Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Semi–Blind Model (In)Validation with Applications to Texture Classification

Mario Sznaier
Department of Electrical Engineering
The Pennsylvania State University
University Park, PA 16802
email: msznaier@frodo.ee.psu.edu

María Cecilia Mazzaro
Automation and Controls Laboratory
GE Global Research
Niskayuna, NY 12309
email: mazzaro@research.ge.com

Octavia Camps
Department of Electrical Engineering
The Pennsylvania State University
University Park, PA 16802
email: camps@psu.edu

Abstract— **This paper addresses the problem of model (in)validation of linear discrete–time (LTI) models subject to unstructured LTI uncertainty, using frequency–domain data corrupted by additive noise. Contrary to the case usually considered in the (deterministic) invalidation literature, here the input to the system has an unknown phase. This problem arises naturally for instance in the context of validating systems subject to unknown time–delays, or in cases where only the spectral power density of the (in this case stochastic) input is known. It can be shown that this leads to a generically NP hard minimization problem. The main result of this paper is an efficient, LMI based convex relaxation of the problem. These results are illustrated with a non–trivial problem: classification of textured images.**

I. Introduction

This paper considers the problem of semi–blind frequency–domain (in)validation of discrete–time, Linear Time Invariant (LTI) models subject to unstructured LTI dynamic uncertainty entering the model in a Linear Fractional Transformation (LFT) form. In general terms, this problem can be formally stated as follows: Given (i) a priori information consisting of a candidate model, and set descriptions $\mathcal{N}$, $\boldsymbol{\Delta}$ and $\mathcal{U}$ of the measurement noise, model uncertainty and experimental inputs, and (ii) experimental data consisting of frequency–domain measurements, corrupted by additive noise, to an unknown input in $\mathcal{U}$, find whether the *a posteriori* experimental data is consistent with the *a priori* information, that is whether the candidate model together with some combination of admissible uncertainty, input and noise could have generated this data. If the answer is negative, then the model is said to be invalidated and should be rejected; otherwise, is said to be not invalidated by the available experimental evidence.

Model (in)validation of LTI systems in a Robust Control setting has been extensively addressed in the past decade (see for instance [10], [7], [2], [1], [5], [9], [17] and references therein). The main result ([2], [1]) shows that in the case of a completely known input and unstructured LTI uncertainty entering the plant as an LFT, model (in)validation reduces to a LMI feasibility problem that can be efficiently solved. However, this framework cannot be directly applied here, where only a set description of the input is available. This situation arises in many practical cases. Examples are the validation of plants subject to unknown time delays or when the only information available about the input is its spectral power density.

As we will show in the paper, semi–blind (in)validation leads to a (generically NP–hard) Bilinear Matrix Inequality (BMI) minimization problem. However, an efficient convex relaxation can be obtained by recasting the problem into a *structured* invalidation form, with two uncertainty blocks. While it has been shown in [16] that (in)validation with structured LTI blocks is NP–hard in the number of uncertainty blocks, as we shown in the sequel, in the case of uncertainty structures with two blocks, *necessary and sufficient* convex LMI based invalidation conditions can be obtained. These conditions are precisely the LTI counterpart of those recently introduced in [14] for the case of slowly linear time varying uncertainty. Our approach is also related to that in [10], [5], in the sense that it recasts model (in)validation into a robust performance form, albeit in the $\mathcal{H}_2$ rather than $\mathcal{L}_\infty$ sense.

II. Preliminaries

Below we summarize the notation used in this paper:

$\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$	set of integer, real and complex numbers respectively.
x, x^*	complex–valued column vector and its conjugate transpose row vector.
$\|x\|$	euclidean norm of vector $x \in \mathbf{C}^m$: $\|x\| \doteq (x^*x)^{\frac{1}{2}}$.
A^*	conjugate transpose of matrix A.
$\overline{\sigma}(A)$	maximum singular value of matrix A.
$A > 0$ $(A \leq 0)$	$A = A^*$ is positive definite (negative semidefinite).
$I, 0$	identity and null matrices of compatible dimensions (when omitted).
$\mathcal{BX}(\gamma)$	γ-ball in a normed space $\mathcal{X}$: $\mathcal{BX}(\gamma) = \{x \in \mathcal{X} \colon \|x\|_{\mathcal{X}} \leq \gamma\}$.
$\mathcal{BX}$	(closed) unit ball in $\mathcal{X}$.
ℓ_2^m	Hilbert space of vector–valued sequences $\{x_i\}_{i \in \mathbf{Z}}$, equipped with the inner product: $\langle x, y \rangle \doteq \sum_{i \in \mathbf{Z}} x_i^* y_i.$ and norm $\|x\|_2 \doteq \langle x, x \rangle^{\frac{1}{2}}$

This work was supported in part by AFOSR under grant FA9550–05–1–0437 and NSF, under grants IIS–0117387, ECS–0221562 and ITR–0312558.

