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Robust Model Predictive Control for Time-Varying Systems

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Abstract— **This paper extends robust model predictive control to problems involving linear time-varying systems with constraints and subject to persistent, unknown but bounded disturbances. The new controller guarantees constraint satisfaction and feasibility of the optimizations. Robust feasibility is achieved by tightening the constraints in the online optimization, with the advantage that the complexity of the optimization is not changed. This method is applicable to the control of constrained nonlinear systems by linearizing about a reference trajectory. The uncertainty introduced by linearization error can be accommodated in the disturbance model, and constraint satisfaction for the nonlinear system is then guaranteed.**

I. Introduction

This paper extends robust Model Predictive Control (MPC) [1] to accommodate constrained systems with linear time-varying (LTV) dynamics. The new MPC guarantees constraint satisfaction and feasibility despite the action of unknown but bounded disturbances. This extends previous work in which robust control of linear time-invariant systems was achieved by tightening the constraints in the planning optimization in a monotonic sequence [2]–[5]. The contribution of this paper is the extension of the constraint-tightening method to LTV systems.

Linear time-varying systems are of interest because they can capture the dynamics of nonlinear systems relative to reference trajectories. For example, the motion of a spacecraft relative to a highly-elliptical reference orbit can be expressed as an LTV system [6]. Previous work on MPC for LTV systems includes nominal stability analysis by Nevistic [7] and robustness results for randomly-varying systems [8]–[13]. However, the latter results do not make use of explicit knowledge of how the system varies as a function of time, which is available in the case of linearization about a known reference. The contribution of this paper is a controller for systems whose dynamics are known, but subjected to an affine, unknown but bounded, persistent disturbance. This class of uncertainty can include the linearization error, provided it can be analytically bounded [14], and the effects of uncertain state knowledge [15].

The approach of this paper is to use *constraint tightening* [2]–[5], in which robustness is achieved by tightening the constraints in a monotonic sequence. The key idea is to retain a “margin” for future feedback action, which becomes available to the MPC optimization as time progresses. Since robustness follows only from the constraint modifications, only nominal predictions are required, avoiding both the large growth in problem size associated with incorporating multivariable uncertainty in the prediction model and the conservatism associated with worst case cost predictions, a common alternative, *e.g.* [8], [16].

This paper considers only robust feasibility, that is the guarantee that every optimization can be solved. Feasibility implies constraint satisfaction, so this method is directly applicable to problems where the primary objective is to satisfy the constraints for all time. This paper does not directly consider the problem of converging to a particular target set, typically achieved by proving monotonicity of the optimal cost. However, this can be achieved by suitable choice of cost function within the MPC optimization [5]. Also, an example is shown in Section V in which time-varying constraints are employed to enforce convergence to a target.

Section II formally defines the problem statement. Section III presents the MPC algorithm and proves its robustness. Section IV shows how to apply the new MPC to the control of constrained nonlinear systems. Section V presents the results of simulations demonstrating the new controller at work.

II. Problem Statement

The aim is to control a linear time-varying system with discretized dynamics

$$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) + \mathbf{w}(k) \tag{1}$$

where $\mathbf{x}(k) \in \Re^{N_x}$ is the state vector, $\mathbf{u}(k) \in \Re^{N_u}$ is the input, $\mathbf{w}(k) \in \Re^{N_x}$ is the disturbance vector. Assume the systems $(\mathbf{A}(k), \mathbf{B}(k))$ are stabilizable for all k and the complete state $\mathbf{x}$ is accessible. (See Ref. [15] for modifications to this method to account for imperfect state information.) The disturbance lies in a bounded set but is otherwise unknown

$$\mathbf{w}(k) \in \mathcal{W}(k) \subset \Re^{N_x}, \ \forall k \tag{2}$$

The control is required to keep an output $\mathbf{y}(k) \in \Re^{N_y}$ within a bounded set for all disturbances. The form of the output constraints

$$\begin{aligned} \mathbf{y}(k) &= \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k) && (3)\\ \mathbf{y}(k) &\in \mathcal{Y}(k) \subset \Re^{N_y}, \ \forall k && (4) \end{aligned}$$

