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Stochastic H_2/H_∞ Control with (x,u,v)-dependent Noise

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Abstract— **In this paper, finite and infinite horizon stochastic H_2/H_∞ control problems with all state, control input and external disturbance-dependent noise are studied. It is shown that the existence condition of finite horizon stochastic H_2/H_∞ control is equivalent to the solvability of four coupled matrix-valued differential equations, and that of infinite horizon stochastic H_2/H_∞ control is equivalent to the solvability of four coupled matrix-valued algebraic equations.**

I. INTRODUCTION

Stochastic H_∞ theory that the system state governed by Ito's differential equation has become an attractive area since the the systematic work [3], the reader can refer to [2], [3], [9]-[12] and the references therein for the recent development of this subject. In [2], necessary and sufficient conditions for finite and infinite horizon stochastic H_2/H_∞ with only state-dependent noise were respectively presented in terms of a pair of coupled generalized differential Riccati equations (GDREs) and coupled generalized algebraic Riccati equations (GARE). It is well-known that most natural phenomena are expressed by stochastic Ito system with not only state but also control and external disturbance-dependent noise, see [3] and [10], so how to extend the results of [2] to more general models becomes without doubt a significant work. Now our goal in this paper is to extend the result of [2] to the system with all state, control and external disturbance or (x,u,v)-dependent noise, which is very valuable in practice. Some previous results, such as those of [2] and [4] are as corollaries of our main theorems. We'll show that for our general systems, the existence of finite horizon H_2/H_∞ control is equivalent to the solvability of four coupled matrix-valued differential equations, while the existence of infinite horizon stochastic H_2/H_∞ control is equivalent to the solvability of four coupled matrix-valued algebraic equations. It should be first emphasized that, to address the infinite horizon stochastic H_2/H_∞ control, we introduce a new concept called "exact detectability", which is an extension of complete detectability of deterministic systems. Moreover, by means of exact detectability, we generalize an important property of Lyapunov equation to generalized Lyapunov-type equation (GLE), which has a direct application in Subsection III.B. Second, although we present necessary and sufficient conditions for both finite and infinite horizon H_2/H_∞ control problems, how to solve these four cross-coupled matrix-valued differential and algebraic equations still remains a challenging and valuable mathematical problem, which merit further study.

The outline of this paper is as follows. In Section II, finite horizon stochastic H_2/H_∞ control is discussed. In Subsection III.A, exact detectability is first introduced and for which a stochastic Popov-Belevith-Hautus (PBH) criterion is presented. Moreover, based on exact detectability, an important property of GLE is presented. Subsection III.B treats with the infinite horizon stochastic H_2/H_∞ control and generalizes the results of [8]. Section IV ends this paper with some comments.

For convenience, we adopt the following traditional notations in this paper.

A' ($Ker(A)$): the represents transpose (Kernel space) of a matrix A;

$A \geq 0 (A > 0)$: A is a positive semidefinite (positive definite) matrix;

I: identity matrix;

${}^2([0,T], \ {}^l)(\ {}^2(\ _+, \ {}^l)$: space of nonanticipative stochastic processes $x(t) \in \ {}^l$ with respect to an increasing σ-algebra $\ _{t\geq 0}$ satisfying $E\int_0^T |x(t)|^2\,dt < \infty$ $(E\int_0^\infty |x(t)|^2\,dt < \infty)$.

Finally, we make an assumption throughout this paper that all systems treated with are real-valued.

II. FINITE HORIZON H_2/H_∞ CONTROL

Consider the following linear stochastic Ito system (the time variable t is suppressed)

$$\left\{ \begin{array}{l} dx = (A_0x + B_2u + B_1v)\,dt + (A_1x + C_2u + C_1v)\,dw \\ z = \left[\begin{array}{c} C_0x \\ Du \end{array} \right], x(0) = x_0 \in \ {}^n \end{array} \right. \tag{1}$$

where $x(t), z(t), u(t)$ and $v(t)$ are respectively the system state, controlled output, control input and external disturbance, all coefficient matrices are assumed to be matrix-valued continuous functions of the time variable t. In particular, we assume $D'(t)D(t) \equiv I$ for simplicity. Without loss of generality, we may take $w(t)$ to be one-dimensional standard Wiener process, defined on the filtered probability space $(\Omega, \ , \ _t, \ )$. To introduce the definition of the finite horizon stochastic H_2/H_∞ control, we first define a system with a finite gain.

