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Linear Programming and l_1-Norm Minimization Problems with Convolution Constraints

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Abstract— **We illustrate some recent results on exact solutions to discrete-time l_1-norm minimization problems with convolution constraints. A fixed-point property for this class of problems is introduced. The convolution constraints can be interpreted as a dynamic system with initial conditions. We show by construction that optimal solutions with a rational Z-transform exist for any initial conditions satisfying the fixed-point property. Some fixed-point initial conditions satisfy a further stability property. If there exists a stable fixed point, then for any initial condition in some neighbourhood of the fixed point an optimal solution can be constructed having a rational Z-transform.**

I. INTRODUCTION

The general multi-block l_1-norm minimization problem has both convolution constraints (also termed rank interpolation conditions) and zero interpolation conditions. In this paper we consider problems with convolution constraints only. We shall convert the given infinite dimensional linear programming problem, which we denote $\mathcal{P}$, to an infinite number of linked finite-dimensional sub-problems, all but an initial finite number of these sub-problems having identical optimal bases. We state without proof some results in [1], discuss the ideas and present examples.

Duality was first used as a tool in the investigation of l_1-norm minimization problems in [2] and [3], and has been regularly employed since. It is also the basis for our results. We start with the primal minimization problem $\mathcal{P}(b)$, where b represents initial conditions for the dynamic system that determines the constraints for $\mathcal{P}$. Denote by $\mathcal{D}(b)$ the dual of $\mathcal{P}(b)$. The maximization problem $\mathcal{D}(b)$ was first described in [4]. We will later modify $\mathcal{D}(b)$ by deleting all but a finite number of its constraints, and impose a terminal boundary condition determined by an optimal solution to $\mathcal{D}(\tilde{b})$, where $\tilde{b}$ is a special initial condition for $\mathcal{P}$ that satisfies a certain fixed-point property, to be described in Section III-G. The main result of [1] is a theorem which identifies certain bases with this fixed-point property. These bases play a fundamental role in the construction of optimal solutions to $\mathcal{P}$. We review without proof some of these results, and also give a new result characterizing finite-length optimal solutions. A preliminary version of some of the results discussed here was given in [5].

For the l_1-norm minimization methodology, and applications to model matching, the optimal rejection of persistent bounded disturbances, feedback design to improve stability robustness, and regulation in the presence of unknown bounded reference signals, see for example [6], [3], [7], [8], [9] and [10]. Other approaches that seek to tease out the finite-dimensional structure that seems to lie behind multi-block problems are the delay augmentation method, discussed in [8], and research involving the use of convex optimization and dynamic programming, see [11] for example.

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II. FORMULATION

A. Terminology

Denote by $\mathbb{R}^n$ the n-dimensional real space. The symbol $0_{p\times k}$ denotes a $p\times k$ matrix all of whose elements are zero. A $p\times k$ matrix M will sometimes have its dimension made explicit by the notation $M_{p\times k}$. The $p\times p$ identity matrix is denoted I_p. The set of positive integers is denoted $\mathcal{N}$. The l_1-norm of a vector sequence $e=(e_k)_{k=1}^{\infty}$ is defined as $\|e\|_1 = \sum_{k=1}^{\infty}|e_k|$ whenever the series exists. The Banach space of absolutely-summable sequences, equipped with the l_1-norm, is denoted l_1. The space of continuous linear functionals on l_1, that is the dual of l_1, is denoted l_∞; it is the space of bounded sequences with the norm $\|e\|_\infty := \sup_k |e_k|$. The Z-transform of an arbitrary sequence $e=(e_k)_{k=1}^{\infty}$ is defined to be $\hat{e}(z)=\sum_{k=1}^{\infty} e_k z^{k-1}$, where z lies within the radius of convergence of the series. If, for $e=(e_k)_{k=1}^{\infty}\in l_\infty$, there is an integer n such that $e_n\neq 0$ and $e_k=0$ for $k>n$, then e is said to be of *length* n. Otherwise $e\neq 0$ has *infinite length.* Given a vector e and any $s\in\mathcal{N}$, $t\in\mathcal{N}$ satisfying $s<t$, we denote $(e_s, e_{s+1},\ldots,e_t)$ by $e_{(s:t)}$. If $s,t,q,r\in\mathcal{N}$, $1<s<t$ and $1<q<r$ then $M_{(s:t,q:r)}$ is a matrix composed of row s to row t, and of columns q to r, of the matrix M having at least t rows and at least r columns.

B. Problem description

The decision vectors for our restricted version of the general l_1 problem are two vectors in l_1, denoted e and u. The cost function is $K_1\|e\|_1 + K_2\|u\|_1$, where K_1 and K_2 are given positive real numbers.

The convolution constraints are given in the form

$$\hat{d}\hat{e}+\hat{n}\hat{u}=\hat{b} \tag{1}$$

where $\hat{n}$, $\hat{d}$ and $\hat{b}$ are polynomials with real coefficients,

$$\begin{aligned}\hat{n}(z) &= n_1+n_2z+n_3z^2+\cdots+n_{l+1}z^l \\ \hat{d}(z) &= d_1+d_2z+d_3z^2+\cdots+d_{l+1}z^l \\ \hat{b}(z) &= b_1+b_2z+\cdots+b_lz^{l-1},\end{aligned} \tag{2}$$

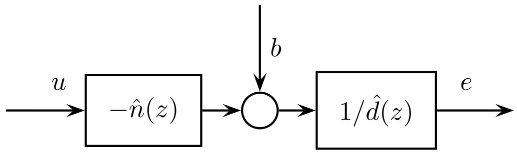

Fig. 1. The constraints for $\mathcal{P}$.

n_{l+1} and d_{l+1} are not both zero, and $l \geq 1$ is a positive integer. We assume that neither $\hat{n}(z)$ nor $\hat{d}(z)$ have zeros lying on the unit circle in the complex plane, and that $\hat{n}(z)$ and $\hat{d}(z)$ have no zeros in common.

