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LMI conditions for the existence of polynomially parameter-dependent Lyapunov functions assuring robust stability

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Abstract— **The robust stability of uncertain systems in polytopic domains is investigated by means of homogeneous polynomially parameter-dependent Lyapunov (HPPDL) functions which are quadratic with respect to the state variables. A systematic procedure to construct linear matrix inequality (LMI) conditions whose solutions assure the existence of HPPDL functions of increasing degree is given. For each degree, a sequence of relaxations based on real algebraic methods provides sufficient LMI conditions of increasing precision for the existence of an HPPDL function which tend asymptotically to the necessity. As a result, families of LMI conditions parametrized on the degree of the HPPDL functions and on the relaxation level provide efficient numerical tests of different complexities to assess the robust stability of both continuous and discrete-time uncertain systems.**

I. INTRODUCTION

Undoubtedly, an important point in the development of robust control theory was the quadratic stability, that allowed many robust control and stability analysis problems to be cast as linear matrix inequality (LMI) optimization problems [1]. LMI conditions are simple to be implemented and can be solved by efficient semi-definite programming algorithms [2, 3].

As a first attempt to reduce the conservatism provided by the quadratic stability, i.e., $x'Px$ with a fixed $P = P' > 0$ matrix, parameter-dependent LMI conditions based on quadratic Lyapunov functions $x'P(\alpha)x$ with $P(\alpha)$ being a Lyapunov matrix that depends affinely on the uncertain parameters have appeared in [4–10]. In this context, it is worth to mention the contribution presented in [5] (continuous-time case) and [6] (discrete-time case), where additional variables allowed the decoupling between the Lyapunov matrices and the system matrices. Another interesting idea was presented in [9] (continuous-time case) and [8] (discrete-time case), where the stability analysis conditions are treated from a algebraic point of view in terms of the uncertain parameter α. These two methods were combined and generalized to cope with any convex region in the complex plane ($\mathcal{D}$-stability) in [10], providing less conservative results for robust stability of time-invariant uncertain systems in polytopic domains. However, an exact characterization of the robust stability of linear uncertain systems based on affine parameter-dependent Lyapunov functions is not known.

To reduce the conservativeness, robust stability analysis methods based on polynomially parameter-dependent Lyapunov functions appeared quite naturally as the next step in the characterization of robust stability domains. In [11], LMI conditions for robust stability analysis of affine uncertain continuous-time systems based on a quadratic Lyapunov function whose Lyapunov matrix depends polynomially on the uncertain parameter were given. The conditions are necessary and sufficient in the sense that, as the degree of the polynomial increases, the characterization of robust stability becomes more precise and, if the system is robustly stable, a finite degree exists for which the LMIs provide a feasible solution. The main drawback is the computational burden demanded as the complexity (number of states, uncertain parameters and degree of the polynomial) grows.

Another approach based on homogeneous Lyapunov functions, polynomially dependent of arbitrary degree on the parameters, appeared in [12] where the robust stability conditions were expressed through a complete square matrix representation of homogeneous matrix forms, that are linear on the uncertain parameters, thus providing families of sufficient LMI conditions of increasing precision as the degree of the Lyapunov function grows.

Concerning robust stability conditions not directly based on the Lyapunov approach, it is worth to mention some recent results based on the optimization of positive polynomials over compact sets [13]. Necessary and sufficient conditions for stability of linear affine uncertain systems are presented in [14], where a family of LMI conditions of increasing precision were given, but the global convergence is not assured. A similar approach appeared in [15] transforming the robust stability test into the minimization of a multivariate polynomial over a compact set by means of an associate Hermite matrix. The conditions are given in terms of a sequence of LMI relaxations with guaranteed convergence. The two methods above were formulated for affine uncertain systems (nominal system affinely affected by uncertain parameters) and can deal with both continuous and discrete-time stability.

The aim of this paper is to investigate the robust stability of uncertain systems in polytopic domains (more general representation than affine uncertainty). Robust stability LMI conditions based on parameter-dependent Lyapunov functions which are quadratic with respect to the state variables and homogeneous polynomially dependent of arbitrary degree on the uncertain parameters are proposed. A systematic procedure to construct LMI conditions that assure the existence of HPPDL functions of increasing degree is given. For each degree, real algebraic geometry properties [16] are used to construct a sequence of relaxations that converges asymptotically, assuring the existence of an HPPDL function

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that guarantees robust stability. These relaxations are based on Pólya's Theorem [16] and have already been used in [17] in the context of multiplier approximations, copositive programming [18] (see also [19]). As a result, families of LMI conditions parametrized on the degree of the HPPDL function and on the relaxation level provide efficient numerical tests of different complexities to assess the robust stability of both continuous and discrete-time uncertain systems.

II. NOTATION

The symbol $(')$ indicates transpose; $P > 0$ (≥ 0) means that P is symmetric positive (semi) definite. $\lambda_{max}(P)$ means the maximum and $\lambda_{min}(P)$ the minimum eigenvalue of matrix P. $\mathbb{R}$ represents the set of real numbers, $\mathbb{Z}_+$ the set of nonnegative integers $\{0, 1, 2, \ldots\}$ and $M!$ denotes factorial. N is used to denote the number of vertices of a polytope and also the dimension of vector α associated to a generic matrix inside the polytope.

III. PRELIMINARIES

Consider the uncertain linear time-invariant system

$$\delta[x(t)] = A(\alpha)x(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $\delta[\cdot]$ denotes the time derivative operator for continuous-time and the shift operator for discrete-time systems. Matrix $A(\alpha) \in \mathbb{R}^{n\times n}$ is not precisely known, but belongs to a convex bounded (polytope type) uncertain domain $\mathcal{A}$ given by

$$\mathcal{A} = \big\{A(\alpha) : A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i, \alpha \in \Delta_N\big\};$$

$$\Delta_N = \big\{\alpha \in \mathbb{R}^N, \sum_{i=1}^{N} \alpha_i = 1; \alpha_i \geq 0\big\} \tag{2}$$

Any uncertain matrix $A(\alpha) \in \mathcal{A}$ can be written as a convex combination of the vertices A_i, $i = 1, \ldots, N$ of the polytope.