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For the linear time-varying stochastic perturbed system

$$\begin{cases} dx = (Ax + Bv)dt + (Cx + Dv)dw, \\ z = Mx \end{cases} \tag{2}$$

let $x(t, v, x_0)$ denotes the solution of (2) starting from initial state x_0 at time 0. We define the perturbed operator as

$$_{[0,T]} : v \in {}^2([0,T], {}^{n_v}) \mapsto Mx(t, v, 0), T \geq 0$$

with the norm of $_{[0,T]}$ as

$$\begin{aligned} \| {}_{[0,T]} \| &= \sup_{v \in {}^2([0,T], {}^{n_v}), v \neq 0, x_0 = 0} \frac{\|z\|_{[0,T]}}{\|v\|_{[0,T]}} \\ &= \sup_{v \in {}^2([0,T], {}^{n_v}), v \neq 0, x_0 = 0} \frac{\{E \int_0^T z'z\, dt\}^{1/2}}{\{E \int_0^T v'v\, dt\}^{1/2}} \end{aligned}$$

Definition 2.1. System (2) is said to be a finite gain system, if $\| {}_{[0,T]} \| < \gamma$ for all $T \geq 0$ with a prescribed $\gamma > 0$.

In analogy with Definition 5 of [2], based on Definition 2.1, finite horizon mixed H_2/H_∞ control can be expressed as follows:

Definition 2.2 For a given disturbance attenuation $\gamma > 0$, $T > 0$, the so-called finite horizon mixed H_2/H_∞ control is to find a state feedback control $u_T^*(t,x) = K_2(t)x(t) \in {}^2([0,T], {}^{n_u})$, such that

i) The closed-loop system

$$\begin{cases} dx = [(A_0 + B_2K_2)x + B_1 v]\, dt + [(A_1 + C_2K_2)x \\ \qquad\qquad + C_1 v]\, dw \\ z = \begin{bmatrix} C_0 x \\ DK_2 x \end{bmatrix}, x(0) = x_0 \in {}^n \end{cases}$$

is a finite gain system, or equivalently, $\| {}^{K_2}_{[0,T]} \| < \gamma$, where ${}^{K_2}_{[0,T]}$ is the perturbed operator of (3).

ii) When the worst case disturbance $v_T^*(t,x) = K_1(t)x(t) \in {}^2([0,T], {}^{n_v})$, if existing, is implemented in (1), $u_T^*(t,x)$ also minimizes the output energy

$$J_2^T(u, v_T^*) = \|z\|^2_{[0,T]} = E \int_0^T |z|^2\, dt.$$

If the above (u_T^*, v_T^*) exist, then we say that (u_T^*, v_T^*) solve the finite horizon mixed H_2/H_∞ control of (1). As said in [2], if we define

$$J_1^T(u, v) = E \int_0^T (\gamma^2 v'v - z'z)\, dt,$$

then finite horizon H_2/H_∞ control is equivalent to looking for the following Nash equilibrium point:

$$J_1^T(u_T^*, v_T^*) \leq J_1^T(u_T^*, v), J_2^T(u_T^*, v_T^*) \leq J_2^T(u, v_T^*).$$

Our main result of this paper is the following theorem.

Theorem 2.1 For given disturbance attenuation $\gamma > 0$, finite horizon mixed H_2/H_∞ control has a pair of solutions (u_T^*, v_T^*) with

$$u_T^*(t,x) = K_2(t)x(t), v_T^*(t,x) = K_1(t)x(t)$$

if and only if (iff) the following four coupled matrix-valued equations admit solution $(P_1, P_2; K_1, K_2)$ with $P_1 \leq 0$ and $P_2 \geq 0$.

$$\begin{cases} \dot{P}_1 + \tilde{Q} - (P_1 B_1 + \tilde{A}_1' P_1 C_1)(\gamma^2 I + C_1' P_1 C_1)^{-1} \\ \quad (B_1' P_1 + C_1' P_1 \tilde{A}_1) + \tilde{A}_1' P_1 \tilde{A}_1 + P_1 \tilde{A}_0 + \tilde{A}_0' P_1 = 0, \\ \gamma^2 I + C_1' P_1 C_1 > 0, \forall t \in [0,T], P_1(T) = 0 \end{cases}$$

$$K_1 = -(\gamma^2 I + C_1' P_1 C_1)^{-1}(B_1' P_1 + C_1' P_1 \tilde{A}_1) \tag{3}$$