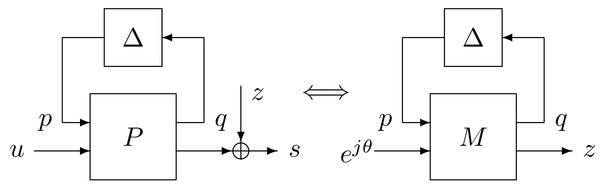


Fig. 1. Model (In)Validation Set–up

$\mathcal{L}_\infty$ Lebesgue space of complex–valued matrix functions $X(z)$ essentially bounded on the unit circle, equipped with the norm: $\|X\|_\infty \doteq \text{ess } \sup_{|z|=1} \overline{\sigma}\,(X(z))$.

$\mathcal{H}_\infty$ subspace of functions in $\mathcal{L}_\infty$ with bounded analytic continuation inside the unit disk, equipped with the norm: $\|X\|_\infty \doteq \text{ess } \sup_{|z|<1} \overline{\sigma}\,(X(z))$.

$\mathcal{RL}_\infty$ $(\mathcal{H}_\infty)$ subspace of $\mathcal{L}_\infty$ $(\mathcal{H}_\infty)$ of rational functions.

$\mathcal{L}_2^m$ Hilbert space of Lebesgue square integrable vector functions $x(\omega)$ equipped with the norm $||x||_2 \doteq \int_o^{2\pi} \text{trace}\,[x(\omega)x(\omega)^*]\, \frac{d\omega}{2\pi}$.

$x(e^{j\omega})$ Fourier transform of a real–valued sequence in ℓ_2^m: $x(e^{j\omega}) \doteq \sum_{i\in\mathbf{Z}} x_i e^{-j\omega i}$.

$X(z)$ $\mathcal{Z}$-transform of a real–valued matrix sequence $\{X_i\}_{i\in\mathbf{Z}}$: $X(z) = \sum_{i\in\mathbf{Z}} X_i z^{-i}$.

$M \star \Delta$ Upper linear fractional transformation:

$$M\star\Delta = M_{21}\Delta(I-M_{11}\Delta)^{-1}M_{12}+M_{22}.$$

III. Problem Statement

Consider the problem of invalidating a model of the form shown in Fig. 1 on the left, consisting of the upper linear fractional interconnection $P \star \Delta$ of a discrete–time, causal, stable, LTI candidate model P:

$$\begin{aligned} q(e^{j\omega}) &= P_{11}(e^{j\omega})p(e^{j\omega}) + P_{12}(e^{j\omega})u(e^{j\omega}) \\ s(e^{j\omega}) &= P_{21}(e^{j\omega})p(e^{j\omega}) + P_{22}(e^{j\omega})u(e^{j\omega}) + z(e^{j\omega}) \end{aligned} \tag{1}$$

and an unstructured uncertainty block $\Delta \in \mathcal{BH}_\infty(\gamma)$

The block P consists of a nominal model of the actual system P_{22} and some description of how the uncertainty affects the model, given by the blocks P_{11}, P_{12} and P_{21}. Furthermore, we assume that model P has a rational transfer function $P(z) \in \mathcal{RH}_\infty$ and that $\|P_{11}\|_\infty < \gamma^{-1}$ so that the interconnection $P \star \Delta$ is robustly ℓ_2 stable. The signals $u \in \ell_2$, $s \in R^{n_z}$, $z \in \mathcal{N} \doteq \mathcal{B}\ell_2^{n_z}(\epsilon)$, represent the input to the plant, the measured output and measurement noise, respectively. In the sequel we will assume that the only information known about the signal u is the magnitude of its Fourier transform, $|u(j\omega)|$, that is

$$u \in \mathcal{U} \doteq \{u \in \ell_2 \colon |u(e^{j\omega})| \text{ given}\}. \tag{2}$$

The goal is, given the measurements $s(e^{j\omega})$ to determine whether the candidate model P together with an admissible triple $(u, \Delta, z) \in \mathcal{U} \times \mathcal{BH}_\infty(\gamma) \times \mathcal{N}$ could have generated this output pair, i.e. whether:

$$s = (P \star \Delta)u + z, \quad \text{for some } (\Delta, z, u). \tag{3}$$

If the answer is affirmative, then the model is said to be not invalidated by the experimental evidence. On the contrary, if no such triple (Δ, z, u) exists, the model should be discarded.