The problem addressed here is to determine if $\mathcal{A}$ is Hurwitz stable (i.e. all matrices $A \in \mathcal{A}$ have eigenvalues with negative real part) for the continuous-time case and if $\mathcal{A}$ is Schur stable (i.e. all matrices $A \in \mathcal{A}$ have eigenvalues with absolute value less than one) for the discrete-time case. The following lemmas give equivalent necessary and sufficient conditions for the Hurwitz (Schur) stability of $\mathcal{A}$.

Lemma *1:* The set $\mathcal{A}$ is Hurwitz stable if and only if there exists a symmetric positive definite parameter-dependent matrix $P(\alpha) \in \mathbb{R}^{n\times n}$ such that one of the following equivalent conditions holds $\forall\ \alpha \in \Delta_N$:

(a) $\Gamma(\alpha) \triangleq A(\alpha)'P(\alpha) + P(\alpha)A(\alpha) < 0$

(b) $\Gamma_d(\alpha) \triangleq (\alpha_1+\alpha_2+\ldots+\alpha_N)^d\Gamma(\alpha) < 0; \quad \forall\ d \in \mathbb{Z}_+$

Condition (a) is straightforwardly obtained through the use of $v(x) = x'P(\alpha)x$ as a Lyapunov function associated to the differential equation $\dot{x} = A(\alpha)x$. For any fixed $\alpha \in \Delta_N$ and for all $d \in \mathbb{Z}_+$, the equivalence between (a) and (b) is immediate since $\alpha \in \Delta_N$ implies $(\sum_{i=1}^{N} \alpha_i)^d = 1$ for all $d \in \mathbb{Z}_+$. Note that $P(\alpha)$ in Lemma 1 does not have a special structure and the verification of stability is based on the existence of a positive definite Lyapunov matrix for any choice of $\alpha \in \Delta_N$, which is a well known result.

The aim here is to investigate necessary and sufficient conditions for the existence of the quadratic Lyapunov function $v(x) = x'P(\alpha)x$ which depends polynomially on the uncertain parameters α, more precisely, the matrix $P(\alpha)$ is a homogeneous polynomial matrix valued function of arbitrary degree on α. The algebraic properties of condition (b) of Lemma 1, which defines a family of polynomials whose number of monomials is parametrized on $d \in \mathbb{Z}_+$, will be used to provide a complete characterization of the existence of $P(\alpha)$ given by (2) assuring the Hurwitz stability of $\mathcal{A}$ in terms of LMIs formulated only at the vertices of $\mathcal{A}$.

Lemma *2:* The set $\mathcal{A}$ is Schur stable if and only if there exists a symmetric positive definite parameter-dependent matrix $P(\alpha) \in \mathbb{R}^{n\times n}$ such that one of the following equivalent conditions holds $\forall\ \alpha \in \Delta_N$:

(a) $\Upsilon(\alpha) \triangleq A(\alpha)'P(\alpha)A(\alpha) - P(\alpha) < 0$

(b) $\Upsilon_d(\alpha) \triangleq (\alpha_1+\alpha_2+\ldots+\alpha_N)^d\Upsilon(\alpha) < 0; \quad \forall\ d \in \mathbb{Z}_+$

The same remarks about the equivalence between (a) and (b) of Lemma 1 also apply to Lemma 2.

Before presenting the main results, some definitions and preliminaries are needed. Define $\mathcal{K}(g)$ as the set of N-tuples obtained as all possible combinations of $k_1k_2\cdots k_N$, $k_i \in \mathbb{Z}_+$, $i = 1, \ldots, N$ such that $k_1 + k_2 + \cdots + k_N = g$. The number of N-tuples in $\mathcal{K}(g)$ is given by $J(g) = (N + g - 1)!/(g!(N-1)!)$. The element $\mathcal{K}_\ell(g)$ is the ℓ-th N-tuple of $\mathcal{K}(g)$, which is lexically ordered, $\ell = 1, \ldots, J(g)$. As an example, consider $N = 2$, $g = 5$, which yields $J(5) = 6$ and $\mathcal{K}(5) = \{05, 14, 23, 32, 41, 50\}$, with $\mathcal{K}_1(5) = 05$, $\mathcal{K}_2(5) = 14$ etc. Sum and subtraction between two N-tuples $\mathcal{K}_i(g)$ and $\mathcal{K}_j(g)$ are done element-wise, e.g. $\mathcal{K}_4(5)+\mathcal{K}_3(5) = 32+23 = 55$. Each element $\mathcal{K}_\ell(g)$ defines a set $\mathcal{G}$ obtained from the following operation: For a given $d \in \mathbb{Z}_+$, $\mathcal{G} = \mathcal{K}_\ell(g) - \mathcal{K}_r(d)$, $r = 1, \ldots, J(d)$. Clearly, when $d = 0$, $\mathcal{G} = \mathcal{K}_\ell(g)$. The N-tuples $\mathcal{G}_r,\ r = 1, \ldots, J(d)$ with non-negative k_i's are used to generate the LMIs in the theorems proposed in the sequel (the N-tuples with negative k_i's are discarded). Associated to the N-tuple $\mathcal{G}_r$, define: *1)* the set $\mathcal{I}_r$ with elements given by subsets of i, $i \in \{1, 2, \ldots, N\}$, with k_i's nonzero. *2)* $\mathcal{G}_r^i$ as being equal to $\mathcal{G}_r$ but with $k_i > 0$ replaced by $k_i - 1$. Note that $\mathcal{G}_r^i$ computed from $\mathcal{G} = \mathcal{K}_\ell(g + d + 1) - \mathcal{K}_r(d)$, $r = 1, \ldots, J(d)$ produces N-tuples belonging to $\mathcal{K}(g)$, as used in the proposed theorems. *3)* the coefficient $\mathcal{C}_r$ given by $d!/(k_1!k_2!\cdots k_N!)$ with $k_1k_2\cdots k_N = \mathcal{K}_r(d)$.