$$\begin{cases} \dot{P}_2 + C_0' C_0 - (P_2 B_2 + \bar{A}_1' P_2 C_2)(I + C_2' P_2 C_2)^{-1} \\ \quad (B_2' P_2 + C_2' P_2 \bar{A}_1) + \bar{A}_1' P_2 \bar{A}_1 + P_2 \bar{A}_0 + \bar{A}_0' P_2 = 0, \\ I + C_2' P_2 C_2 > 0, \forall t \in [0,T], P_2(T) = 0 \end{cases}$$

$$K_2 = -(I + C_2' P_2 C_2)^{-1}(B_2' P_2 + C_2' P_2 \bar{A}_1) \tag{4}$$

where

$$\tilde{A}_0 = A_0 + B_2 K_2, \tilde{A}_1 = A_1 + C_2 K_2, \tilde{Q} = -(C_0' C_0 + K_2' K_2)$$

$$\bar{A}_0 = A_0 + B_1 K_1, \bar{A}_1 = A_1 + C_1 K_1.$$

To prove Theorem 2.1, the following lemma is necessary, which indicates the relation between the $_2$-gain and the corresponding GDRE.

Lemma 2.1 [9]**.** For (2) and any given $\gamma > 0$, there exists a solution $P(t) \leq 0$ on $[0,T]$ to the following GDRE

$$\begin{cases} \dot{P} + PA + A'P - (PB + C'PD)(\gamma^2 I + D'PD)^{-1} \\ \qquad (B'P + D'PC) + C'PC - M'M = 0, \\ \gamma^2 I + D'PD > 0, \forall t \in [0,T], P(T) = 0 \end{cases}$$

iff (2) is a finite gain system or $\| {}_{[0,T]} \| < \gamma$.

Proof of Theorem 2.1. Sufficiency. Substituting $u = u_T^* = K_2(t)x(t)$ into (2), it follows

$$\begin{cases} dx = (\tilde{A}_0 x + B_1 v)\, dt + (\tilde{A}_1 x + C_1 v)\, dw \\ z = \begin{bmatrix} C_0 x \\ DK_2 x \end{bmatrix}, x(0) = x_0 \in {}^n. \end{cases} \tag{5}$$

Considering equation (3), using Lemma 2.1 for (5) immediately yields $\| {}^{K_2}_{[0,T]} \| < \gamma$. $v_T^* = K_1(t)x(t)$ is the worst case disturbance can be seen from

$$\begin{aligned} J_1^T(u_T^*, v) &= E \int_0^T [\gamma^2 v'v - z'z]\, dt \\ &= E \int_0^T [\gamma^2 v'v - z'z + d(x'P_1 x)] + x_0' P_1 x_0 \\ &= x_0' P_1 x_0 + E \int_0^T \|v - v_T^*\|^2_{(\gamma, P_1, C_1)}\, dt \\ &\geq J_1^T(u_T^*, v_T^*) = x_0' P_1 x_0 \end{aligned} \tag{6}$$

where

$$\|v - v_T^*\|^2_{(\gamma, P_1, C_1)} := (v - v_T^*)'(\gamma^2 I + C_1' P_1 C_1)(v - v_T^*).$$

Now, substituting v_T^* into (1), it gives

$$\begin{cases} dx = (\bar{A}_0 x + B_2 u)\, dt + (\bar{A}_1 x + C_2 u)\, dw \\ z = \begin{bmatrix} C_0 x \\ Du \end{bmatrix}, x(0) = x_0 \in {}^n. \end{cases} \tag{7}$$

With the constraint of (7), minimizing $J_2^T(u, v_T^*)$ is a standard stochastic linear quadratic optimization problem. Applying a standard completion of square technique together with considering (4), we have

$$\min_{u\in \ _{ad}^{T}} E\int_0^T z'z\,dt = x_0'P_2(0)x_0$$

with the corresponding optimal control $u_T^* = K_2(t)x(t)$ given in (4). The sufficiency is proved.