The l_1-norm minimization problem we investigate is

$$\mathcal{P}: \qquad \begin{cases} \min\limits_{e \in l_1, u \in l_1} K_1 \|e\|_1 + K_2 \|u\|_1 \\ \text{subject to} \\ \hat{d}\hat{e} + \hat{n}\hat{u} = \hat{b}. \end{cases} \tag{3}$$

The vector b specifies initial conditions for the discrete-time dynamic system represented by (1). If $b = 0$ then obviously (1) is satisfied by $e = 0$ and $u = 0$, which is also optimal for $\mathcal{P}$. We assume $b \neq 0$. A pictorial representation of the system being optimized is given in Fig. 1.

We assume that $\hat{n}(z)$ and $\hat{d}(z)$ have no zeros in common and that none of them lie on the unit circle in the complex plane. It can be shown that a solution to $\mathcal{P}$ is guaranteed to exist. There is a stronger conjecture, namely that optimal solution vectors (e, u) for $\mathcal{P}$ have a rational $Z-$transform. This conjecture remains open.

We will later investigate families of problems parameterized by varying b, and sometimes it is convenient to indicate this using the notation $\mathcal{P}(b)$.

C. Linear programming formulation

The constraints (1) have the matrix representation

$$De + Nu = \left[\begin{array}{c} b^T, 0, \dots \end{array}\right]^T \tag{4}$$

where D is the infinite-dimensional lower-triangular toeplitz matrix with $(d_1, d_2, \dots, d_{l+1}, 0, 0, \dots)$ as its first column, and N, defined similarly, has first column $(n_1, n_2, \dots, n_{l+1}, 0, 0, \dots)$. Also $b := [b_1, \dots, b_l]^T$.

Using block matrix notation, the problem $\mathcal{P}$ can be written as

$$\mathcal{P}: \qquad \begin{cases} \min\limits_{e \in l_1, u \in l_1} K_1 \|e\|_1 + K_2 \|u\|_1 \\ \text{subject to} \\ \left[\begin{array}{cc} D & N \end{array}\right] \left[\begin{array}{c} e \\ u \end{array}\right] = \left[\begin{array}{c} b^T, 0, \dots \end{array}\right]^T. \end{cases} \tag{5}$$

D. Four useful submatrices

We shall be concerned with the effect of boundary conditions on various linear programs associated with $\mathcal{P}$ and its dual. Boundary conditions associated with these programs will be described using the four matrices introduced below. They are intimately related to two well-known classical constructions, Sylvester matrices and Bezoutians. The $l \times l$ top left-hand corner submatrix of N is denoted N_{LT},

$$N_{LT} := \left[\begin{array}{cccc} n_1 & 0 & 0 & 0 \\ \vdots & n_1 & 0 & \vdots \\ n_{l-1} & \vdots & n_1 & 0 \\ n_l & n_{l-1} & \dots & n_1 \end{array}\right]_{l \times l}$$

with D_{LT} defined similarly.

The matrices N_{LT} and D_{LT} are lower-triangular toeplitz. Also N_{UT} and D_{UT}, which are upper-triangular toeplitz, are defined by

$$N_{UT} := \left[\begin{array}{cccc} n_{l+1} & n_l & \dots & n_2 \\ 0 & n_{l+1} & \vdots & \vdots \\ 0 & 0 & n_{l+1} & n_l \\ 0 & 0 & 0 & n_{l+1} \end{array}\right]_{l \times l},$$

and similarly for D_{UT}.

Then

$$S(\hat{d}, \hat{n}) := \left[\begin{array}{cc} D_{LT} & N_{LT} \\ D_{UT} & N_{UT} \end{array}\right]$$

is the Sylvester matrix for the polynomials $\hat{d}(z)$ and $\hat{n}(z)$. It is well-known that S is non-singular if and only if $\hat{d}(z)$ and $\hat{n}(z)$ are coprime.

III. Duality

A. $\mathcal{D}$ -the infinite dual of $\mathcal{P}$

We describe a dual to $\mathcal{P}$, here denoted $\mathcal{D}$, or sometimes $\mathcal{D}(b)$, in the notation of this paper. A proof, based on the minimum-norm theorem, that $\mathcal{D}$ really is dual to $\mathcal{P}$ has been presented in [4].