Under the assumptions that both signals (u, s) are the impulse responses of some discrete–time, causal, stable, LTI, rational systems in $\mathcal{RH}_\infty$, and since the *magnitude* of $u(e^{j\omega})$ is known, equations (1) can be rewritten as follows:

$$\begin{aligned} q(e^{j\omega}) &= M_{11}(e^{j\omega})p(e^{j\omega}) + M_{12}(e^{j\omega}) \\ z(e^{j\omega}) &= M_{21}(e^{j\omega})p(e^{j\omega}) + M_{22}, \end{aligned}$$

where

$$\begin{aligned} &M_{11}(e^{j\omega}) \doteq \gamma P_{11}(e^{j\omega}),\ M_{21}(e^{j\omega}) \doteq -\frac{\gamma}{\epsilon}P_{21}(e^{j\omega}), \quad (4) \\ &M_{12}(e^{j\omega}) \doteq P_{12}(e^{j\omega})S_u(e^{j\omega})e^{j\theta(\omega)}, \\ &M_{22}(e^{j\omega}) \doteq \frac{1}{\epsilon}\big(s(e^{j\omega}) - P_{22}(e^{j\omega})S_u(e^{j\omega})e^{j\theta(\omega)}\big) \end{aligned}$$

Here S_u is a stable transfer matrix such that $|u|^2 = S_u^* S_u$, $e^{j\theta(\omega)}$ represents the unknown phase, and (z, Δ) have been normalized so that $z \in \mathcal{BL}_2$, $\Delta \in \mathcal{BH}_\infty$. In this framework, the semi–blind model (in)validation problem can be precisely stated as follows.

Problem 1: Given the output $s(e^{j\omega})$ and the admissible sets of inputs $\mathcal{U}$ and noise $\mathcal{N}$, determine whether there exists at least one pair $z \in \mathcal{N}$, $\Delta \in \mathcal{BH}_\infty$ and a scalar function $\theta(\omega)$ so that equation (1) holds; or equivalently, whether :

$$\mu = \min_{\Delta,\theta} \|M \star \Delta\|_2 \leq 1, \tag{5}$$

where the system M is defined in (4), and we have used the fact that the $\|.\|_2$ of a single input LTI system coincides with the energy of its impulse response.

IV. Main Results

In this section we propose a sufficient condition for solving Problem 1, in terms of frequency–dependent Linear Matrix Inequalities. A difficulty in solving (5) stems from the fact that the problem is not jointly convex in θ, Δ. Indeed, it can be shown using standard Nevanlinna–Pick and Schur complement arguments that even the simple case of multiplicative uncertainty leads to a BMI. The goal of this section is to obtain a tight convex relaxation of the problem.

A. Problem Transformation

The first step in obtaining a convex relaxation of Problem 1, or equivalently, of the optimization problem (5) is to note that:

$$\begin{aligned} \|(P \star \Delta)S_u e^{j\theta(\omega)} - s\|_2 &= \|(P \star \Delta)S_u - s e^{-j\theta(\omega)}\|_2 \\ &= \|M_{\text{aug}} \star \Delta_{\text{aug}}\|_2 \end{aligned} \tag{6}$$

where

$$M_{\rm aug} \doteq \begin{bmatrix} 0 & 0 & 1 \\ 0 & \gamma P_{11}(e^{j\omega}) & P_{12}(e^{j\omega})S_u(e^{j\omega}) \\ \frac{1}{\epsilon}s(e^{j\omega}) & -\frac{\gamma}{\epsilon}P_{21}(e^{j\omega}) & -\frac{1}{\epsilon}P_{22}(e^{j\omega})S_u(e^{j\omega}) \end{bmatrix}$$

$$\Delta_{\rm aug} \doteq \begin{bmatrix} e^{-j\theta(\omega)} & 0 \\ 0 & \Delta \end{bmatrix} \tag{7}$$

In terms of these augmented structures, (5) can be restated as a constrained optimization problem with structured uncertainty:

$$\mu = \min_{\Delta \in \mathbf{\Delta}_{\rm aug}} \|M_{\rm aug} \star \Delta_{\rm aug}\|_2$$
$$\mathbf{\Delta}_{\rm aug} \doteq \left\{ \begin{bmatrix} \delta_1(e^{j\omega}) & 0 \\ 0 & \Delta_2(e^{j\omega}) \end{bmatrix} : \begin{array}{l} |\delta_1| = 1 \\ \Delta_2 \in \mathcal{BH}_\infty \end{array} \right\}$$

Note that the set $\Delta_{\rm aug}$ is not convex, due to the constraint $|\delta_1| = 1$. To address this difficulty and obtain a tractable optimization problem, we will relax the constraint to $\|\delta_1(e^{j\omega})\|_\infty \leq 1$. This leads to the following model (in)validation problem with 2–block LTI structured uncertainty.

$$\mu_{\rm st} = \min_{\Delta \in \mathbf{\Delta}_{\rm st}} \|M_{\rm aug} \star \Delta_{\rm st}\|_2$$
$$\mathbf{\Delta}_{\rm st} \doteq \left\{ \begin{bmatrix} \Delta_1(e^{j\omega}) & 0 \\ 0 & \Delta_2(e^{j\omega}) \end{bmatrix} : \|\Delta_i\|_\infty \leq 1 \right\} \tag{8}$$

Next, we present a necessary and sufficient condition equivalent to $\mu_{\rm st} > 1$.

Theorem 1: Consider a system $M(z) \in \mathcal{RH}_\infty$ and 2-block structured uncertainty $\Delta \in \Delta_{\rm st} = \{{\rm diag}(\Delta_1, \Delta_2) : \|\Delta\|_\infty \leq 1\}$. Then the following conditions are equivalent:

(i) $\inf_{\Delta \in \Delta_{\rm st}} \|M \star \Delta\|_2^2 > 1$.