Necessity. If (u_T^*, v_T^*) solves the finite horizon stochastic H_2/H_∞ control with

$$u_T^*(t,x) = K_2(t)x(t), v_T^*(t,x) = K_1(t)x(t)$$

where K_1, and K_2 is to be determined, then substituting $u_T^*(t,x) = K_2(t)x(t)$ into (1) results in (5). By Definition 2.2 $\| \ _{[0,T]}^{K_2}\| < \gamma$, Again, Lemma 2.1 concludes that (3) admits a solution $P_1 \le 0$ with $v_T^* = K_1(t)x(t)$, where K_1 is defined by (3). Likewise, if we implement v_T^* in (1), it deduces (7). While GDRE (4) always exists a solution $P_2 \ge 0$ for fixed K_1, see [1]. As discussion in the sufficiency part, in this case, $u_T^* = K_2(t)x(t)$ with K_2 defined by (4).

Theorem 2.1 may concludes many special results. For example, if we take $C_2 \equiv 0, C_1 \equiv 0$, then Theorem 5 of [2] is derived; If we take $C_1 \equiv 0$, then the finite horizon stochastic H_2/H_∞ control with state and control-dependent noise is available, which is viewed as an open problem in [2]; If we let $A_1 \equiv C_2 \equiv C_1 \equiv 0$, $x_0 \in \ ^n$, i.e., (1) comes down to a deterministic system, then the related consequences of [4] are followed. We leave the other special discussions to the reader.

III. INFINITE HORIZON H_2/H_∞ CONTROL

A. Preliminaries

From now on, we assume all coefficient matrices are real-valued constant, i.e., the system is stochastic linear time-invariant (SLTI). To discuss the infinite horizon stochastic H_2/H_∞ control, we need to introduce some definitions.

Definition 3.1 [7]**.** Consider the following unforced stochastic system with measurement equation

$$\begin{cases} dx = Ax\,dt + Cx\,dw, x(0) = x_0 \in \ ^n, \\ y = Qx. \end{cases}$$

We call $x(0) \in \ ^n$ an unobservable state, if for any $T > 0$, the corresponding output response always equals zero, i.e., $y(t) \equiv 0, a.s.t \in [0,T], \forall T \ge 0$. (8) or $[A,C|Q]$ is called exactly observable, if there is no unobservable state except zero initial state.

Remark 3.1. If we let

$$\begin{aligned} P_0 \ &= \ [Q', A'Q', C'Q', A'C'Q', C'A'Q', \\ &\quad (A')^2Q', (C')^2Q', \cdots]', \end{aligned}$$

then all the unobservable states consist of a subspace, called an unobservable subspace. We denote it by $\ _0$, and $\ _0 = Ker(P_0)$, see [7]. Obviously, $[A,C|Q]$ is exactly observable, iff $\ _0 = \{\theta\}$. From $\ _0 = Ker(P_0)$, it is easy to show that (Q,A) is completely observable implies that $[A,C|Q]$ is exactly observable, but the inverse is not true.

It should be pointed out that there doesn't have any implication between exact observability and stochastic detectability (see Definition 3 of [2]) of $[A,C|Q]$, see the following examples.

Example 3.1. Taking

$$A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

one can easily see that $[A,C|Q]$ is exactly observable using Theorem 4 of [8]. But $[A,C|Q]$ is not stochastically detectable, because $(A', Q', C', 0)$ is not stabilizable. To see this fact, we quote Theorem 1 of [6], which says that $(A', Q', C', 0)$ is stabilizable iff the following GARE

$$PA' + AP + CPC' - PQ'QP + I_{2\times 2} = 0 \tag{8}$$

admits a positive definite solution $P > 0$, but for our given data, the solutions of GARE (8) is

$$P_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \le 0, \quad P_2 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} < 0.$$

Example 3.2. Taking $Q = 0$, $A = C = -I_{2\times 2}$, we can also see that $[A,C|Q]$ is stochastically detectable, but it is not exactly observable.

Below, we introduce a slightly more weak concept than exact observability called "exact detectability" as follows.

Definition 3.2. (8) or $[A,C|Q]$ is said to be exactly detectable, if $y(t) = Qx(t) \equiv 0$ $a.s.$, $t \in [0,T]$, $\forall T \ge 0$, implies $\lim_{t\to\infty} E|x(t)|^2 = 0$.

Proposition 3.1. If $[A,C|Q]$ is exactly detectable, $P_0 \ge 0$ solves GLE

$$PA + A'P + C'PC = -Q'Q \tag{9}$$

then the unobservable subspace $\ _0$ of $[A,C|Q]$ satisfies $\ _0 = Ker(P_0)$.