The dual to $\mathcal{P}$ can be expressed in the form

$$\mathcal{D}: \quad \begin{cases} \max\limits_{e^* \in l_\infty, u^* \in l_\infty} J_{\mathcal{D}}(e^*, u^*) \\ \text{subject to } \|e^*\|_\infty \leq K_1, \ \|u^*\|_\infty \leq K_2 \\ \text{and} \\ D^T u^* = N^T e^* \end{cases}.$$

It is shown in [1] that the cost function for $\mathcal{D}$, $J_{\mathcal{D}}(e^*, u^*)$, can be represented as

$$J_{\mathcal{D}}(e^*, u^*) = [e_1^*, \dots, e_l^*, u_1^*, \dots, u_l^*] \, S^{-1} b. \tag{6}$$

The optimal values of $\mathcal{P}$ and $\mathcal{D}$ are equal. It is shown in [4] that if (e, u) and (e^*, u^*) are feasible for $\mathcal{P}$ and $\mathcal{D}$, respectively, then a necessary and sufficient condition that they both be optimal is that they be *complementary*; that is, for all $k \in \mathcal{N}$,

$$\begin{array}{c} e_k > 0 \Longrightarrow e_k^* = K_1, \ u_k > 0 \Longrightarrow u_k^* = K_2 \\ e_k < 0 \Longrightarrow e_k^* = -K_1, \ u_k < 0 \Longrightarrow u_k^* = -K_2 \\ |e_k^*| < K_1 \Longrightarrow e_k = 0, \ |u_k^*| < K_2 \Longrightarrow u_k = 0. \end{array} \tag{7}$$

This special case of a complementarity relation between optimal primal and dual vectors also occurs in all of the finite-dimensional dual pairs of programs we shall consider.

Definition 1: Vectors (e, u) and (e^*, u^*), both in $\mathbb{R}^p \times \mathbb{R}^p$, $p \in \mathcal{N}$, are said to be *complementary* if (7) holds for $k = 1, \dots, p$.

B. Imposing periodicity on the dual

We seek initial conditions, b, such that there are vectors e^* and u^* optimal for $\mathcal{D}(b)$ that are periodic of period p, so we set up a modified dual program of dimension p, termed $MD(b,p)$, whose feasible vectors have infinite periodic extensions feasible for $\mathcal{D}(b)$. The cost function is kept the same for $MD(b,p)$ and $\mathcal{D}(b)$.

It is natural to look to circulant matrices in order to characterize periodicity. For $p > l$ define $D_C(p)$ to be the circulant matrix of dimension $p \times p$ whose first column is $(d_1, d_2, \ldots, d_{l+1}, 0, \ldots, 0)$. That is, $D_C(p)$ is defined as

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 & d_{l+1} & \cdots & d_3 & d_2 \\ d_2 & d_1 & 0 & & \cdots & d_{l+1} & \cdots & d_3 \\ d_3 & d_2 & d_1 & & & 0 & d_{l+1} & d_l \\ \vdots & d_3 & d_2 & \ddots & \ddots & & & d_{l+1} \\ d_{l+1} & \vdots & \ddots & & \ddots & \ddots & 0 & \ddots \\ 0 & d_{l+1} & \ddots & & & & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & d_2 & d_1 & 0 \\ 0 & 0 & \ddots & & d_{l+1} & \cdots & d_2 & d_1 \end{bmatrix}_{p \times p}$$

Thus $D_C(p)$ is a truncated version of D, with D_{UT} inserted into the top right-hand corner. Similarly $N_C(p)$ is defined to be the circulant matrix of dimension $p \times p$ whose first column is $(n_1, n_2, \ldots, n_{l+1}, 0, \ldots, 0)$.

C. $MD(b,p)$ -a finite-dimensional modification of $\mathcal{D}(b)$

Let $p \geq 2l$ be an integer. We construct a convex programming problem related to $\mathcal{D}(b)$ that has $2p$ variables and p equality constraints. The constraints for $MD(b,p)$, the problem $\mathcal{D}$ constrained in a manner consistent with its variables e^* and u^* being periodic of period p, can be written $D_C^T(p)u^* = N_C^T(p)e^*$.

$$MD(b,p) : \begin{cases} \max\limits_{e^* \in \mathbb{R}^p, u^* \in \mathbb{R}^p} [e_1^*, \ldots, e_l^*, u_1^*, \ldots, u_l^*]\, S^{-1} b \\ \text{subject to } D_C^T(p)u^* = N_C^T(p)e^* \text{ and} \\ \|e^*\|_\infty \leq K_1, \;\; \|u^*\|_\infty \leq K_2. \end{cases}$$

It is clear that the optimal value of the program $MD(b,p)$, denoted $\left(J_{MD(b,p)}\right)_{opt}$, must be less than or equal to the optimal value for the program $\mathcal{D}(b)$, denoted $\left(J_{\mathcal{D}(b)}\right)_{opt}$. We seek initial conditions b and integers p such that $\left(J_{MD(b,p)}\right)_{opt} = \left(J_{\mathcal{D}(b)}\right)_{opt}$.

D. The dual of $MD(b,p)$, denoted $DMD(b,p)$

For $p \geq 2l$, a dual of $MD(b,p)$ can be constructed in the form

$$DMD(b,p) : \begin{cases} \min\limits_{e \in \mathbb{R}^p, u \in \mathbb{R}^p} \sum\limits_{k=1}^{p} K_1 |e_k| + K_2 |u_k| \\ \text{subject to} \\ D_C(p)e + N_C(p)u = b. \end{cases}$$

The optimal values of $DMD(b,p)$ and $MD(b,p)$ are equal. Furthermore, if (e,u) and (e^*,u^*) are feasible for $DMD(b,p)$ and $MD(b,p)$, respectively, then a necessary and sufficient condition that they both be optimal solutions is that they be complementary. A proof is given in [1].