(ii) There exists a Hermitian matrix $X(\omega) \geq 0$ and a real transfer function $y(\omega) \geq 0$, such that $\forall \omega$ in $[0, 2\pi)$ the following inequalities hold:

$$M(e^{j\omega})^* \begin{bmatrix} X(\omega) & 0 \\ 0 & -1 \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X(\omega) & 0 \\ 0 & -y(\omega) \end{bmatrix} \leq 0, \tag{9}$$

$X(\omega) = {\rm diag}\big(x_1(\omega)I_1, x_2(\omega)I_2\big)$ and

$$\int_0^{2\pi} y(\omega)\frac{d\omega}{2\pi} > 1. \tag{10}$$

Proof: The proof, given in the appendix, follows from the losslessness of the S–procedure for up to 3 Hermitian forms in a complex linear space. ■

B. A Convex Sufficient Condition for Semi-Blind Invalidation

Next, we use the results from the previous section to obtain a sufficient condition for semi-blind identification. To this effect, note that, since $\Delta_{\rm aug} \subset \Delta_{\rm st}$, then $\mu > \mu_{\rm st}$. Thus, if $\mu_{\rm st} > 1$, then the model is invalidated by the experimental data. This leads to the following algorithm:

Algorithm 1: Given a candidate model P, the experimental data $s(e^{j\omega})$ and candidate input, noise and uncertainty sets $\{\mathcal{U}, \mathcal{N}, \mathcal{BH}_\infty(\gamma)\}$:

1) Form the system $M_{\rm aug}$ defined in (7).
2) Evaluate at each frequency
$$\hat{y}(\omega) \doteq \sup\{y\colon \text{conditions (9) hold}\} \tag{11}$$
and compute the integral $I(\hat{y}) \doteq \int_0^{2\pi} \hat{y}(\omega)\frac{d\omega}{2\pi}$.
3) If $I(\hat{y}) > 1$ then the model is invalidated by the experimental data.

Note that in this case, the condition is no longer necessary, since even if $\mu_{\rm st} \leq 1$ for some $\tilde{\Delta} = {\rm diag}\{\tilde{\Delta}_1, \tilde{\Delta}_2\} \in \mathbf{\Delta}_{\rm st}$, $\tilde{\Delta}_1$ may not satisfy the constraint $|\tilde{\Delta}_1| = 1$. However, as we argue next, solutions of (5) with minimal norm of $\|\Delta\|_2$ will tend to have $|\tilde{\Delta}_1| \sim 1$. Thus, conservatism can be reduced by searching over γ to minimize $\|\Delta\|_2$, subject to $\mu_{\rm st} \leq 1$.

Consider first the case where indeed the nominal model $P_{22}S_u$ matches the actual plant up to an unknown phase shift, e.g. $P_{22}S_u = se^{j\theta}$. In this case it is easy to see that solutions to (8) with $\|\Delta_2\| = 0$ indeed satisfy the constraint $|\Delta_1(e^{j\omega})| = 1,\ \forall\omega$. Next, we consider a more general scenario where $P_{22}S_u$ and s do not match exactly. For simplicity, we will assume multiplicative uncertainty, that is, $P_{11} = 0, P_{12} = I, P_{21}S_u = P_{22}S_u = P$, and that all the transfer functions involved are scalar. In this case it is not hard to show that, in order for the model not to be invalidated by the experimental data, the following condition must hold frequency by frequency[1]:

$$|\Delta_2(e^{j\omega})| \geq 1 - \frac{1}{|P(e^{j\omega})|} - \frac{|s(e^{j\omega})|}{|P(e^{j\omega})|}|\Delta_1(e^{j\omega})| \tag{12}$$

Thus, assuming that $|s| \sim |P_{22}S_u|$ then, in order to minimize $\|\Delta_2\|_\infty$, $|\Delta_1(e^{j\omega})|$ should be close to its maximum at all frequencies. Hence replacing the condition $\mu > 1$ by $\mu_{st} > 1$, should not entail too much conservatism. This observation has been experimentally substantiated.

Note that in principle applying the test above requires having experimental data at all frequencies. However, due to the continuity of $M(e^{j\omega})$, which in turns implies continuity of $X(\omega)$ and $y(\omega)$, the integral (10) can be approximated with arbitrary precision by a sum and thus the (in)validation test requires only a finite (albeit possibly large) number of experimental data points.