Proof. With the constraint (8), applying Ito's formula, it follows for any $T \ge 0$,

$$\begin{aligned} 0 \ &\le \ E\int_0^T y'y\,dt = E\int_0^T x'Q'Qx\,dt \\ &= E\int_0^T x'Q'Qx\,dt + x_0'P_0x_0 \\ &\quad + E\int_0^T d(x'P_0x) - Ex'(T)P_0x(T) \\ &= \ E\int_0^T x'(P_0A + A'P_0 + C'P_0C + Q'Q)x\,dt \\ &\quad + x_0'P_0x_0 - Ex'(T)P_0x(T) \\ &= \ x_0'P_0x_0 - Ex'(T)P_0x(T) \end{aligned} \tag{10}$$

Obviously, for any $x_0 \in Ker(P_0)$, (10) yields $y(t) \equiv 0, a,s.\forall t \in [0,T]$, i.e., $x_0 \in \ _0$, i.e., $x_0 \in \ _0$. Conversely, $\forall x_0 \in \ _0$, it follows $\lim_{T\to\infty} Ex'(T)P_0x(T) = 0$ from exact detectability. Therefore, $0 \le x_0'P_0x_0 = E\int_0^\infty y'y\,dt = 0$, and $x_0 \in Ker(P_0)$. In conclusion, $\ _0 = Ker(P_0)$.

Similar to the proof of Theorem 4 in [8], we have the following stochastic PBH criterion.

Theorem 3.1 (Stochastic PBH Criterion). $[A,C|Q]$ is exactly detectable iff there does not exist nonzero $Z \in \mathbb{S}_n$ such that

$$AZ + ZA' + CZC' = \lambda Z, \quad QZ = 0, \quad Re\lambda \geq 0. \qquad (11)$$

Proof. Using the PBH Criterion on complete detectability of deterministic systems, which says that (A,C) is completely detectable iff there does not exist nonzero $\zeta \in \mathbb{C}^n$ such that

$$A\zeta = \lambda\zeta, C\zeta = 0, Re\lambda \geq 0$$

then this theorem can be showed identically as the proof of Theorem 4 in [8].

Remark 3.2. By comparing Corollary 3.1 with Theorem 3.1, it can be seen that stochastic detectability implies exact detectability, but the converse is not true.

As known in stability theory, if $(A, Q^{1/2})$ is detectable, $P \geq 0$ solves $PA + A'P = -Q$ with $Q \geq 0$, then A is stable. Below we extend this consequence to GLE

$$PA + A'P + C'PC = -Q, \ Q \geq 0 \qquad (12)$$

by means of exact detectability.

Theorem 3.2. If $[A,C|Q]$ is exactly detectable, $P \geq 0$ is a solution to (12), then (A,C) is stable.

Proof. If P is strictly positive definite, then $[A,C|Q]$ is exactly observable by Proposition 3.1, while the stability of (A,C) is followed from Theorem 6 of [8] immediately. Otherwise, $Ker(P)$ is not empty, and for any $\xi \neq 0 \in Ker(P)$, it is easy to test that $\xi \in Ker(Q)$, i.e., $Ker(P) \subset Ker(Q)$. Moreover, $Ker(P)$ is an invariant subspace with respect to both A and C. Suppose S is an orthogonal matrix such that

$$S'PS = \begin{bmatrix} 0 & 0 \\ 0 & P_2 \end{bmatrix}, \quad det(P_2) \neq 0$$

then

$$S'QS = \begin{bmatrix} 0 & 0 \\ 0 & Q_2 \end{bmatrix}, \ S'AS = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \ S'CS = \begin{bmatrix} C_1 & C_{12} \\ 0 & C_2 \end{bmatrix}.$$

Pre- and post-multiplying S' and S on both sides of (12), it follows that

$$S'PS \cdot S'AS + S'A'S \cdot S'PS + S'C'S \cdot S'PS \cdot S'CS = -S'QS.$$

which is equivalent to

$$P_2A_2 + A_2'P_2 + C_2'P_2C_2 = -Q_2, \quad Q_2 \geq 0$$

In addition, applying Ito's formula to $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = S'x = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}' x$, it gives

$$\begin{cases} d\eta_1 = (A_1\eta_1 + A_{12}\eta_2)\,dt + (C_1\eta_1 + C_{12}\eta_2)\,dw \\ d\eta_2 = A_2\eta_2\,dt + C_2\eta_2\,dw \end{cases}$$