Considering $\mathcal{K}(5)$ given above and $d = 2$, one has $J(2) = 3$, $\mathcal{K}(2) = \{02, 11, 20\}$. For $\ell = 1$ (the first N-tuple of $\mathcal{K}(5)$), one has $\mathcal{K}_1(5) - \mathcal{K}_r(2),\ r = 1, \ldots, 3$, which yields $\mathcal{G} = \{05\} - \{02, 11, 20\} = \{03, \Diamond 4, \Diamond 5\}$, where $\Diamond$ means a negative integer. The N-tuples associated with $r = 2, 3$ are ignored when generating the LMIs. The set $\mathcal{I}_1$ is $\{2\}$, $\mathcal{G}_1^2 = \{02\}$ and $\mathcal{C}_1 = 2!/(0!2!) = 1$. Now, consider $\ell = 4$ (forth N-tuple), $\mathcal{K}_4(5) - \mathcal{K}_r(2),\ r = 1, \ldots, 3$ which yields $\mathcal{G} = \{32\} - \{02, 11, 20\} = \{30, 21, 12\}$. The associated sets are $\mathcal{I}_1 = \{1\}$, $\mathcal{I}_2 = \{1, 2\}$, $\mathcal{I}_3 = \{1, 2\}$, the elements of $\mathcal{G}$ are $\mathcal{G}_1^1 = \{20\}$, $\mathcal{G}_2^1 = \{11\}$, $\mathcal{G}_2^2 = \{20\}$, $\mathcal{G}_3^1 = \{02\}$,

$\mathcal{G}_3^2 = \{11\}$, and $\mathcal{C}_1 = 2!/(0!2!) = 1$, $\mathcal{C}_2 = 2!/(1!1!) = 2$, $\mathcal{C}_3 = 2!/(2!0!) = 1$. The other N-tuples $\mathcal{K}_\ell(g)$ are handled similarly.

The elements of the each N-tuple $\mathcal{K}_\ell(g)$ define subscripts $k_1 k_2 \cdots k_N$ of the Lyapunov constant symmetric matrices $P_{k_1 k_2 \cdots k_N} \triangleq P_{\mathcal{K}_\ell(g)}$, $\ell = 1, \ldots, J(g)$ to construct a HPPDL function $P_g(\alpha)$ given by

$$P_g(\alpha) = \sum_{\ell=1}^{J(g)} \alpha_1^{k_1} \cdots \alpha_N^{k_N} P_{\mathcal{K}_\ell(g)};\ k_1 \cdots k_N = \mathcal{K}_\ell(g) \quad (3)$$

In what follows, a sufficient condition that tends asymptotically to the necessity as d increases assuring the existence of $P_g(\alpha) > 0$ of arbitrary degree given by (3) such that condition (b) of Lemma 1 holds is given.

IV. MAIN RESULTS

Theorem *1:* An HPPDL matrix of arbitrary degree $P_g(\alpha)$ given by (3) assures the Hurwitz stability of $\mathcal{A}$ if and only if there exist symmetric matrices $P_{\mathcal{K}_j(g)} \in \mathbb{R}^{n \times n}$, $\mathcal{K}_j(g) \in \mathcal{K}(g)$, $j = 1, \ldots, J(g)$, and a sufficiently large $d \in \mathbb{Z}_+$ such that the following LMIs hold

$$T_\ell = \sum_{r=1}^{J(d)} \left(\sum_{i \in \mathcal{I}_r} \mathcal{C}_r \left(A_i' P_{\mathcal{G}_r^i} + P_{\mathcal{G}_r^i} A_i \right) \right) < 0\ ;$$
$$\mathcal{G} = \mathcal{K}_\ell(g+d+1) - \mathcal{K}_r(d),\ r = 1, \ldots, J(d),$$
$$\ell = 1, \ldots, J(g+d+1) \quad (4)$$

$$R_p = \sum_{r=1}^{J(d)} \left(\mathcal{C}_r P_{\mathcal{G}_r} \right) > 0\ ; \mathcal{G} = \mathcal{K}_p(g+d) - \mathcal{K}_r(d),$$
$$r = 1, \ldots, J(d), \quad p = 1, \ldots, J(g+d) \quad (5)$$

Moreover, for a fixed d, if the LMIs (4)-(5) are fulfilled for a given degree $\hat{g}$, then the LMIs corresponding to any degree $g > \hat{g}$ are also satisfied. Similarly, for a given g, if the LMIs (4)-(5) provide a feasible solution for $\hat{d}$, then the LMIs for $d > \hat{d}$ also have feasible solutions.