E. Notation for a basis

Consider the set of equations $Ae + Bu = b$, or in block matrix notation

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix} = b, \tag{8}$$

where A and B are any real $p \times p$ matrices, and e, u and b are real p-dimensional column vectors. Let $\begin{bmatrix} A & B \end{bmatrix}_{\mathcal{B}}$ be any non-singular $p \times p$ sub-matrix made up of the columns of the $p \times 2p$ matrix $\begin{bmatrix} A & B \end{bmatrix}$. Thus the p integers $i_1, i_2, \ldots, i_p$ from $1, 2, \ldots, 2p$ identify the columns of $\begin{bmatrix} A & B \end{bmatrix}$ that have been retained in $\begin{bmatrix} A & B \end{bmatrix}_{\mathcal{B}}$. The set $\{i_1, i_2, \ldots, i_p\}$ determines the basis $\begin{bmatrix} A & B \end{bmatrix}_{\mathcal{B}}$ and the notation $\mathcal{B} = [i_1, i_2, \ldots, i_p]$ will be used to identify the basis. Sometimes, if A and B are clear from context, the basis is denoted simply by $\mathcal{B}$. If the p components of $\begin{bmatrix} e \\ u \end{bmatrix}$ not associated with columns of $\begin{bmatrix} A & B \end{bmatrix}_{\mathcal{B}}$ are set equal to zero, the solution to the resulting set of equations is said to be a *basic solution* to (8) with respect to the basis columns, $\mathcal{B}$, denoted $\begin{bmatrix} e \\ u \end{bmatrix}_{\mathcal{B}}^{bsol}$, a $2p \times 1$ vector. The components of $\begin{bmatrix} e \\ u \end{bmatrix}$ associated with the basis columns $\mathcal{B}$ are called *basic variables.*

For any integer $p \geq 2l$, suppose we are given a p-dimensional vector of basis columns, $\mathcal{B}$, for which $\begin{bmatrix} D_{(1:p,1:p)} & N_{(1:p,1:p)} \end{bmatrix}_{\mathcal{B}}$ is non-singular. Then we define

$$\begin{aligned} Z(\mathcal{B}) &:= \begin{bmatrix} D_{(1:p,1:p)} & N_{(1:p,1:p)} \end{bmatrix}_{\mathcal{B}} \\ Y(\mathcal{B}) &:= \begin{bmatrix} D_C(p) & N_C(p) \end{bmatrix}_{\mathcal{B}} \\ F(p) &:= \begin{bmatrix} D_{(1:p,1:p)} & N_{(1:p,1:p)} \end{bmatrix} - \begin{bmatrix} D_C(p) & N_C(p) \end{bmatrix} \\ A(\mathcal{B}) &:= I_p - YZ^{-1} = [F(p)]_{\mathcal{B}} Z^{-1} \\ G(\mathcal{B}) &:= A_{(1:l,1:l)} \end{aligned} \tag{9}$$

If $\mathcal{B}$ is clear from context, or an explicit representation is not required, we sometimes write, as above for example, Z instead of $Z(\mathcal{B})$.

F. Optimal solutions of finite length

For some problems $\mathcal{P}(b)$ the optimal vectors e and u are both of finite length.

Proposition 2: Suppose that, for some integer $p \geq 2l$, an optimal basis, $\mathcal{B}$, for $DMD(b,p)$ has the property that

$$Z(\mathcal{B}) = Y(\mathcal{B}). \tag{10}$$

Then $\mathcal{P}(b)$ has an optimal solution, $(\bar{e}, \bar{u})$, given by

$$\begin{aligned} \bar{e} &= \left(e_1^{bsol}, \ldots, e_{p-l}^{bsol}, 0, 0, \ldots\right) \\ \bar{u} &= \left(u_1^{bsol}, \ldots, u_{p-l}^{bsol}, 0, 0, \ldots\right), \end{aligned} \tag{11}$$

where $(e^{bsol}, u^{bsol}) \in \mathbb{R}^p \times \mathbb{R}^p$ is the optimal basic feasible solution for $DMD(b,p)$ with respect to the basis $\mathcal{B}$.

Proof: If (10) holds then $(\bar{e}, \bar{u}) \in l_1 \times l_1$ is feasible for $\mathcal{P}(b)$, and since $MD(b,p)$ is dual to $DMD(b,p)$ it follows that, for $(\tilde{e}^*, \tilde{u}^*) \in \arg\max MD(b,p)$,

$$\sum_{k=1}^{p} K_1 |\bar{e}_k| + K_2 |\bar{u}_k| = [\tilde{e}_1^*, \ldots, \tilde{e}_l^*, \tilde{u}_1^*, \ldots, \tilde{u}_l^*]\, S^{-1} b.$$

Define $\tilde{e}^*_{ext} \in l_\infty$ and $\tilde{u}^*_{ext} \in l_\infty$ to be the infinite periodic extensions of $(\tilde{e}^*_1, \ldots, \tilde{e}^*_p)$ and $(\tilde{u}^*_1, \ldots, \tilde{u}^*_p)$, respectively. Then $(\tilde{e}^*_{ext}, \tilde{u}^*_{ext})$ is feasible for $\mathcal{D}(b)$, and by (6)

$$J_{\mathcal{D}}(\tilde{e}^*_{ext}, \tilde{u}^*_{ext}) = [\tilde{e}^*_1, \ldots, \tilde{e}^*_l, \tilde{u}^*_1, \ldots, \tilde{u}^*_l]\, S^{-1} b.$$