can capture both input and state constraints, or mixtures thereof, such as limited control magnitude or state error box limits [17]. The matrices $\mathbf{C}(k)$ and $\mathbf{D}(k)$ and the sets $\mathcal{Y}(k)$ are all chosen by the designer. The objective is to minimize the cost function

$$J = \sum_{k=0}^{\infty} \ell\left(\mathbf{u}(k), \mathbf{x}(k), k\right) \tag{5}$$

where $\ell(\cdot)$ is a stage cost function. Typically, this would be a quadratic function, resulting in a quadratic program solution, or a convex piecewise linear function (*e.g.* $|\mathbf{u}|+|\mathbf{x}|$) that can be implemented with slack variables in a linear program [18]. The robust feasibility result requires no assumptions concerning the nature of this cost.

III. MPC Formulation

The online optimization approximates the complete problem in Section II by solving it over a finite horizon of N steps. The key features of the optimization are:

- predictions are made using the *nominal* system model, *i.e.* (1) without the disturbance term;
- the embedded system dynamics model is *time-varying*;
- the applied constraints vary both with the time step to which they apply and with the step at which planning takes place.

Define the MPC optimization problem as $\mathsf{P}(k)$

$$J^*(k) = \min_{\mathbf{u},\mathbf{x},\mathbf{y}} \sum_{j=0}^{N} \ell\left(\mathbf{u}(k+j|k), \mathbf{x}(k+j|k), k+j\right)$$

subject to $\forall j \in \{0 \dots N\}$

$$\begin{aligned}
\mathbf{x}(k+j+1|k) &= \mathbf{A}(k+j)\mathbf{x}(k+j|k) \\
&\quad + \mathbf{B}(k+j)\mathbf{u}(k+j|k) && \text{(6a)}\\
\mathbf{y}(k+j|k) &= \mathbf{C}(k+j)\mathbf{x}(k+j|k) \\
&\quad + \mathbf{D}(k+j)\mathbf{u}(k+j|k) && \text{(6b)}\\
\mathbf{x}(k|k) &= \mathbf{x}(k) && \text{(6c)}\\
\mathbf{x}(k+N+1|k) &= \mathbf{0} && \text{(6d)}\\
\mathbf{y}(k+j|k) &\in \mathcal{Y}(k+j|k) && \text{(6e)}
\end{aligned}$$

where the double index $(k+j|k)$ denotes a prediction j steps ahead from time k. The constraint sets for the plan $\mathcal{Y}(k+j|k)$ are tightened for robustness using the following recursions

$$\begin{aligned}
\mathcal{Y}(k|k) &= \mathcal{Y}(k) && \text{(7a)}\\
\mathcal{Y}(k+j|k) &= \mathcal{Y}(k+j|k+1) && \text{(7b)}\\
\sim \left[\mathbf{C}(k+j) + \mathbf{D}(k+j)\mathbf{K}(k+j|k+1)\right]&\mathbf{L}(k+j|k+1)\mathcal{W}(k)
\end{aligned}$$

where $\mathbf{K}(k+j|k)$ and $\mathbf{L}(k+j|k)$ are the controller and state transition matrices, respectively, for a candidate control policy that renders the system nilpotent, *i.e.* the control $\mathbf{u}(k) = \mathbf{K}(k|n)\mathbf{x}(k),\ k \geq n$ applied from any state $\mathbf{x}(n)$ at time n would drive the nominal system to the origin in a finite number of steps M. This policy is calculated using a finite-horizon LQR method (10)–(14), where the nilpotency horizon $M < N-1$ and the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ are chosen by the designer. In practice, the infinite terminal cost in (10) is replaced by a very large terminal weighting, orders of magnitude higher than the other weightings, and it can be verified numerically that $\mathbf{L}(k+M|k) = \mathbf{0}\ \forall k$ within working tolerance.