Proof: Sufficiency. Since $\Gamma_d(\alpha)$ with $P_g(\alpha)$ given by (3) can be written as

$$\Gamma_d(\alpha) = \left(\sum_{r=1}^{J(d)} \left(\sum_{i \in \mathcal{I}_r} \mathcal{C}_r \left(A_i' P_{\mathcal{G}_r^i} + P_{\mathcal{G}_r^i} A_i \right) \right) \right) \alpha_1^{k_1} \cdots \alpha_N^{k_N};$$
$$\mathcal{G} = \mathcal{K}_\ell(g+d+1) - \mathcal{K}_r(d),\ r = 1, \ldots, J(d),$$
$$\ell = 1, \ldots, J(g+d+1),\ k_1 \cdots k_N = \mathcal{K}_\ell(g+d+1) \quad (6)$$

it is straightforward to show that, if there exist symmetric matrices $P_{\mathcal{K}_j(g)}$, $\mathcal{K}_j(g) \in \mathcal{K}(g)$, $j = 1, \ldots, J(g)$ and $d \in \mathbb{Z}_+$ such that (4) holds for $\ell = 1, \ldots, J(g+d+1)$ and (5) holds for $p = 1, \ldots, J(g+d)$ then, from (6), one can conclude that the conditions of Lemma 1 are verified for all $\alpha \in \Delta_N$.

Necessity: Defining

$$L \triangleq \max_{\ell=1,\ldots,J(g+d+1)} \lambda_{max}(-T_\ell); \quad \kappa \triangleq \min_{\alpha \in \Delta_N} \lambda_{min}(-\Gamma(\alpha))$$

it is clear that for any vector w such that $w'w = 1$ one has

$$L \geq \max_{\ell=1,\ldots,J(g+d+1)} w'(-T_\ell)w; \quad \min_{\alpha \in \Delta_N} w'(-\Gamma(\alpha))w \geq \kappa$$

The choice of $d \in \mathbb{Z}_+$ such that $d \geq g(g+1)L/2\kappa - g$, [20, Theorem 1] assures that all coefficients $w'(-T_\ell)w$, $\ell = 1, \ldots, J(g+d+1)$ of the polynomial $w'(-\Gamma_d(\alpha))w$ are positive. Since w is arbitrary, the conclusion is that all LMIs T_ℓ, $\ell = 1, \ldots, J(g+d+1)$ are negative definite. A similar analysis can be applied to the constraints $R_p > 0$.

Suppose the LMIs of (4)-(5) are fulfilled for a fixed d and a certain $\hat{g}$, that is, there exist $J(\hat{g})$ symmetric matrices $P_{\mathcal{K}_j(\hat{g})}$, $j = 1, \ldots, J(\hat{g})$ such that (4)-(5) holds, i.e., $P_{\hat{g}}(\alpha)$ is an HPPDL positive definite matrix assuring the robust stability of the system. Then, the terms of the polynomial matrix $P_{\hat{g}+1}(\alpha) = (\alpha_1 + \cdots + \alpha_N) P_{\hat{g}}(\alpha)$ satisfy the LMIs of Theorem 1 corresponding to the degree $\hat{g}+1$, which can be obtained in this case by linear combination of the LMIs of Theorem 1 for $\hat{g}$. As the LMIs of (4)-(5) for a given $\hat{d} > 0$ are linear combinations of the LMIs of (4)-(5) for $\hat{d}-1$ (fixed g) it is straightforward to show that if the LMIs of (4)-(5) are feasible for a given $\hat{d}$, then the LMIs for $d > \hat{d}$ are also feasible (combination of terms with same sign). ■

The LMIs (5) in Theorem 1 assure that $P_g(\alpha)$ given by (3) is positive definite for all $\alpha \in \Delta_N$. Note that for $g = 0$ ($P_g(\alpha) = P$, i.e. quadratic stability) and for $g = 1$ ($P_g(\alpha)$ affine on α), condition (5) is also necessary to ensure $P_g(\alpha) > 0$ (stability of the vertices of the polytope) and there is no need to use $d > 0$ in (5). For $g > 1$, the relaxations must be applied in both (4) and (5) to produce asymptotically necessary conditions. See [21] for details concerning Pólya's Theorem applied to robust stability conditions based on affine parameter-dependent Lyapunov matrices.

Theorem *2:* An HPPDL matrix of arbitrary degree $P_g(\alpha)$ given by (3) assures the Schur stability of $\mathcal{A}$ if and only if there exist symmetric matrices $P_{\mathcal{K}_j(g)} \in \mathbb{R}^{n \times n}$, $\mathcal{K}_j(g) \in \mathcal{K}(g)$, $j = 1, \ldots, J(g)$, and a sufficiently large d such that the LMIs (5) and the following LMIs hold

$$\sum_{r=1}^{J(d)} \left(\sum_{i \in \mathcal{I}_r} \mathcal{C}_r \begin{bmatrix} -P_{\mathcal{G}_r^i} & P_{\mathcal{G}_r^i} A_i \\ \star & -P_{\mathcal{G}_r^i} \end{bmatrix} \right) < 0;$$
$$\mathcal{G} = \mathcal{K}_\ell(g+d+1) - \mathcal{K}_r(d),\ r = 1, \ldots, J(d),$$
$$\ell = 1, \ldots, J(g+d+1) \quad (7)$$

Moreover, for a fixed d, if the LMIs (5) and (7) are fulfilled for a given degree $\hat{g}$, then the LMIs corresponding to any degree $g > \hat{g}$ are also satisfied. Similarly, for a given g, if the LMIs (5) and (7) provide a feasible solution for $\hat{d}$, then the LMIs for $d > \hat{d}$ also have feasible solutions.

Proof: Similar to the proof of Theorem 1. ■

Theorem 2 exploits the Schur complement form of the discrete-time Lyapunov inequality presented in Lemma 2 for sake of compactness, to use the same notation and sets defined for the continuous-time case. Similar results could be obtained directly from the inequality $A(\alpha)'P(\alpha)A(\alpha) - P(\alpha) < 0$ but the triple product $A(\alpha)'P(\alpha)A(\alpha)$ would generate different sets and new coefficients.

LMI conditions which are equivalent to the conditions of Lemmas 1 (Hurwitz case) and 2 (Schur case) but present a larger number of decision variables can be formulated through the use of the Finsler's Lemma [22].