Obviously, $y = Qx \equiv 0, a.s.$ iff $Q_{22}\eta_2 \equiv 0, a.s$, and for which a sufficient condition is $\eta_2 \equiv 0$. Hence, by the definition of exact detectability, we have that (A_1, C_1) or equivalently, the following system

$$d\eta_1 = A_1\eta_1\,dt + C_1\eta_1\,dw$$

is stable in mean square sense. Below, we further show that (A_2, C_2) is stable. Applying Ito's formula to $\eta_2'P_2\eta_2$, it follows for any fixed $T \geq 0$

$$\begin{aligned} 0 < E\int_0^T \eta_2'(t)Q_2\eta_2(t)dt \\ = \eta_2'(0)P_2\eta_2(0) - E\eta_2'(T)P_2\eta_2(T) \end{aligned} \qquad (13)$$

As discussed in the proof of Theorem 6 [8], if we let $V(\eta_2(t)) = E\eta_2'(t)Q_2\eta_2(t)$, $t_n = nT, n = 1,2,\cdots$, then $\lim_{t\to\infty} V(x(t))$ exists. Moreover,

$$E\int_0^T \eta_2'(t)Q_2\eta_2(t)\,dt = \eta_2'(0)H(0)\eta_2(0)$$

where $H(\cdot) \geq 0$ is the unique solution to the following differential Riccati equation [1]

$$-\dot{H}(t) = H(t)A_1 + A_1'H(t) + C_1'H(t)C_1 + Q, \ H(T) = 0$$

We assert $H(0) > 0$. Otherwise, there exists $\eta_2(0) \neq 0 \in \mathbb{R}^{n_2}$, such that $H(0)\eta_2(0) = 0$, which deduces

$$E\int_0^T \eta_2'(t)Q_2\eta_2(t)\,dt = 0 \quad \Rightarrow Q_2\eta_2(t) = 0 \quad a.s. \ t \in [0,T]$$

The above implies that $y(t) = QS\eta(t) = S\begin{bmatrix} 0 & 0 \\ 0 & Q_2 \end{bmatrix}\eta(t) = 0 \quad a.s. t \in [0,T]$, which yields $\lim_{t\to\infty} E|x(t)|^2 = \lim_{T\to\infty} E\eta_2'(T)P_2\eta_2(T) = 0$ because of exact detectability of $[S'AS, S'CS|QS] = [S^{-1}AS, S^{-1}CS|QS]$. Therefore, (13) concludes $\eta_2'(0)P_2\eta_2(0) = 0$, which contradicts $P_2 > 0$, and $H(0) > 0$ is proved. The rest treatment for showing the stability of (A, C_2) is similar to that of Theorem 6 in [8] and is omitted. Now, from the stability of (A_1, C_1) and (A_2, C_2), we have $\lim_{t\to\infty} E|\eta(t)|^2 = 0$ (see [3]) , which is equivalent to $\lim_{t\to\infty} E|x(t)|^2 = 0$, the proof of this theorem is completed.

Theorem 3.2 improves Lemma 2 of [2] with exact detectability replacing stochastic detectability.

B. Main Results

Based on the above theories, especially the theory of exact detectability developed in Subsection III.A, we are able to extend the results of [2] on infinite horizon stochastic H_2/H_∞ control to the more general models with state-and control-dependent noise, which is taken as an open problem in [2]. In this subsection, we don't intend to treat with (x,u,v) but only (x,u)-dependent noise for the sake of simplicity.

Consider the following SLTI system

$$\begin{cases} dx = (A_0x + B_2u + B_1v)\,dt + (A_1x + C_2u)\,dw, \\ x(0) = x_0 \\ z = \begin{bmatrix} C_0x \\ Du \end{bmatrix}, \quad D'D = I. \end{cases} \qquad (14)$$

The following definition on stochastic H_2/H_∞ control is adapted from Definition 4 of [2].

Definition 3.3. Given a disturbance attenuation $\gamma > 0$, the infinite horizon mixed stochastic H_2/H_∞ control for system (14) is to find a constant feedback $u^* = K_2 x \in \ ^2(\ _+,\ ^{n_u})$ such that

1)

$$\|\ \|_\infty = \sup_{v\in\ ^2(\ _+,\ ^{n_v}),v\neq 0,x_0=0} \frac{\{E\int_0^\infty (x'C_0'C_0x + u^{*\prime}u^*)dt\}^{1/2}}{\{E\int_0^\infty v'vdt\}^{1/2}} < \gamma \tag{15}$$

where

$$: v \in\ ^2(\ _+,\ ^{n_u}) \mapsto z = \begin{bmatrix} C_0 x \\ Du^* \end{bmatrix}, x_0 = 0$$

associated with (14) is called the perturbation operator of (14).