The operator "$\sim$" denotes the Pontryagin difference [17] defined as

$$\mathcal{A} \sim \mathcal{B} = \{\mathbf{a} \mid \mathbf{a}+\mathbf{b} \in \mathcal{A}\ \forall \mathbf{b} \in \mathcal{B}\} \tag{8}$$

with the resulting useful property

$$\mathbf{c} \in \mathcal{A} \sim \mathcal{B} \Rightarrow \mathbf{c}+\mathbf{b} \in \mathcal{A}\ \forall \mathbf{b} \in \mathcal{B} \tag{9}$$

Under the reasonable assumption that $\mathbf{0} \in \mathcal{W}(k)$, the recursion (7b) implies $\mathcal{Y}(k+j|k) \subseteq \mathcal{Y}(k+j|k+1)$, where $\mathcal{Y}(k+j|k)$ represents the constraints applied upon step $k+j$ in the plan at time k. This is the analog of the monotonic sequence seen in other constraint tightening methods [3], [4]. Intuitively, it means that there is some "margin" in the constraints that is retained for future feedback action and "returned" to the optimization as time progresses.

The terminal constraint (6d) here is for the trajectory to end at the origin. A point constraint, as opposed to a robust invariant set, is compatible with robustness since the candidate control policy is forced to be nilpotent [4], [5] by the penalty (10). In the case of a nonlinear system linearized about a reference, the origin terminal constraint implies that the plan for the nonlinear system terminates on the reference trajectory. If the reference trajectory is optimized, this is a good approximation to the infinite horizon problem.

$$\begin{aligned}
\mathbf{P}(k+M|k) &= \infty\mathbf{I} && \text{(10)}\\
\mathbf{P}(k+j|k) &= \mathbf{Q} + \mathbf{A}(k+j)^T\mathbf{P}(k+j+1|k)\mathbf{A}(k+j) && \text{(11)}\\
&\quad - \mathbf{A}(k+j)^T\mathbf{P}(k+j+1|k)\mathbf{B}(k+j)\left[\mathbf{R}+\mathbf{B}(k+j)^T\mathbf{P}(k+j+1|k)\mathbf{B}(k+j)\right]^{-1}\mathbf{B}(k+j)^T\mathbf{P}(k+j+1|k)\mathbf{A}(k+j)\\
\mathbf{K}(k+j|k) &= \begin{cases} -\left[\mathbf{R}+\mathbf{B}(k+j)^T\mathbf{P}(k+j+1|k)\mathbf{B}(k+j)\right]^{-1}\mathbf{B}(k+j)^T\mathbf{P}(k+j+1|k)\mathbf{A}(k+j) & 0 \leq j < M \\ \mathbf{0} & M \leq j < N \end{cases} && \text{(12)}\\
\mathbf{L}(k|k) &= \mathbf{I} && \text{(13)}\\
\mathbf{L}(k+j|k) &= \left(\mathbf{A}(k+j-1)+\mathbf{B}(k+j-1)\mathbf{K}(k+j-1|k)\right)\mathbf{L}(k+j-1|k) && \text{(14)}
\end{aligned}$$

The optimization $\mathsf{P}(k)$ is employed in the following algorithm.

Algorithm 1 (Robustly Feasible MPC)

1) Solve problem $\mathsf{P}(k)$
2) Apply control $\mathbf{u}(k) = \mathbf{u}^*(k|k)$ from the optimal sequence
3) Increment k. Go to Step 1

Theorem 1 (Robust Feasibility): If $\mathsf{P}(0)$ has a feasible solution then the system (1) controlled by Algorithm 1 and subjected to disturbances obeying (2) robustly satisfies the constraints (4) and all subsequent optimizations $\mathsf{P}(k)$ are feasible for all $k > 0$,

Proof: This is an extension of the results in Refs. [4] and [5]. It is based on recursion, showing that feasibility of $\mathsf{P}(k_0)$ implies feasibility of $\mathsf{P}(k_0+1)$ for any k_0 and $\mathbf{w}(k_0) \in \mathcal{W}(k_0)$. Feasibility of $\mathsf{P}(k_0+1)$ is proven by showing feasibility of a particular candidate solution.