We have constructed a feasible vector pair for $\mathcal{P}(b)$ having the same cost as a vector pair feasible for $\mathcal{D}(b)$. These vector pairs must therefore both be optimal. ∎

Example Consider the problem $\mathcal{P}(1)$ with $(2+z)\hat{e} + (-6+z)\hat{u} = 1$, $K_1 = 1$ and $K_2 = 6$. Then $\hat{d} = (2+z)$, $\hat{n} = (-6+z)$, $\hat{b} = 1$, $l = 1$ and, putting $p = 2$,

$$D_{(1:2,1:2)} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \quad N_{(1:2,1:2)} = \begin{bmatrix} -6 & 0 \\ 1 & -6 \end{bmatrix},$$
$$D_C(2) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad N_C(2) = \begin{bmatrix} -6 & 1 \\ 1 & -6 \end{bmatrix},$$
$$S = \begin{bmatrix} 2 & -6 \\ 1 & 1 \end{bmatrix}.$$

The unique optimal basic solution to $DMD(1,2)$ is $e^{bsol} = (1/8, 0)$, $u^{bsol} = (-1/8, 0)$, corresponding to the optimal basis $Y(\mathcal{B}) = \begin{bmatrix} 2 & -6 \\ 1 & 1 \end{bmatrix}$. By Proposition 2 the optimal solution for $\mathcal{P}(1)$ must be $\bar{e} = (1/8, 0, 0, \ldots)$, $\bar{u} = (-1/8, 0, 0, \ldots)$. For $MD(1,2)$ the unique optimizing vectors are $\tilde{e}^* = (1, -5/8)$ and $\tilde{u}^* = (-6, 43/8)$. Optimizing vectors for $\mathcal{D}(1)$ are $e^* = (1, -5/8, 1, -5/8, \ldots)$ and $u^* = (-6, 43/8, -6, 43/8, \ldots)$. These are not unique.

G. A fundamental fixed point property

Vectors $\tilde{b}$ of length $p \geq 2l$ that satisfy the following fixed-point scheme play a fundamental role in constructing optimal solutions for $\mathcal{P}$ having infinite length:

Does $DMD(\tilde{b}, p)$ possess a unique optimal basis,
$Y(\tilde{\mathcal{B}}) = \begin{bmatrix} D_C(p) & N_C(p) \end{bmatrix}_{\tilde{\mathcal{B}}}$,
for which $Z(\tilde{\mathcal{B}}) = \begin{bmatrix} D_{(1:p,1:p)} & N_{(1:p,1:p)} \end{bmatrix}_{\tilde{\mathcal{B}}}$
is non-singular?
↓ yes
Is $\tilde{b}$ an eigenvector of $G(\tilde{\mathcal{B}}) := \left[I_p - YZ^{-1}\right]_{(1:l,1:l)}$
with corresponding simple eigenvalue $\lambda \in (0,1)$?
↓ yes
The vector $\tilde{b}$ is said to be
a fixed point of period p for the program $\mathcal{P}$.

Definition 3: If there exists an integer $p \geq 2l$ and a vector $\tilde{b}$ that satisfies the above fixed-point scheme, then $\tilde{b}$ is said to be *a fixed point of period* p for the program $\mathcal{P}$. The optimal basis, $\tilde{\mathcal{B}}$, for the program $DMD(\tilde{b}, p)$ is termed *the fixed-point basis* corresponding to $\tilde{b}$.

Definition 4: Suppose that $\tilde{b}$ is a fixed point of period p. If there is a unique optimal basic solution for $MD(\tilde{b}, p)$ it will be termed *the fixed-point dual vector* corresponding to $\tilde{b}$, and will be denoted $(\tilde{e}^*, \tilde{u}^*)$. Thus $(\tilde{e}^*, \tilde{u}^*) \in \arg\max MD(\tilde{b}, p)$.

The infinite periodic extensions of $\tilde{e}^*$ and $\tilde{u}^*$ are denoted $\tilde{e}^*_{ext}$ and $\tilde{u}^*_{ext}$, respectively. That is $\tilde{e}^*_{ext} := [\tilde{e}^{*T}, \tilde{e}^{*T}, \ldots]^T$ and $\tilde{u}^*_{ext} := [\tilde{u}^{*T}, \tilde{u}^{*T}, \ldots]^T$.

Definition 5: The *period* of a fixed point $\tilde{b}$ is the (fundamental) period of the infinite periodic extension of $\tilde{e}^*$.

In the following Definition $\rho(G)$ denotes the spectral radius of G. The eigenvalue λ is the Perron-Frobenius eigenvalue of G.

Definition 6: If in the above scheme a fixed point $\tilde{b}$ satisfies the additional property that $\rho(G(\tilde{\mathcal{B}})) = \lambda$, where $\lambda \in (0,1)$ is the eigenvalue of $G(\tilde{\mathcal{B}})$ associated with $\tilde{b}$, then $\tilde{b}$ is said to be a *stable fixed point of period* p for the program $\mathcal{P}$.

IV. Optimal solutions of infinite length

We now consider the case where $\mathcal{P}$ has optimizing vectors e and u both of infinite length. We construct finite programs whose constraint equations are expressed in terms of truncated Toeplitz matrices.