V. Application: Texture Classification

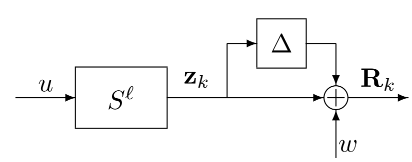


Fig. 2. The Texture Recognition Set-up

In this section we illustrate the use of the proposed framework by applying it to the problem of texture classification. This problem has been the subject of intense

[1] If this condition fails, then $|P| - |P\Delta_2| - |s\Delta_1| > 1$ which implies $|P(1+\Delta_2) - s\Delta_1| > 1$ for all ω. Note also that $|P| > 1$, for the problem to be non-trivial, since otherwise the data is validated by $\Delta_1 = 0$.

research in the computer vision and image processing communities, with application ranging from medical diagnosis to object recognition and image database retrieval. Most texture recognition schemes are stochastic in nature, relying on representations in terms of statistics of the responses to a collection of filters [3], [12]. In this paper we propose a different approach, based upon recasting the problem into a robust model (in)validation form. To this effect, motivated by the work in [11] (for dynamic texture) and [15] (for static images), we will postulate that all images corresponding to realizations of a given texture $\mathcal{T}$ are realizations of a second order stationary random process. Thus, they can be obtained as the output of a linear shift invariant operator S to white noise, or, in a deterministic setting, to a signal $u(e^{j\omega}) \in \ell_2,\ |u(e^{j\omega})| = 1$[2]. This leads to the set-up shown in Figure 2, where S^ℓ represents a nominal model of a particular texture, $\mathbf{z}_k$ and $\mathbf{R}_k$ denote the rows of the ideal and actual images, respectively, and where the (unknown) operator $\Delta \in \boldsymbol{\Delta}$ describes the mismatch between these two images, i.e.:

$$\mathbf{R}_k = \left[(\Delta + I)S^\ell u\right]_k + w$$

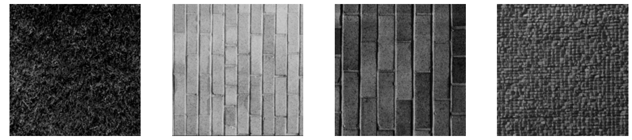
In this framework, the texture recognition problem can be solved as follows. Given an unknown image $\mathbf{R}$ with n rows $\mathbf{R}_i$ and a set of nominal models $\{S^\ell\}$:

- Find for each S^ℓ an input $u, |u(e^{j\omega})| = 1$ and an admissible uncertainty operator Δ of minimum size γ^ℓ_{opt}:
$$\gamma^\ell_{\text{opt}} \doteq \min_{\Delta \in \boldsymbol{\Delta}, u} \{\|\Delta\|_* \colon \mathbf{R}_k = \left[(\Delta + I)S^\ell u\right]_k + \omega_k\}. \quad (13)$$
where $\|.\|_*$ denotes some norm of interest.
- Let $j \doteq \arg\min_\ell \gamma^\ell_{\text{opt}}$. Assign the image $\mathbf{R}$ to the texture represented by model S^j.

Depending on the choice of the admissible uncertainty set $\boldsymbol{\Delta}$, one gets different conditions that solve (13). In the case of texture recognition, it can be argued from physical considerations that the operator Δ should not be causal (to account for interactions amongst all pixels of the image). On the other hand, *linearity* should be retained, to preserve invariance with respect to input scaling. Finally, we are interested in quantifying the difference between images in terms of the (relative) sum of the squared pixel errors, i.e. $(R_k - Z_k)^T(R_k - Z_k)/(Z_k^T Z_k)$. Thus, Δ should be characterized in terms of its induced ℓ_2 norm. Based on these considerations, in the sequel we will assume that $\Delta \in \mathcal{BH}_\infty(\gamma)$, and search for the smallest value of γ so that the interconnection (S, Δ) can reproduce the given image. This can be accomplished by simply performing a sequence of (in)validation tests using Algorithm 1 for increasing values of γ, until the model becomes not invalidated by the data, i.e., until the value of $I(\hat{y})$ falls below 1.

The above approach was tested on slices taken from the textures shown in Figure 3. The nominal models S^ℓ where obtained applying the identification procedure outlined in [15] to the 64×64 pixels upper left sub–image of each 512×512 image. The experimental frequency-domain data was obtained by applying the Discrete Fourier Transform to displaced 64×64 sub-images[3], corrupted by additive noise $w \in \mathcal{B}\ell_2(\epsilon)$, where ϵ represents a 5% of the energy of each nominal image. Note that an unknown spatial displacement translates to an unknown phase shift in the frequency domain, leading precisely to the type of semi-blind problems addressed in section IV.

A sample[4] of the results obtained is shown in Table V. The first column displays information on the slices used for classification purposes, namely the corresponding (known) texture and the position of their upper left corner within the original larger image (x_0, y_0)[5]. The second, third and fourth columns display the minimum size –in the $\mathcal{H}_\infty$ norm– of the LTI uncertainty operator, γ^ℓ_{opt}, that is required for each model $\ell = 1, 2, 10, 11$ to reproduce the given image. As shown there, assigning each sample to the category corresponding to the minimum uncertainty value $\{\gamma^\ell_{\text{opt}}\}$, (indicated by the † symbol), leads to a correct classification.



Text. 1 Text. 2 Text. 10 Text. 11

Fig. 3. Sample textures used for recognition.