Assume that at some time step k_0, the problem $\mathsf{P}(k_0)$ is feasible and has a solution with states $\mathbf{x}^*(k_0+j|k_0)$, controls $\mathbf{u}^*(k_0+j|k_0)$ and outputs $\mathbf{y}^*(k_0+j|k_0)$ satisfying (6). Consider the following candidate solution, denoted by $\hat{\cdot}$, for problem $\mathsf{P}(k_0+1)$

$$\begin{aligned}\hat{\mathbf{u}}(k_0+j|k_0+1) &= \mathbf{u}^*(k_0+j|k_0) \\ &\quad + \mathbf{K}(k_0+j|k_0+1)\mathbf{L}(k_0+j|k_0+1)\mathbf{w}(k_0) \\ &\quad \forall j \in \{1 \dots N\}\end{aligned} \tag{15a}$$

$$\hat{\mathbf{u}}(k_0+N+1|k_0+1) = \mathbf{0} \tag{15b}$$

$$\begin{aligned}\hat{\mathbf{x}}(k_0+j|k_0+1) &= \mathbf{x}^*(k_0+j|k_0) \\ &\quad + \mathbf{L}(k_0+j|k_0+1)\mathbf{w}(k_0) \\ &\quad \forall j \in \{1 \dots N+1\}\end{aligned} \tag{15c}$$

$$\hat{\mathbf{x}}(k_0+N+2|k_0+1) = \mathbf{A}\hat{\mathbf{x}}(k_0+N+1|k_0+1) \tag{15d}$$

$$\begin{aligned}\hat{\mathbf{y}}(k_0+j|k_0+1) &= \mathbf{C}\hat{\mathbf{x}}(k_0+j|k_0+1) \\ &\quad + \mathbf{D}\hat{\mathbf{u}}(k_0+j|k_0+1) \\ &\quad \forall j \in \{1 \dots N+1\}\end{aligned} \tag{15e}$$

This solution is formed by shifting the previous solution by one step, *i.e.* removing the first step, adding one step of zero control at the end, and adding perturbations representing the rejection of the disturbance by the candidate controller $\mathbf{K}$ and the associated state transition matrices $\mathbf{L}$.

To prove feasibility, it is necessary to show that the candidate solution satisfies all of the constraints (6) for time k_0+1.

Dynamics constraints (6a) Feasibility at time k implies that the original solution $\mathbf{x}^*(k_0+j|k_0), \mathbf{u}^*(k_0+j|k_0)$ satisfies the dynamics constraints. Since (14) defines $\mathbf{L}$ to be the state transition matrix, the perturbation sequence $\mathbf{L}(k_0+j|k_0+1)\mathbf{w}(k_0), \mathbf{K}(k_0+j|k_0+1)\mathbf{L}(k_0+j|k_0+1)\mathbf{w}(k_0)$ must also satisfy the dynamics model. Therefore by superposition, the summed sequences (15a) and (15c) satisfy the dynamics constraints, and the final step (15b) and (15d) satisfy (6a) by construction.

Output equality constraints (6b) are satisfied by construction from (15e).

Initial condition constraints (6c) The true state at k_0+1 is found using the true system (1) giving

$$\mathbf{x}(k_0+1) = \mathbf{A}(k_0)\mathbf{x}(k_0) + \mathbf{B}(k_0)\mathbf{u}(k_0) + \mathbf{w}(k_0) \tag{16}$$

Feasibility at time k_0 implies $\mathbf{x}(k_0+1|k_0) = \mathbf{A}(k_0)\mathbf{x}(k_0) + \mathbf{B}(k_0)\mathbf{u}(k_0)$ and substituting into (15c) along with (13) gives

$$\hat{\mathbf{x}}(k_0+1|k_0+1) = \mathbf{A}\mathbf{x}(k_0) + \mathbf{B}\mathbf{u}(k_0) + \mathbf{w}(k_0) \tag{17}$$

Comparing (16) and (17) shows that $\mathbf{x}(k_0+1) = \hat{\mathbf{x}}(k_0+1|k_0+1)$, satisfying the constraint (6c).