2) u^* stabilizes system (14) internally. That is

$$dx = (A_0 + B_2K_2)xdt + (A_1 + C_2K_2)xdw$$

is asymptotically mean square stable, which is also called $(A_0 + B_2K_2, A_1 + C_2K_2)$ stable for short.

3) When the worst case disturbance $v^* = K_1 x \in\ ^2(\ _+,\ ^{n_v})$ with K_1 a constant matrix, if existing, is applied to system (14), u^* minimizes the output energy

$$\|z\|_2^2 = E\int_0^\infty (x'C_0'C_0x + u'u)dt.$$

We say that the infinite horizon stochastic H_2/H_∞ control admits a pair of solutions if the above (u^*, v^*) exists.

Our destination is to extend Theorems 1-4 of [2] to system (14). To this end, we must generalize Lemmas 3-5 of [2] as follows.

Lemma 3.1. Assume $\gamma \neq 0$, $\tilde{K} = (I + C_2'P_2C_2)^{-1}(B_2'P_2 + C_2'P_2A_1)$,

$$\tilde{C_2} = \begin{bmatrix} C_0 \\ \gamma^{-1}B_1'P_1 \\ \tilde{K} \end{bmatrix}, \quad \tilde{A_3} = \begin{bmatrix} C_0 \\ \tilde{K} \end{bmatrix}.$$

(i) If $[A_0, A_1|C_0]$ is exactly observable, then so is $[A_0 - B_2\tilde{K}, A_1 - C_2\tilde{K}|\tilde{C_2}]$.

(ii) If $[A_0 - \gamma^{-2}B_1B_1'P_1, A_1|C_0]$ is exactly observable, then so is $[A_0 - \gamma^{-2}B_1B_1'P_1 - B_2\tilde{K}, A_1 - C_2\tilde{K}|\tilde{A_3}]$.

Proof. This lemma can be proved by Theorem 4.2 of [7] (see Appendix B of [2]) or by applying Theorem 4 of [8], the detailed proof is omitted.

Lemma 3.2. Under the conditions of Lemma 5.1, we have

(1) If $[A_0, A_1|C_0]$ is exactly detectable, then so is $[A_0 - B_2\tilde{K}, A_1 - C_2\tilde{K}|\tilde{C_2}]$.

(2) If $[A_0 - \gamma^{-2}B_1B_1'P_1, A_1|C_0]$ is exactly detectable, then so is $[A_0 - \gamma^{-2}B_1B_1'P_1 - B_2\tilde{K}, A_1 - C_2\tilde{K}|\tilde{A_3}]$.

Proof. If $[A_0 - B_2\tilde{K}, A_1 - C_2\tilde{K}|\tilde{C_2}]$ is not exactly detectable, then Theorem 3.1 tells us that there exist nonzero $X \in\ _n$ such that

$$\begin{aligned} X(A_0 - B_2\tilde{K})' + (A_0 - B_2\tilde{K})X \\ +(A_1 - C_2\tilde{K})X(A_1 - C_2\tilde{K})' = \lambda X, \\ \tilde{C_2}X = 0,\ Re(\lambda) \geq 0 \end{aligned} \tag{16}$$

However, $\tilde{C_2}X = 0$ implies $\tilde{K}X = 0$. Therefore, (16) concludes

$$XA_0' + A_0X + A_1XA_1' = \lambda X, \quad C_0X = 0$$

which contradicts the exact detectability of $[A_0, A_1|C_0]$, thus the proof of (1) is completed. By the same discussion, (2) can also be shown.

Remark 3.3. Note that in order to extend Lemma 4 of [2] to Lemma 3.2, we have replaced stochastic detectability used in [2] with a weaker condition-"exact detectability". Obviously, under the assumption of stochastic detectability, we don't know whether Lemma 3.2 holds, which prevents us from generalizing Theorems 3 and 4 to the system (14).