Lemma *3:* The set $\mathcal{A}$ is Hurwitz (Schur) stable if and only if there exists a symmetric positive definite parameter-dependent matrix $P(\alpha) \in \mathbb{R}^{n\times n}$ and parameter-dependent matrices $\mathcal{X}(\alpha) \in \mathbb{R}^{2n\times n}$ such that one of the following equivalent conditions holds $\forall\ \alpha \in \Delta_N$:

(a) $\Theta(\alpha) \triangleq \mathcal{Q}(\alpha) + \mathcal{X}(\alpha)\mathcal{B}(\alpha) + \mathcal{B}(\alpha)'\mathcal{X}(\alpha)' < 0$

(b) $\Theta_d(\alpha) = (\alpha_1+\alpha_2+\ldots+\alpha_N)^d\Theta(\alpha) < 0; \quad \forall\, d \in \mathbb{Z}_+$

with $\mathcal{B}(\alpha) = \begin{bmatrix} A(\alpha) & -\mathbf{I} \end{bmatrix}$ and $\mathcal{Q}(\alpha)$ given by

$$\mathcal{Q}_H(\alpha) = \begin{bmatrix} \mathbf{0} & P(\alpha) \\ P(\alpha) & \mathbf{0} \end{bmatrix};\ \mathcal{Q}_S(\alpha) = \begin{bmatrix} -P(\alpha) & \mathbf{0} \\ \mathbf{0} & P(\alpha) \end{bmatrix}$$

for the Hurwitz and Schur cases, respectively.

The equivalence between (a) and (b) is straightforward. For a fixed α, the equivalence between Lemma 3 and Lemmas 1 and 2 can be proved by using the Finsler's Lemma [22]. If a special structure of $P(\alpha)$ is considered, as for instance an HPPDL matrix as in (3), from the conditions of Lemma 3 less conservative LMI tests for evaluating the Hurwitz (Schur) stability of $\mathcal{A}$ can be obtained as follows.

Theorem *3:* An HPPDL matrix of arbitrary degree $P_g(\alpha)$ given by (3) assures the Hurwitz (Schur) stability of $\mathcal{A}$ if and only if there exist symmetric matrices $P_{\mathcal{K}_j(g)} \in \mathbb{R}^{n\times n}$, $\mathcal{K}_j(g) \in \mathcal{K}(g)$, $j = 1,\ldots,J(g)$, matrices $X_{\mathcal{K}_j(g)} \in \mathbb{R}^{2n\times n}$, $\mathcal{K}_j(g) \in \mathcal{K}(g)$, $j = 1,\ldots,J(g)$, and a sufficiently large d such that the LMIs (5) and the following LMIs hold

$$\sum_{r=1}^{J(d)} \left(\sum_{i\in\mathcal{I}_r} \mathcal{C}_r \left(Q_{\mathcal{G}_r^i} + X_{\mathcal{G}_r^i}B_i + B_i'X_{\mathcal{G}_r^i}' \right) \right) \ < \ 0\ ;$$
$$\mathcal{G} = \mathcal{K}_\ell(g+d+1) - \mathcal{K}_r(d),\ r = 1,\ldots,J(d),$$
$$\ell = 1,\ldots,J(g+d+1) \qquad (8)$$

where $Q_{\mathcal{G}_r^i}$ are respectively given, for Hurwitz and Schur cases, by

$$Q^H_{\mathcal{G}_r^i} = \begin{bmatrix} \mathbf{0} & P_{\mathcal{G}_r^i} \\ P_{\mathcal{G}_r^i} & \mathbf{0} \end{bmatrix}; \quad Q^S_{\mathcal{G}_r^i} = \begin{bmatrix} -P_{\mathcal{G}_r^i} & \mathbf{0} \\ \mathbf{0} & P_{\mathcal{G}_r^i} \end{bmatrix}$$

and $B_i = [A_i \quad -\mathbf{I}]$. Moreover, for a fixed d, if the LMIs of (5) and (8) are fulfilled for a given degree $\hat{g}$, then the LMIs corresponding to any degree $g > \hat{g}$ are also satisfied. Similarly, for a given g, if the LMIs (5) and (8) provide a feasible solution for $\hat{d}$, then the LMIs for $d > \hat{d}$ also have feasible solutions.

Proof: The proof is very similar to the proof of Theorem 1. With $P_g(\alpha) > 0$ given by (3), $X_g(\alpha)$ given by

$$X_g(\alpha) = \sum_{j=1}^{J(g)} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_N^{k_N} X_{\mathcal{K}_j(g)}; \quad k_1k_2\cdots k_N = \mathcal{K}_j(g)$$

and $A(\alpha) \in \mathcal{A}$ as in (2), $\Theta_d(\alpha)$ in Lemma 3 (b) can be written as a homogeneous polynomially form of degree $g + d + 1$ with matrix valued coefficients given by (8). Again, for a fixed d, if the conditions (5) and (8) are fulfilled for a given $\hat{g}$, then $P_{\hat{g}+1}(\alpha) = (\sum_{i=1}^N \alpha_i)P_{\hat{g}}(\alpha)$, $X_{\hat{g}+1}(\alpha) = (\sum_{i=1}^N \alpha_i)X_{\hat{g}}(\alpha)$ are a feasible solution to the LMIs (5) and (8) for $g = \hat{g} + 1$. If a feasible solution exists for $\hat{d}$ then there are also feasible solutions for $d > \hat{d}$. ∎

The extra matrix variables $X_{\mathcal{K}_j(g)}, j = 1,\ldots,J(g)$ in Theorem 3 provide less conservative results than the ones obtained from Theorems 1 and 2 for fixed values of d and g. For a detailed discussion about this aspect with $d = 0$ and $g = 1$, see [10]. Moreover, these extra variables allows a faster convergence (as d grows) towards the necessary condition for the existence of an HPPDL matrix of a given degree g, as illustrated by means of numerical experiments.