Terminal constraints (6d) Since the candidate solution is forced to be nilpotent by (10), then $\mathbf{L}(k_0+j|k_0+1) = \mathbf{0}$ for $j > M$. Therefore, (15c) implies $\hat{\mathbf{x}}(k_0+N+1|k_0+1) = \mathbf{x}(k_0+N+1|k_0)$. Feasibility at time k_0 implies that $\mathbf{x}(k_0+N+1|k_0) = \mathbf{0}$ thus implying $\hat{\mathbf{x}}(k_0+N+1|k_0+1) = \mathbf{0}$. Therefore, using the final step of the candidate solution (15d) gives $\hat{\mathbf{x}}(k_0+N+2|k_0+1) = \mathbf{0}$ which satisfies the terminal constraint (6d).

Output constraints (6e) First consider steps $j = 1 \dots N$ of the output sequence (15e). Substituting (15a), (15c) and (6b) with $k = k_0$ into (15e) gives

$$\begin{aligned}&\hat{\mathbf{y}}(k_0+j|k_0+1) = \mathbf{y}(k_0+j|k_0) \\ &+ [\mathbf{C}(k_0+j) + \mathbf{D}(k_0+j)\mathbf{K}(k_0+j|k_0+1)]\,\mathbf{L}(k_0+j|k_0+1)\mathbf{w}(k_0)\end{aligned}$$

Substituing this relationship and the constraint tightening recursion (7b) into the property (9) of the Pontryagin difference shows that if $\mathbf{y}(k_0+j|k_0) \in \mathcal{Y}(k_0+j|k_0)$, which follows from feasibility at time k_0, then $\hat{\mathbf{y}}(k_0+j|k_0+1) \in \mathcal{Y}(k_0+j|k_0+1)$, satisfying the constraints (6e) at time k_0+1. The final step $j = N+1$ is trivial as $\hat{\mathbf{y}}(k_0+N+1|k_0+1) = \mathbf{0}$.

In summary, under the assumption that a feasible solution is found at time k_0 for problem $\mathsf{P}(k_0)$, then a solution can be constructed for time k_0+1 that is feasible for problem $\mathsf{P}(k_0+1)$ for any k_0 and disturbance $\mathbf{w}(k_0)$ obeying (2). Hence feasibility at k_0 implies feasibility at k_0+1, and the theorem is proven by recursion. ■

Remark 1 (Feasibility and Optimality): Theorem 1 does not assume that an optimal solution is found at each step. The result holds as long as a feasible solution is found. Furthermore, since the method relies on a known candidate solution being feasible, that solution can be constructed and used to initialize a search procedure, and if no better solution is found in the time available, the candidate solution can be employed for control. This means that the optimization can be run with an arbitrary computation time limit.

Remark 2 (Computation): The calculation of the candidate policies (10)–(14) and the constraint sets $\mathcal{Y}(k+j|k)$ using (7) can be performed offline. Tools for computing the Pontryagin difference for polytopes are available [19]. The only online computation is the solution of $\mathsf{P}(k)$, an optimization problem of the same size as its nominal counterpart but with modified constraints (6e).

Remark 3 (Set Approximation): Since the result of Theorem 1 depended on the property (9) of the Pontryagin difference, and not its definition, any sequence of sets for which the property (9) holds can be used for the constraints. (By definition, the Pontryagin difference (8) is the largest set for which (9) holds.) For example, a constant sequence of constraint sets can be employed $\mathcal{Y}(k+j|k) = \tilde{\mathcal{Y}}(j)\ \forall k$ given by

$$\begin{aligned}\tilde{\mathcal{Y}}(0) &= \bigcap_k \mathcal{Y}(k)\\ \tilde{\mathcal{Y}}(j) &= \bigcap_k \Big\{ \tilde{\mathcal{Y}}(j-1)\\ &\sim \left[\mathbf{C}(k+j)+\mathbf{D}(k+j)\mathbf{K}(k+j|k+1)\right]\mathbf{L}(k+j|k+1)\mathcal{W}(k)\Big\}\end{aligned}$$

If the system is periodic, the intersection only need consider one period of repetition. The resulting MPC optimization has time-invariant constraints, reducing the computation overhead, but retains the time-varying prediction model, hence it still uses knowledge of system variation. This would be a good approximation for a time-varying system with constant constraints.

IV. Control of Constrained Nonlinear Systems

This section describes how to apply the method in Section III for the control of a nonlinear system.