A. A Modified Version of $MD(b,p)$, termed $M^2D(b,p)$

The program $MD(b,p)$ will now have a terminal boundary condition imposed. Let $(\tilde{e}^*, \tilde{u}^*)$ be the fixed-point dual vector corresponding to some fixed point $\tilde{b}$. The $l-$dimensional column vector $b^*(\tilde{e}^*, \tilde{u}^*)$ defined by

$$b^{*T}(\tilde{e}^*, \tilde{u}^*) := [\tilde{e}^*_1, \ldots, \tilde{e}^*_l, \tilde{u}^*_1, \ldots, \tilde{u}^*_l] \begin{bmatrix} -N_{UT} \\ D_{UT} \end{bmatrix} \quad (12)$$

is used as a terminal boundary condition. The resulting modified version of $MD(b,p)$ is denoted $M^2D(b, b^*(\tilde{e}^*, \tilde{u}^*), p)$:

$$\max_{e^* \in \mathbb{R}^p, u^* \in \mathbb{R}^p} [e^*_1, \ldots, e^*_l, u^*_1, \ldots, u^*_l]\, S^{-1} \begin{bmatrix} b_{l\times 1} \\ 0_{l\times 1} \end{bmatrix}$$
subject to
$$N^T_{(1:p,1:p)} e^* - D^T_{(1:p,1:p)} u^* = \begin{bmatrix} 0_{(p-l)\times 1} \\ b^*(\tilde{e}^*, \tilde{u}^*) \end{bmatrix}$$
$$\|e^*\|_\infty \leq K_1, \|u^*\|_\infty \leq K_2.$$

Since optimizing vectors for $MD(b,p)$ are feasible for $M^2D(b,p)$, the optimal value for the program $M^2D(b,p)$ cannot be smaller than the optimal value for the program $MD(b,p)$.

B. The dual of $M^2D(b, b^*(\tilde{e}^*, \tilde{u}^*), p)$

The following result is proved in [1].

Theorem 7: The dual of $M^2D(b, b^*(\tilde{e}^*, \tilde{u}^*), p)$ is denoted $DM^2D(b, b^*(\tilde{e}^*, \tilde{u}^*), p)$, and can be expressed in the form

$$\min_{e \in \mathbb{R}^p, u \in \mathbb{R}^p} J_{DM^2D}(e,u)$$
subject to
$$D_{(1:p,1:p)} e + N_{(1:p,1:p)} u = \begin{bmatrix} b_{l\times 1} \\ 0_{(p-l)\times 1} \end{bmatrix}, \quad (13)$$

where $J_{DM^2D}(e,u) := \sum_{k=1}^{p} K_1 |e_k| + K_2 |u_k| + L_{(1:l)} F_{(1:l,1:2p)} \begin{bmatrix} e \\ u \end{bmatrix}$, $L := [\tilde{e}^*_1, \ldots, \tilde{e}^*_l, \tilde{u}^*_1, \ldots, \tilde{u}^*_l]\, S^{-1}$ and $F := \begin{bmatrix} D_{(1:p,1:p)} & N_{(1:p,1:p)} \end{bmatrix} - \begin{bmatrix} D_C(p) & N_C(p) \end{bmatrix}$. The optimal value for the program $M^2D(b, b^*(\tilde{e}^*, \tilde{u}^*), p)$

is equal to the optimal value for the program $DM^2D(b, b^*(\tilde{e}^*, \tilde{u}^*), p)$. Furthermore, if (e, u) and (e^*, u^*) are feasible for $DM^2D(b, b^*(\tilde{e}^*, \tilde{u}^*), p)$ and $M^2D(b, b^*(\tilde{e}^*, \tilde{u}^*), p)$, respectively, then a necessary and sufficient condition that they both be optimal is that they be complementary, in the sense of Definition 1.

It can be shown that, for all initial condition vectors b in some neighbourhood of a stable $\tilde{b}$, solving $DM^2D(b, b^*(\tilde{e}^*, \tilde{u}^*), p)$ where $(\tilde{e}^*, \tilde{u}^*)$ is the fixed point dual vector corresponding to $\tilde{b}$, essentially solves $\mathcal{P}(b)$.

The underlying idea is as follows. Suppose $\tilde{b}$ is a fixed point of period p for the program $\mathcal{P}$. Then for any $n \in \mathcal{N}$, $G^n(\tilde{\mathcal{B}})\tilde{b}$ is a scalar multiple of $\tilde{b}$. To see why this is significant, consider the infinite sequence of programs

$$\begin{gathered} DM^2D(G^n(\tilde{\mathcal{B}})\tilde{b}, b^*(\tilde{e}^*, \tilde{u}^*), p) \\ \min_{e \in \mathbb{R}^p, u \in \mathbb{R}^p} J_{DM^2D}(e, u) \\ \text{subject to} \\ D_{(1:p,1:p)}e + N_{(1:p,1:p)}u = \begin{bmatrix} G^n(\tilde{\mathcal{B}})\tilde{b} \\ 0_{(p-l)\times 1} \end{bmatrix}. \end{gathered}$$

The optimal solution to $\mathcal{P}(\tilde{b})$ is the concatenation for $n = 0, 1, \ldots$ of the optimal solutions to the programs $DM^2D(G^n(\tilde{\mathcal{B}})\tilde{b}, b^*(\tilde{e}^*, \tilde{u}^*), p)$. This is summarized in the following Proposition, which is a special case of a theorem proved in [1].