Image, (x_0, y_0)	γ^1_{opt}	γ^2_{opt}	γ^{10}_{opt}	γ^{11}_{opt}
1, $(1, 4)$	$0.01^\dagger$	0.6	0.5	0.5
2, $(1, 3)$	0.7	$0.01^\dagger$	0.7	0.4
10, $(1, 2)$	0.7	0.3	$0.01^\dagger$	0.3
11, $(1, 2)$	0.7	0.3	0.4	$0.01^\dagger$

TABLE I

TEXTURE RECOGNITION RESULTS

VI. CONCLUSIONS AND FURTHER RESEARCH

Many problems of practical interest require validating models in the presence of only partially known inputs. Examples of these situations include systems subject to unknown shifts (either spatial or temporal), or cases where only the power spectral density of the input is known. Unfortunately, these situations lead to non-convex, generically NP-hard optimization problems.

In this paper, we propose a tractable convex relaxation, based upon the idea of including the (unknown) input phase $\theta(\omega)$ into an augmented uncertainty structure and relaxing the constraint $|\theta(\omega)| = 1$ to $\|\theta\|_\infty \leq 1$. As we argue in the paper, the conservativeness of this relaxation can

[2]This can be assumed without loss of generality, by absorbing, if necessary, the spectral characteristics of the input in the model.

[3]The one dimensional DFT, i.e., applied to the sequence of rows $\mathbf{R}_k$.

[4]Further details and the complete dataset can be obtained by contacting the authors.

[5]$y_0 > 0$ indicates a displacement from top to bottom; $x_0 > 0$ indicates a displacement from left to right.

be minimized by seeking minimum norm solutions to the resulting (convex) optimization problem.

In the case of unstructured uncertainty, the relaxation above leads to an invalidation problem with structured, 2-block, LTI uncertainty. As an intermediate result, in this paper we obtained an LMI based necessary and sufficient condition for these structures to be (in)validated by the experimental data. As expected these conditions are exactly the LTI counterpart of those recently introduced in [14] for the case of slowly time varying uncertainty.

These results were illustrated in the problem of texture classification. The main idea here is to represent all images corresponding to a given texture as the output of a LTI system to an input with unity magnitude and unknown phase, recasting the problem into a semi–blind validation form.

Efforts are currently underway to generalize the results here to cases involving time varying and slowly time varying uncertainty structures. This will require developing necessary and sufficient invalidation conditions for mixed LTI/LTV and LTI/SLTV structures.

APPENDIX

Proof: [Sufficiency] Assume conditions (9) and (10) hold, i.e., for all $\omega \in [0, 2\pi)$, there exist $X(\omega) = X(\omega)^* \geq 0$ and a positive transfer function $y(\omega)$ so that:

$$M(e^{j\omega})^* \begin{bmatrix} X(\omega) & 0 \\ 0 & -I \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X(\omega) & 0 \\ 0 & -y(\omega) \end{bmatrix} \leq 0 \quad (14)$$

and $\int_0^{2\pi} y(\omega) \frac{d\omega}{2\pi} > 1$. Factor $X(\omega) = D(e^{j\omega})^* D(e^{j\omega})$.

Multiplying (14) from the left and from the right by $r(e^{j\omega})^* = [p(e^{j\omega})^* \; 1]^*$, and $r(e^{j\omega})$ respectively, rearranging terms and integrating over $[0, 2\pi]$ yields:

$$\left[\|Dq\|_2^2 - \|Dp\|_2^2\right] + \int_0^{2\pi} y(\omega) \frac{d\omega}{2\pi} \leq \|z\|_2^2. \quad (15)$$

Since by construction $X(e^{j\omega})$ commutes with the uncertainty Δ and $\|\Delta\|_\infty \leq 1$, the term between brackets on the left hand side of the above equation is non–negative. Hence,

$$1 < \int_0^{2\pi} y(\omega) \frac{d\omega}{2\pi} \leq \|z\|_2^2 = \|(M \star \Delta)\|_2^2,$$

for any $\Delta \in \mathcal{BH}_\infty$. ■

Before proceeding with the necessity part of the proof, we need the following preliminary result:

Lemma 1: If the following LMI:

$$M^* \begin{bmatrix} X & 0 \\ 0 & -I \end{bmatrix} M - \begin{bmatrix} X & 0 \\ 0 & -1 \end{bmatrix} < 0 \quad (16)$$

does not have a positive semi-definite solution X, then there exist signals $\mathbf{r} = [\mathbf{p}^* \; v]^*$ and $\mathbf{s} = [\mathbf{q}^* \; \mathbf{z}^*]^*$ such that:

$$\mathbf{s} = M\mathbf{r}, \quad \|\mathbf{q}_k\|^2 \geq \|\mathbf{p}_k\|^2 \;\; k = 1, 2, \quad \|\mathbf{z}\|^2 \leq |v|^2. \quad (17)$$