Finally, note that the conditions of Theorem 3 can be easily extended to cope with any convex region in the complex plane following the lines depicted in [7, 10].

V. STUDY OF CASE $N = 2$

Consider the conditions of Theorem 1 with $g = 2$ applied to a polytope $\mathcal{A}$ with $N = 2$ vertices. In this case, $R_1 = P_{20}$, $R_2 = P_{11}$, $R_3 = P_{02}$,

$$T_1 = A_1'P_{20} + P_{20}A_1; \qquad T_2 = A_2'P_{02} + P_{02}A_2$$
$$T_3 = A_1'P_{02} + P_{02}A_1 + A_2'P_{11} + P_{11}A_2$$
$$T_4 = A_2'P_{20} + P_{20}A_2 + A_1'P_{11} + P_{11}A_1$$

Then, for $d = 0$, the number of LMIs is $J(3) + J(2) = 7$, $\mathcal{K}(3) = \{03, 12, 21, 30\}$, and the LMIs are

$$T_1 < 0; \qquad T_2 < 0; \qquad R_1 > 0; \qquad R_3 > 0 \qquad (9)$$

which are necessary conditions (stability of the vertices) and

$$T_3 < 0; \qquad T_4 < 0; \qquad R_2 > 0 \qquad (10)$$

For $d = 1$, there are $J(4) + J(3) = 9$ LMIs, $\mathcal{K}(4) = \{04, 13, 22, 31, 40\}$, $\mathcal{K}(1) = \{01, 10\}$. The LMIs are (9) and

$$T_1 + T_4 < 0; \quad T_3 + T_4 < 0; \quad T_2 + T_3 < 0$$
$$R_1 + R_2 > 0; \quad R_2 + R_3 > 0 \qquad (11)$$

Note that a feasible solution to (9)-(10) is also feasible to (9)-(11), but the converse is not true, since the constraint (10) is more restrictive than (11). Note also that, differently from (10), the LMIs (11) do not impose $R_2 = P_{11} > 0$. For $d = 2$, the number of LMIs is $J(5) + J(4) = 11$, $\mathcal{K}(5) = \{05, 14, 23, 32, 41, 50\}$, $\mathcal{K}(2) = \{02, 11, 20\}$ and the LMIs are (9) and

$$2T_1 + T_4 < 0; \quad T_1 + T_3 + 2T_4 < 0; \quad T_2 + 2T_3 + T_4 < 0$$
$$2T_2 + T_3 < 0; \quad 2R_1 + R_2 > 0; \quad R_1 + 2R_2 + R_3 > 0$$
$$R_2 + 2R_3 > 0$$

As d increases, the new LMIs become easier to be fulfilled and, if an HPPDL function assuring the Hurwitz stability exists, the necessity is attained asymptotically.

TABLE I

NUMBER OF SCALAR VARIABLES K AND NUMBER OF LMI ROWS L IN THEOREMS 1 (T1), 2 (T2) AND 3 (T3).

Method	K	L
T1	$n(n+1)J(g)/2$	$n(J(g+d)+J(g+d+1))$
T2	$n(n+1)J(g)/2$	$nJ(g+d)+2nJ(g+d+1)$
T3	$n(5n+1)J(g)/2$	$nJ(g+d)+2nJ(g+d+1)$

VI. COMPLEXITY ISSUES

The complexity of an LMI optimization problem can be estimated from the number of scalar variables K and the number of LMI rows L. Table I shows the complexities associated to Theorems 1 (T1), 2 (T2) and 3 (T3).

As it can be seen, K and L grow polynomially with d and g for all theorems. Note also that the number of scalar variables does not depend on d, showing that the sequence of relaxations is less costly than the increase of the degree g. If an LMI solver based on interior point methods is used, as for instance the LMI Control Toolbox [2], the complexity can be estimated as being proportional to K^3L whereas the solver SeDuMi [3] yields $K^2L^{2.5}+L^{3.5}$. The surface depicted in Figure 1 illustrates the complexity associated to the conditions of Theorem 1 using $\log_{10}(K^3L)$ for $d \in [0, 20]$, $g \in [1, 15]$ and considering a continuous-time uncertain system with $n = 3$, $N = 3$.

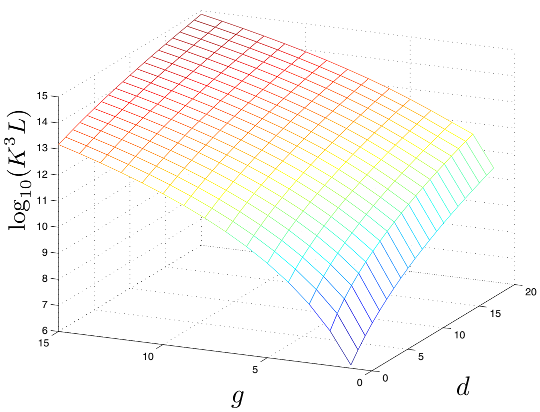

Fig. 1. Complexity associated to the conditions of Theorem 1 using $\log_{10}(K^3L)$ for $d \in [0, 20]$, $g \in [1, 15]$ and considering a continuous-time uncertain system with $n = 3$, $N = 3$.

As expected, to increase g demands more computational effort than to increase d. The best choices of g and d to obtain accurate results with less computational burden are analyzed in next section.

VII. NUMERICAL EXPERIMENTS

All numerical tests have been performed in a Pentium IV 2.6 GHz, 512 MB RAM, using the LMI Control Toolbox [2] and SeDuMi [3] within the MatLab environment.