Below we further generalize Lemma 5 of [2] which is called "stochastic bounded real lemma" (SBRL) to the system (2) with constant coefficients. As in [2], we define the perturbed operation $\tilde{\ }$ of 2) as

$$\tilde{\ } : v \in\ ^2(\ _+,\ ^{n_v}) \mapsto z = Mx, x_0 = 0$$

$$\begin{aligned} \|\tilde{\ }\|_\infty &= \sup_{v\in\ ^2(\ _+,\ ^{n_v}),v\neq 0,x_0=0} \frac{\|z\|_2}{\|v\|_2} \\ &= \sup_{v\in\ ^2(\ _+,\ ^{n_v}),v\neq 0,x_0=0} \frac{\{E\int_0^\infty x'M'Mxdt\}^{1/2}}{\{E\int_0^\infty v'vdt\}^{1/2}}. \end{aligned}$$

Lemma 3.3[9] **(linear SBRL).** Assume system (2) is internally stable, then $\|\tilde{\ }\|_\infty < \gamma$ for some $\gamma > 0$ iff GARE

$$\begin{cases} PA + A'P + C'PC - (PB + C'PD) \\ \times(\gamma^2 I + D'PD)^{-1}(B'P + D'PC) - M'M = 0 \\ \gamma^2 I + D'PD > 0. \end{cases}$$

has a solution $P \leq 0$, and $(A + B\tilde{K}, C + D\tilde{K})$ is stable, where $\tilde{K} = -(\gamma^2 I + D'PD)^{-1}(B'P + D'PC)$.

As soon as Lemmas 3.1-3.3 as well as Theorem 3.2 are obtained, it is an easy thing to generalize Theorems 1-4 of [2] to the more general system (14). However, since the procedures are analogous to those employed in [2], to avoid some unnecessary repeat, the detailed proofs of the following theorems are omitted.

Theorem 3.3. For system (14), suppose the coupled GAREs

$$P_1\tilde{A}_0 + \tilde{A}_0'P_1 + \tilde{A}_1'P_1\tilde{A}_1 + \tilde{Q} - \gamma^{-2}P_1B_1B_1'P_1 = 0 \tag{17}$$

$$\begin{cases} P_2\bar{A}_0 + \bar{A}_0'P_2 + A_1'P_2A_1 + C_0'C_0 \\ \quad -(P_2B_2 + A_1'P_2C_2)(I + C_2'P_2C_2)^{-1} \\ \quad\quad \times(B_2'P_2 + C_2'P_2A_1) = 0 \\ I + C_2'P_2C_2 > 0 \end{cases}$$

have a pair of solutions (P_1, P_2) with $P_1 \leq 0, P_2 \geq 0$ ($P_1 < 0, P_2 > 0$) with $K_1 = -\gamma^{-2}B_1'P_1$, $K_2 = -(I + C_2'P_2C_2)^{-1}(B_2'P_2 + C_2'P_2A_1)$.In addition, assume $[A_0, A_1|C_0]$ and $[A_0 - \gamma^{-2}B_1B_1'P_1, A_1|C_0]$ are exactly detectable (exactly observable), then the stochastic H_2/H_∞ control admits a pair of solutions $u^* = -(I + C_2'P_2C_2)^{-1}(B_2'P_2 + C_2'P_2A_1), v^* = -\gamma^{-2}B_1'P_1$.

Theorem 3.4. If the infinite horizon stochastic H_2/H_∞ control exists a pair of solutions (u^*, v^*) as

$$u^* = K_2 x, \quad v^* = K_1 x$$

with K_2 and K_1 constant matrices of appropriate dimensions. If $[A_0 + B_1 K_1, A_1 | C_0]$ is exactly detectable (exactly observable), then the coupled GAREs (17) and (18) admit solutions $P_1 \leq 0$ and $P_2 \geq 0$ $(P_2 > 0)$, respectively.

Remark 3.4. If we take $C_2 = 0$ in (14), then the coupled GAREs (17) and (18) deduce to (14) and (15) of [2], respectively.

Remark 3.5. A discrete-time stochastic H_2/H_∞ control problem was recently discussed in [5] based on the Nash game theory, which can be viewed as the continuous version of [8].

IV. CONCLUSIONS

This note have presented a necessary and sufficient condition for finite/infinite horizon stochastic H_2/H_∞ control with (x, u, v)-dependent noise, which generalizes the corresponding consequence of [2]. More important, we believe that Theorem 3.2 will have many applications in the study of stochastic control theory such as stochastic linear quadratic optimization and the related GARE.

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