \quad \text{(VI.2)} \end{aligned}$$

where $\mathbb{E}\{\mathbf{w}(k)\mathbf{e}(h)^\top\} = 0$ for all $k, h \in \mathbb{T}$. The estimate $\hat{\mathbf{x}}(k) = \mathbb{E}[\,\mathbf{x}(k) \mid \mathbf{y}\,]$ can be computed efficiently by either *Double sweep* algorithms of the Rauch-Tung-Striebel type or by causal-anticausal decomposition filters. No Riccati equation is required to be solved for computing the filters. We refer to [9, Sect. VI] for details.

VII. Conclusions and further work

In this paper we have just touched upon a simple instance of the problem of modeling and identification of stationary processes admitting a reciprocal realization. Much remains to be done both at a general theoretical level to cover the (general) case where $\mathbf{y}$ is itself not reciprocal and also at the practical level of assessing the usefulness of the procedures delineated in this paper for Texture analysis and synthesis.

Appendix

In this appendix we report the expressions of the maps Ψ_M and Ψ_F in (IV.3). For ease of exposition we first recall that the reciprocal model (II.3) can be also written in the form

$$\mathbf{x}(t) = F_- \mathbf{x}(t-1) + F_+ \mathbf{x}(t+1) + \mathbf{d}(t)$$

where the matrices F_- and F_+ and the noise sequence $\mathbf{d}(t)$ are related to M, F and $\mathbf{e}(t)$ by the relations

$$\begin{aligned} F_- &= M^{-1}F^\top \\ F_+ &= M^{-1}F \\ \mathbf{d}(t) &= M^{-1}\mathbf{e}(t) \end{aligned} \quad \text{(A.1)}$$

Hence the noise $\mathbf{d}(t)$ is still a first order MA process with variance $Var\{\mathbf{d}(t)\} = M^{-1}$.

Defining, for ease of notation, $P := \Sigma(0)^{-1}$, the matrices F_- and F_+ can be computed from $A := \Sigma(1)\Sigma(0)^{-1} = \Sigma(1)P$ as

$$F_- = \left(P - A^\top P A\right) A \left(P - A^\top A^\top P A A\right)^{-1}$$

$$F_+ = \left(P - APA^\top\right) A^\top \left(P - AAPA^\top A^\top\right)^{-1}$$

and the noise variance

$$Var\{\mathbf{d}(t)\} = P - F_- A^\top - F_+ A$$

from which

$$M = \left(P - F_- A^\top - F_+ A\right)^{-1}.$$

The matrix F consequently can be computed inverting (A.1)

$$\begin{aligned} F &= MF_+ \\ &= \left(P - F_- A^\top - F_+ A\right)^{-1} \cdot \\ &\quad \cdot \left(P - APA^\top\right) A^\top \left(P - AAPA^\top A^\top\right)^{-1} \end{aligned}$$

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