Example 1: This example illustrates how the relaxation procedure evolves as d increases. Consider a continuous-time uncertain system with $n = 3$ and $N = 2$, given by

$$A_1 = \begin{bmatrix} -0.1938 & 0.3961 & -0.7104 \\ 0.0374 & 0.0988 & -0.9082 \\ 0.4803 & -0.2257 & -0.4496 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.6343 & 0.1343 & -0.9079 \\ -0.7179 & -0.6443 & -0.2978 \\ 0.3733 & -0.4191 & 0.3495 \end{bmatrix}$$

Choosing $g = 1$, i.e. an affine parameter-dependent Lyapunov function, Theorem 1 provides the sequence of relaxations for $d = \{0, 1, 2, 3, 4, 5\}$ shown in Table II.

TABLE II

EVOLUTION OF THE MAXIMUM EIGENVALUES OF THE LMIS GIVEN BY (4), $\ell = 1, \ldots, J(d+1+1)$, FOR $d = \{0, 1, 2, 3, 4, 5\}$ IN THE STABILITY ANALYSIS OF EXAMPLE 1.

	$\lambda_{max}(T_\ell)$					
ℓ	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
1	-0.039	-0.039	-0.039	-0.039	-0.039	-0.039
2	1908.58	1024.06	273.88	92.67	27.81	-0.03
3	-0.004	823.43	511.81	258.75	53.84	-21.06
4		-0.004	198.02	222.42	62.54	-64.22
5			-0.004	63.65	41.83	-57.69
6				-0.004	18.48	-15.95
7					-0.004	-0.19
8						-0.004

The convergence is attained for $d = 5$, assuring that an affine parameter-dependent Lyapunov function is necessary and sufficient to check the robust stability of this uncertain system ($K = 12$ scalar variables and $L = 30$ LMI rows have been used in Theorem 1 for $d = 5$).

Example 2: Consider the continuous-time uncertain system given in [12, Example 2]. Table III shows the number of scalar variables and LMI rows needed to attain the optimum value of $\hat{\rho}$, as well as the elapsed time required by LMI Control Toolbox [2] and SeDuMi [3], for theorems 1, 3 and for the methods proposed in [12] and [15].

TABLE III

COMPARISON OF THE RESULTS OF THEOREMS 1 AND 3 WITH [12] AND [15] FOR THE SECOND EXAMPLE IN [12] ($n = 3$, $N = 2$ AND $\hat{\rho} = 3.551$) IN TERMS OF COMPUTATIONAL BURDEN (K SCALAR VARIABLES AND L LMI ROWS).

Method	K	L	Time [2]	Time [3]
$[12]_{m=2}$	69	21	4.6 s	0.24 s
$[15]_{k=4}$	44	65	–	0.71 s
$T3_{g=2,d=0}$	72	33	0.13 s	0.60 s
$T3_{g=2,d=1}$	72	42	0.21 s	0.45 s
$T3_{g=2,d=2}$	72	51	0.30 s	0.39 s
$T3_{g=2,d=3}$	72	60	0.38 s	0.25 s
$T1_{g=2,d=12}$	18	93	0.64 s	0.47 s
$T1_{g=3,d=1}$	24	33	0.06 s	0.19 s
$T1_{g=4,d=0}$	30	33	0.04 s	0.16 s

Example 3: Consider a discrete-time uncertain system ($n = 2$, $N = 4$) given by

$$A_1 = \begin{bmatrix} -0.468 & 0.845 \\ 0.272 & -0.423 \end{bmatrix}; \; A_2 = \begin{bmatrix} 0.825 & 0.427 \\ 0.299 & -0.346 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.744 & 0.214 \\ 1.242 & 0.545 \end{bmatrix}; \quad A_4 = \begin{bmatrix} 0.330 & -1.140 \\ -0.322 & 0.309 \end{bmatrix}$$

Table IV shows a numerical comparison with the positive polynomial approach from [15] and the results presented in this paper. The best result (i.e. the one that demands less computational burden) is provided by Theorem 3 with $g = 1$, $d = 0$, assuring that an affine parameter-dependent Lyapunov function is necessary and sufficient to guarantee the robust stability.

TABLE IV

COMPARISON OF THE RESULTS OF THEOREMS 2 AND 3 WITH THE ONES PRESENTED IN [15] FOR THE EXAMPLE 3 ($n = 2$, $N = 4$).

Method	K	L	Time [2]	Time [3]
$[15]_{k=4}$	494	385	–	$9.76\ s$
$\mathrm{T3}_{g=1,d=0}$	44	48	$0.09\ s$	$0.10\ s$
$\mathrm{T2}_{g=1,d=3}$	12	232	$2.73\ s$	$0.25\ s$
$\mathrm{T2}_{g=2,d=2}$	30	294	$5.20\ s$	$0.32\ s$
$\mathrm{T2}_{g=3,d=0}$	60	180	$0.31\ s$	$0.21\ s$

Example 4: Consider now a continuous-time uncertain system of larger dimension ($n = 3$, $N = 4$) given by

$$A_1 = \begin{bmatrix} -0.789 & -0.533 & 0.353 \\ -0.469 & -0.390 & 0.676 \\ -0.970 & -0.914 & 0.053 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1.091 & -0.349 & 0.498 \\ 0.498 & -0.772 & 0.223 \\ -0.113 & 0.640 & -0.493 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.419 & 0.896 & -0.854 \\ -0.198 & -0.417 & 0.592 \\ 0.574 & 0.113 & -0.970 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -0.646 & -0.875 & -0.997 \\ -0.732 & -0.993 & -0.126 \\ 0.707 & 0.289 & 0.019 \end{bmatrix}$$

Table V shows a numerical comparison of the results from Theorems 1 and 3 with the ones presented in [12] and with the polynomial approach from [15]. Clearly, Theorems 1 and 3 present the best results (significantly less computational effort).