Proof: Define the following Hermitian forms:

$$\begin{aligned} \sigma_o(\mathbf{r}) &= |v|^2 - \|\mathbf{z}\|^2 \\ \sigma_1(\mathbf{r}) &= \|\mathbf{q}_1\|^2 - \|\mathbf{p}_1\|^2 \\ \sigma_2(\mathbf{r}) &= \|\mathbf{q}_2\|^2 - \|\mathbf{p}_2\|^2 \end{aligned} \quad (18)$$

If (17) fails, then for all $\mathbf{r}, \|\mathbf{r}\| \leq 1$ such that $\sigma_1(\mathbf{r}) \geq 0$ and $\sigma_2(\mathbf{r}) \geq 0$, we must have $\sigma_o(\mathbf{r}) \leq 0$. Since the S–procedure is lossless for the case of 3 Hermitian forms in a complex space [4], this implies that there exist $x_1 \geq 0, x_2 \geq 0$ such that:

$$\begin{aligned} & 0 \geq \sigma_o(\mathbf{r}) + x_1\sigma_1(\mathbf{r}) + x_2\sigma_2(r), \; \forall \|\mathbf{r}\| \leq 1 \Rightarrow \\ & 0 \geq \mathbf{r}^* \left[M^* \begin{bmatrix} X & 0 \\ 0 & -I \end{bmatrix} M - \begin{bmatrix} X & 0 \\ 0 & -1 \end{bmatrix} \right] \mathbf{r} \\ & X \doteq \begin{bmatrix} x_1 I_1 & 0 \\ 0 & x_2 I_2 \end{bmatrix}, \; \forall \|\mathbf{r}\| \leq 1 \Rightarrow \\ & 0 \geq M^* \begin{bmatrix} X & 0 \\ 0 & -I \end{bmatrix} M - \begin{bmatrix} X & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \quad (19)$$

which contradicts the hypothesis that the LMI (16) did not admit a solution $X \geq 0$. ■

Remark 1: From robust stability of the interconnection (M_{11}, Δ), it follows that in the proof above $v \neq 0$. Thus, $\mathbf{r}$ can be scaled to have its last component equal to 1.

Proof: [Necessity] Following [13], define at each frequency ω:

$$\hat{y}(\omega) \doteq \sup\{y\colon \text{conditions (9) hold}\}$$

Note that since $y \leq M_{22}^* M_{22}$, $\hat{y}$ is well defined. Moreover, note that if $(X(\omega), y(\omega))$ solve the LMI (9), then so do $X_\alpha(\omega) \doteq \alpha X(\omega)$ and $y_\alpha(\omega) \doteq \alpha y(\omega)$ for any $\alpha \in (0, 1)$ [6]. Thus, it follows that $\hat{y}(\omega) \geq 0$. Assume that condition (10) fails, i.e. $\int_0^{2\pi} \hat{y}(\omega) \frac{d\omega}{2\pi} \leq 1$ and define frequency by frequency the system:

$$\hat{M}(e^{j\omega}) \doteq M(e^{j\omega}) \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{\hat{y}(e^{j\omega}) + \epsilon}} \end{bmatrix}$$

where $\epsilon > 0$ is arbitrary. By assumption the following LMI:

$$\begin{aligned} & \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{\hat{y}(e^{j\omega}) + \epsilon}} \end{bmatrix} \left(M(e^{j\omega})^* \begin{bmatrix} X(e^{j\omega}) & 0 \\ 0 & -I \end{bmatrix} M(e^{j\omega}) - \right. \\ & \left. \begin{bmatrix} X(w) & 0 \\ 0 & -(\hat{y}(e^{j\omega}) + \epsilon) \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{\hat{y}(e^{j\omega}) + \epsilon}} \end{bmatrix} \\ = \; & \hat{M}(e^{j\omega})^* \begin{bmatrix} X(e^{j\omega}) & 0 \\ 0 & -I \end{bmatrix} \hat{M}(e^{j\omega}) - \begin{bmatrix} X(e^{j\omega}) & 0 \\ 0 & -1 \end{bmatrix} < 0 \end{aligned}$$

is not feasible. Applying Lemma 1, there exist an input/output pair, $r(e^{j\omega}) = [(p(e^{j\omega}))^* \;\; 1]^*$ and $s(e^{j\omega}) =$

[6]This follows from noting that $\forall (p, q, v, z)$ and $\alpha \in (0, 1)$:

$$\begin{aligned} 0 \geq & |X(\omega)^{\frac{1}{2}} q(e^{j\omega})|^2 - |X(\omega)^{\frac{1}{2}} p(e^{j\omega})|^2 + y(\omega)|v(e^{j\omega})|^2 \\ & - |z(e^{j\omega})|^2 > |X(\omega)^{\frac{1}{2}} q(e^{j\omega})|^2 - |X(\omega)^{\frac{1}{2}} p(e^{j\omega})|^2 \\ & + y(\omega)|v(e^{j\omega})|^2 - \frac{1}{\alpha}|z(e^{j\omega})|^2. \end{aligned}$$