Proposition 8: Suppose $\tilde{b}$ is a fixed point of period p for $\mathcal{P}$; as usual $(\tilde{e}^*, \tilde{u}^*)$ denotes the fixed-point dual vector and $\tilde{\mathcal{B}}$ is the optimal basis for $DMD(\tilde{b}, p)$. Suppose that $\rho(G(\tilde{\mathcal{B}})) < 1$. Then the optimal solution for $\mathcal{D}(\tilde{b})$ is the infinite periodic extension of $(\tilde{e}^*, \tilde{u}^*)$, the fixed-point dual vector corresponding to $\tilde{b}$. The optimal value for the program $\mathcal{P}(\tilde{b})$ is $[\tilde{e}_1^*, \ldots, \tilde{e}_l^*, \tilde{u}_1^*, \ldots, \tilde{u}_l^*]\, S^{-1}b$. Denote by $(e^{bsol(n)}, u^{bsol(n)})$ the basic solution, with respect to the basis $\tilde{\mathcal{B}}$, to the equations $D_{(1:p,1:p)}e + N_{(1:p,1:p)}u = \begin{bmatrix} G^n(\tilde{\mathcal{B}})\tilde{b} \\ 0_{(p-l)\times 1} \end{bmatrix}$. Then an optimizing vector for $\mathcal{P}(\tilde{b})$, denoted (e_{opt}, u_{opt}), is given by

$$e_{opt} := \begin{bmatrix} e^{bsol(0)} \\ e^{bsol(1)} \\ \vdots \end{bmatrix}, \; u_{opt} := \begin{bmatrix} u^{bsol(0)} \\ u^{bsol(1)} \\ \vdots \end{bmatrix}.$$

By extending the argument sketched above, the following theorems can be proved.

Theorem 9: If a fixed point, $\tilde{b}$, of period p exists for the program $\mathcal{P}$, then there is an optimal solution to $\mathcal{P}(\tilde{b})$ which satisfies a p'th order recurrence relation.

Theorem 10: If a stable fixed point, $\tilde{b}$, of period p exists for the program $\mathcal{P}$ then there exists a cone, $\mathcal{C}$, in $\mathbb{R}^p$ containing $\tilde{b}$ with the property that, for all $b \in \mathcal{C}$, the optimal solution to $\mathcal{D}(b)$ is the infinite periodic extension of the optimal solution to $MD(b, p)$. For any $b \in \mathcal{C}$, the optimal solution to $\mathcal{P}(b)$ satisfies an n'th order recurrence relation, where n is an integer multiple of p.

Example 11: Consider the problem $\mathcal{P}$ with constraints $(1 - z/4 - z^2/8)\hat{e} + \left(1 + z/15 - 2z^2/15\right)\hat{u} = 1$, with $K_1 = 1$ and $K_2 = 1$. Putting $p = 4$,

$$D_{(1:4,1:4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ -1/8 & -1/4 & 1 & 0 \\ 0 & -1/8 & -1/4 & 1 \end{bmatrix}$$

$$N_{(1:4,1:4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/15 & 1 & 0 & 0 \\ -2/15 & 1/15 & 1 & 0 \\ 0 & -2/15 & 1/15 & 1 \end{bmatrix}$$

$$D_C(4) = \begin{bmatrix} 1 & 0 & -1/8 & -1/4 \\ -1/4 & 1 & 0 & -1/8 \\ -1/8 & -1/4 & 1 & 0 \\ 0 & -1/8 & -1/4 & 1 \end{bmatrix}$$

$$N_C(4) = \begin{bmatrix} 1 & 0 & -2/15 & 1/15 \\ 1/15 & 1 & 0 & -2/15 \\ -2/15 & 1/15 & 1 & 0 \\ 0 & -2/15 & 1/15 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1/4 & 1 & 1/15 & 1 \\ -1/8 & -1/4 & -2/15 & 1/15 \\ 0 & -1/8 & 0 & -2/15 \end{bmatrix}.$$

The optimal basis, $\mathcal{B}$, for the program $DMD(b, p)$ with $p = 4$ and $b = [1, 0]^T$, is $\mathcal{B} = \{1, 3, 5, 7\}$. So

$$\begin{bmatrix} D_{(1:4,1:4)} & N_{(1:4,1:4)} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1/4 & 0 & 1/15 & 0 \\ -1/8 & 1 & -2/15 & 1 \\ 0 & -1/4 & 0 & 1/15 \end{bmatrix},$$

$$\begin{bmatrix} D_C & N_C \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 & -1/8 & 1 & -2/15 \\ -1/4 & 0 & 1/15 & 0 \\ -1/8 & 1 & -2/15 & 1 \\ 0 & -1/4 & 0 & 1/15 \end{bmatrix}.$$

Then

$$G(\mathcal{B}) = \begin{bmatrix} 25/1444 & 5/1444 \\ 0 & 0 \end{bmatrix}$$

and $G(\mathcal{B})$ has only one non-zero eigenvalue, $\lambda = 25/1444$, whose corresponding eigenvector is $[1, 0]^T$, which is equal to b. This shows that $[1, 0]^T$ is a fixed point for $\mathcal{P}$. So we put $\tilde{b} = [1, 0]^T$. Also $\mathcal{B} = \tilde{\mathcal{B}}$, the fixed point basis corresponding to $\tilde{b}$. Then $\tilde{e}^* = [1, 17/165, 1, 17/165]^T$, $\tilde{u}^* = [1, -23/88, 1, -23/88]^T$ and $b^*(\tilde{e}^*, \tilde{u}^*) = [-1/120, 481/1320]^T$. The optimal basic solution to the program $DM^2D(\tilde{b}, b^*(\tilde{e}^*, \tilde{u}^*), 4)$ is found from the basic variables,