TABLE V

COMPARISON OF THE RESULTS OF THEOREMS 1 AND 3 WITH [12] AND [15] FOR EXAMPLE 4 ($n = 3$, $N = 4$).

Method	K	L	Time [2]	Time [3]
$[12]_{m=2}$	414	48	$> 600\ s$	$78.9\ s$
$[15]_{k=4}$	494	385	–	$19.07\ s$
$\mathrm{T3}_{g=2,d=0}$	240	150	$3.99\ s$	$0.61\ s$
$\mathrm{T1}_{g=2,d=3}$	60	420	$5.82\ s$	$0.99\ s$
$\mathrm{T1}_{g=3,d=1}$	120	273	$2.01\ s$	$0.50\ s$
$\mathrm{T1}_{g=4,d=0}$	210	273	$1.35\ s$	$0.65\ s$

VIII. CONCLUSION

A systematic procedure to construct families of LMI conditions to check the existence of a HPPDL function of arbitrary degree assuring robust stability of uncertain time-invariant linear systems in polytopic domains was presented in this paper. Numerical experiments illustrate that the proposed conditions provide a good trade-off between the required computational burden and the accuracy of the results when compared with other methods. The conditions proposed can easily be extended to cope with other robust analysis problems formulated in terms of LMIs.

REFERENCES

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM Studies in Applied Mathematics, 1994.

[2] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, *LMI Control Toolbox User's Guide*. The Math Works Inc., Natick, MA, 1995.

[3] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11–12, pp. 625–653, 1999, URL: http://fewcal.kub.nl/sturm/software/sedumi.html.

[4] P. Gahinet, P. Apkarian, and M. Chilali, "Affine parameter-dependent Lyapunov functions and real parametric uncertainty," *IEEE Trans. Automat. Contr.*, vol. 41, no. 3, pp. 436–442, March 1996.

[5] J. C. Geromel, M. C. de Oliveira, and L. Hsu, "LMI characterization of structural and robust stability," *Lin. Alg. Appl.*, vol. 285, no. 1–3, pp. 69–80, December 1998.

[6] M. C. de Oliveira, J. C. Geromel, and L. Hsu, "LMI characterization of structural and robust stability: the discrete-time case," *Lin. Alg. Appl.*, vol. 296, no. 1–3, pp. 27–38, June 1999.

[7] D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou, "A new robust $\mathcal{D}$-stability condition for real convex polytopic uncertainty," *Syst. Contr. Lett.*, vol. 40, no. 1, pp. 21–30, May 2000.

[8] D. C. W. Ramos and P. L. D. Peres, "A less conservative LMI condition for the robust stability of discrete-time uncertain systems," *Syst. Contr. Lett.*, vol. 43, no. 5, pp. 371–378, August 2001.

[9] ——, "An LMI condition for the robust stability of uncertain continuous-time linear systems," *IEEE Trans. Automat. Contr.*, vol. 47, no. 4, pp. 675–678, April 2002.

[10] V. J. S. Leite and P. L. D. Peres, "An improved LMI condition for robust $\mathcal{D}$-stability of uncertain polytopic systems," *IEEE Trans. Automat. Contr.*, vol. 48, no. 3, pp. 500–504, March 2003.

[11] P.-A. Bliman, "A convex approach to robust stability for linear systems with uncertain scalar parameters," *SIAM J. Control Optim.*, vol. 42, no. 6, pp. 2016–2042, 2004.

[12] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, "Robust stability of polytopic systems via polynomially parameter-dependent Lyapunov functions," in *Proc. 42nd IEEE Conf. Decision Contr.*, Maui, HI, USA, December 2003, pp. 4670–4675.

[13] J. B. Lasserre, "Global optimization with polynomials and the problem of moments," *SIAM J. Control Optim.*, vol. 11, no. 3, pp. 796–817, February 2001.

[14] G. Chesi, "Robust analysis of linear systems affected by time-invariant parametric uncertainty," in *Proc. 42nd IEEE Conf. Decision Contr.*, Maui, HI, USA, December 2003, pp. 5019–5024.

[15] D. Henrion, D. Arzelier, D. Peaucelle, and J. B. Lasserre, "On parameter-dependent Lyapunov functions for robust stability of linear systems," in *Proc. 43rd IEEE Conf. Decision Contr.*, Paradise Island, Bahamas, December 2004, pp. 887–892.

[16] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed. Cambridge, UK: Cambridge University Press, 1952.

[17] C. W. Scherer, "Higher-order relaxations for robust LMI problems with verifications for exactness," in *Proc. 42nd IEEE Conf. Decision Contr.*, Maui, HI, USA, December 2003, pp. 4652–4657.

[18] E. de Klerk and D. V. Pasechnik, "Approximation of the stability number of a graph via copositive programming," *SIAM J. Optim.*, vol. 12, no. 4, pp. 875–892, 2002.

[19] C. W. Scherer, "Relaxations for robust linear matrix inequality problems with verifications for exactness," *SIAM J. Matrix Anal. Appl.*, 2005, to appear.

[20] V. Powers and B. Reznick, "A new bound for Pólya's Theorem with applications to polynomials positive on polyhedra," *Journal of Pure and Applied Algebra*, vol. 164, pp. 221–229, 2001.

[21] R. C. L. F. Oliveira and P. L. D. Peres, "Stability of polytopes of matrices via affine parameter-dependent Lyapunov functions: Asymptotically exact LMI conditions," *Lin. Alg. Appl.*, vol. 405, pp. 209–228, August 2005.

[22] M. C. de Oliveira and R. E. Skelton, "Stability tests for constrained linear systems," in *Perspectives in Robust Control*, ser. Lecture Notes in Control and Information Science, S. O. Reza Moheimani, Ed. New York: Springer-Verlag, 2001, vol. 268, pp. 241–257.