$[(q(e^{j\omega}))^* \ (z(e^{j\omega}))^*]^*$, so that:

$$\begin{aligned} s(e^{j\omega}) &= M(e^{j\omega}) r(e^{j\omega}) \\ \|q(e^{j\omega})_i\|^2 &\geq \|p(e^{j\omega})_i\|^2, \ i = 1, 2 \\ \|z(e^{j\omega})\|^2 &\leq (y(e^{j\omega}) + \epsilon). \end{aligned} \tag{20}$$

Thus [8], there exists a LTI operator $\Delta_o(\epsilon) = \text{diag}\{\Delta_1, \Delta_2\} \in \mathcal{BH}_\infty$ such that $\mathbf{p}_i = \Delta_i \mathbf{q}_i, \ i = 1, 2$. Closing the LFT with this operator and applying the input $v = 1$, yields an output $\mathbf{z}$ such that $\|z\|_2^2 \leq 1 + \epsilon$. Define now (frequency by frequency) the operator:

$$\tilde{\Delta}(e^{j\omega}) \doteq \lim_{\epsilon \to 0} \Delta_o(e^{j\omega}, \epsilon).$$

Note that $\tilde{\Delta}$ is well defined, since for each fixed frequency, $\Delta_o(e^{j\omega}, \epsilon)$ is a sequence of matrices in the (finite–dimensional) compact set $\bar{\sigma}(\Delta) \leq 1$ and hence it contains a convergent subsequence. For this operator we have:

$$\|M \star \tilde{\Delta}\|_2 \leq \lim_{\epsilon \to 0} 1 + \epsilon = 1$$

■

REFERENCES

[1] J. Chen, "Frequency-domain tests for validation of linear fractional uncertain models," *IEEE Transactions on Automatic Control*, vol. 42, no. 6, pp. 748–760, June 1997.

[2] J. Chen and S. Wang, "Validation of linear fractional uncertain models: Solutions via matrix inequalities," *IEEE Transactions on Automatic Control*, vol. 41, no. 6, pp. 844–849, June 1996.

[3] D. Forsyth and J. Ponce. *Computer Vision: A Modern Approach*. Prentice Hall, 2003.

[4] A. L. Fradkoff and V. A. Yakubovich, "The S-procedure and duality theorems for nonconvex problems of quadratic programming," *Vestnik Leningradskogo Universiteta*, vol. 1, pp. 81–87, 1973.

[5] M. P. Newlin and R. S. Smith, "A generalization of the structured singular value and its application to model validation," *IEEE Transactions on Automatic Control*, vol. 43, no. 7, pp. 901–907, July 1998.

[6] F. Paganini, *Sets and Constraints in the Analysis of Uncertain Systems*, Ph. D. thesis, California Institute of Technology, 1996.

[7] K. Poolla, P. Khargonekar, A. Tikku, J. Krause, and K. Nagpal, "A time domain approach to model validation," *IEEE Transactions on Automatic Control*, vol. 39, no. 5, pp. 951–959, May 1994.

[8] K. Poolla and A. Tikku, "Robust Performance Against Time-Varying Structured Perturbations," *IEEE Transactions on Automatic Control*, vol. 40, no. 9, pp. 1589–1602, Sept. 1995.

[9] S. Rangan and K. Poolla, "Model validation for structured uncertainty models," in *American Control Conference*, pp. 629–633, Philadelphia, PA, USA, June 1998.

[10] R. S. Smith and J. C. Doyle, "Model validation: A connection between robust control and identification," *IEEE Transactions on Automatic Control*, vol. 37, no. 7, pp. 942–952, July 1992.

[11] G. Doretto, A. Chiuso, Y. N. Wu, and S. Soatto. Dynamic textures. *Int. J. Computer Vision*, 51(2):91–109, 2003.

[12] M. Sonka, V. Hlavac, and R. Boyle. *Image Processing, Analysis, and Machine Vision*. PWS Publishing, 1999.

[13] M. Sznaier, T. Amishima, P. A. Parrilo and J. Tierno, "A convex approach to robust $\mathcal{H}_2$ performance analysis," *Automatica*, vol. 38, no. 6, pp. 957–966, June 2002.

[14] M. C. Mazzaro and M. Sznaier, "Convex Necessary and Sufficient Conditions for Frequency Domain Model (In)Validation Under SLTV Structured Uncertainty," *IEEE Transactions on Automatic Control,* vol. 49, no. 10, pp. 1683–1692, Oct. 2004.

[15] M. Sznaier, O. Camps, and M. C. Mazzaro. Finite horizon model reduction of a class of neutrally stable systems with applications to texture synthesis and recognition. In *43rd IEEE Conf. Dec. Control*, Paradise Island, Bahamas, Dec. 2004.

[16] O. Toker and J. Chen, "Time domain validation of structured uncertainty model sets," in 35th *IEEE Conference on Decision and Control*, pp. 255–260, Kobe, Japan, Dec. 1996.

[17] D. Xu, Z. Ren, G. Gu, and J. Chen, "LFT uncertain model validation with time and frequency–domain measurements," *IEEE Transactions on Automatic Control*, vol. 44, no. 7, pp. 1435–1441, July 1999.