$$\begin{bmatrix} D_{(1:4,1:4)} & N_{(1:4,1:4)} \end{bmatrix}_{\tilde{\mathcal{B}}}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/19 \\ 10/361 \\ 15/19 \\ 75/722 \end{bmatrix}$$

so $e = (4/19, 0, 10/361, 0)$ and $u = (15/19, 0, 75/722, 0)$ are the unique optimizing vectors. The optimizing vectors for $DM^2D(G^n(\tilde{\mathcal{B}})\tilde{b}, b^*(\tilde{e}^*, \tilde{u}^*), 4)$ are then $e = \lambda^n\,(4/19, 0, 10/361, 0)$ and $u = \lambda^n\,(15/19, 0, 75/722, 0)$.

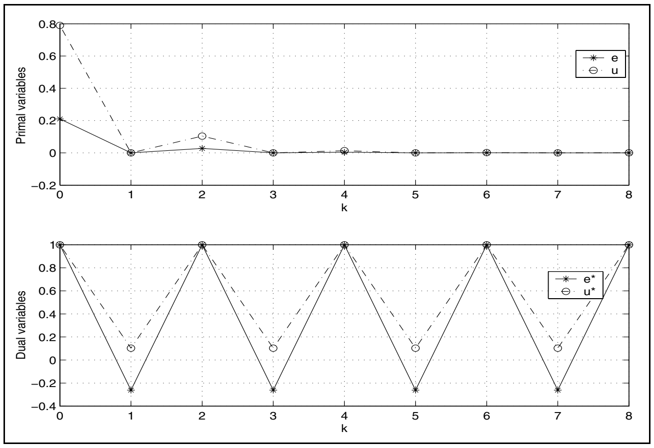

Fig. 2. Optimal solutions for Example 11 with $\hat{b} = 1$

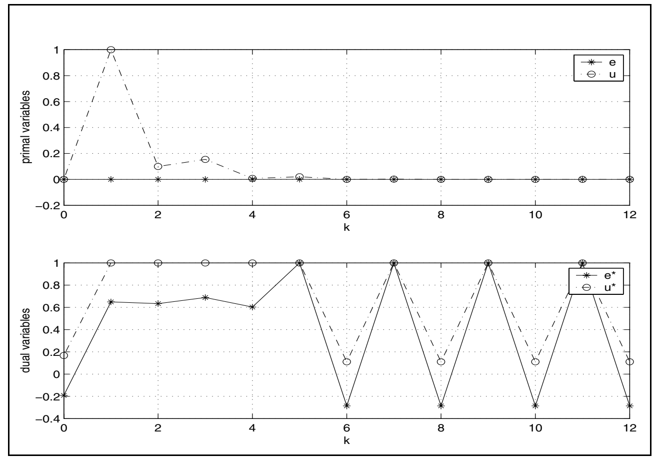

Fig. 3. Example 12 with $\hat{b} = z$, showing an ultimately periodic optimal solution in the dual.

The optimizing vectors for $\mathcal{P}(\tilde{b})$ are therefore $e = \left(4/19, 0, (4/19)\,(5/38)\,, 0, (4/19)\,(5/38)^2\,, 0, \dots\right)$ and $u = \left(15/19, 0, (15/19)\,(5/38)\,, 0, (15/19)\,(5/38)^2\,, 0, \dots\right)$. The optimal value for $\mathcal{P}(\tilde{b})$ is $1/(1-5/38) = 38/33$. The optimal value for $\mathcal{P}(\tilde{b})$ must be the same as that for $DMD(\tilde{b}, p)$:

$$\begin{bmatrix} D_C & N_C \end{bmatrix}_{\mathcal{B}}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 304/1419 \\ 40/1419 \\ 380/473 \\ 50/473 \end{bmatrix}$$

with l_1-norm indeed equal to $38/33$. The optimizing vectors for $\mathcal{P}(\tilde{b})$ are shown in Fig. 2. The primal and dual optimizing vectors are both of infinite length. The dual optimal vectors are periodic of (fundamental) period 2. The restriction $p \geq 2l$ required the concatenated sub-blocks to be of length 4.

It can be shown that, for all b values contained in some cone surrounding $\tilde{b} = [1, 0]^T$, the optimal solution for $\mathcal{D}(b)$ is ultimately periodic of period 2. This is a consequence of $\tilde{b}$ being a *stable* fixed point. The fixed-point vector $\tilde{b}$ is the key to constructing solutions for an infinite number of problems having right hand side vectors b in a cone containing $\tilde{b}$.

Example 12: We conclude with an example showing an ultimately periodic optimal solution for $\mathcal{D}$. See Fig. 3. For this example $\hat{n} = 1 - z/10 - 13z^2/90 + z^3/45$, $\hat{d} = 1 - 9z/20 - 3z^2/40 + z^3/40$, $\hat{b} = z$, $K_1 = K_2 = 1$. Although $b = [0, 1, 0]^T$ is not a fixed point for $\mathcal{P}$, it is in a cone surrounding $\tilde{b} = [(197 + 7\sqrt{241})/54, -1, 0]^T$, a stable fixed point of fundamental period 2 for $\mathcal{P}$. The corresponding fixed-point basis is $\mathcal{B} = \{1, 3, 5, 7\}$. For this problem there are also unstable fixed points.

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