A. Problem Statement for Nonlinear Case

The aim is to control a nonlinear system with discrete-time dynamics

$$\mathbf{z}(k+1) = \mathbf{f}(\mathbf{z}(k), \mathbf{v}(k), k) \tag{18}$$

where $\mathbf{z}(k)$ is the state of the system and $\mathbf{v}(k)$ is the control input. The system is subject to state and input constraints

$$\mathbf{z}(k) \in \mathcal{Z}(k) \tag{19a}$$
$$\mathbf{v}(k) \in \mathcal{V}(k) \tag{19b}$$

Assume that the system (18) is twice differentiable and that the sets in (19) are compact. Also assume that a reference trajectory consisting of states $\tilde{\mathbf{z}}(k)$ and inputs $\tilde{\mathbf{v}}(k)$ is known, satisfying (18) and (19).

B. MPC for Nonlinear Case

This section presents the conversion of the problem statement in Section IV-A to the LTV form in Section II. The method is based on a familiar linearization process, and also includes transformations of the constraints. This section then develops a guarantee of constraint satisfaction for the nonlinear system despite the linearization error introduced.

The system (18) is linearized by taking perturbations around the reference trajectory, giving the state and control of the representative linear system

$$\mathbf{x}(k) = \mathbf{z}(k) - \tilde{\mathbf{z}}(k) \tag{20a}$$
$$\mathbf{u}(k) = \mathbf{v}(k) - \tilde{\mathbf{v}}(k) \tag{20b}$$

Then the evolution of state $\mathbf{x}$ under control $\mathbf{u}$ is governed by an LTV system of the form of (1) with matrices

$$\mathbf{A}(k) = \left.\frac{\partial \mathbf{f}(\mathbf{z},\mathbf{v},k)}{\partial \mathbf{z}}\right|_{\mathbf{z}=\tilde{\mathbf{z}}(k),\mathbf{v}=\tilde{\mathbf{v}}(k)} \tag{21a}$$
$$\mathbf{B}(k) = \left.\frac{\partial \mathbf{f}(\mathbf{z},\mathbf{v},k)}{\partial \mathbf{v}}\right|_{\mathbf{z}=\tilde{\mathbf{z}}(k),\mathbf{v}=\tilde{\mathbf{v}}(k)} \tag{21b}$$

The equivalent constraints to (19) acting on the perturbations (20) can be expressed as a Pontryagin difference and represent a simple shift of the sets $\mathcal{Z}(k)$ and $\mathcal{V}(k)$, easily performed if these sets are polytopes

$$\mathcal{X}(k) = \mathcal{Z}(k) \sim \{\tilde{\mathbf{z}}(k)\} \tag{22a}$$
$$\mathcal{U}(k) = \mathcal{V}(k) \sim \{\tilde{\mathbf{v}}(k)\} \tag{22b}$$

which can then be written in the form of (4) using

$$\mathbf{C}(k) = \begin{bmatrix} \mathbf{I}_{N_x} \\ \mathbf{0}_{N_u \times N_x} \end{bmatrix} \tag{23a}$$
$$\mathbf{D}(k) = \begin{bmatrix} \mathbf{0}_{N_x \times N_u} \\ \mathbf{I}_{N_u} \end{bmatrix} \tag{23b}$$
$$\mathcal{Y}(k) = \mathcal{X}(k) \times \mathcal{U}(k) \tag{23c}$$

The effect of the linearization error on the LTV system is captured in the affine disturbance $\mathbf{w}(k)$ in (1). Under the assumption that the system (18) is twice differentiable and that the sets in (19) are compact, then the linearization error theorem [14] says that for every k, a set $\mathcal{W}(k)$ can be found that bounds the linearization error

$$\begin{aligned}\mathbf{w}(k) &= \mathbf{f}(\mathbf{z}(k),\mathbf{v}(k),k) - \mathbf{f}(\tilde{\mathbf{z}}(k),\tilde{\mathbf{v}}(k),k) \qquad (24)\\ &\quad - \mathbf{A}(k)(\mathbf{z}(k)-\tilde{\mathbf{z}}(k)) - \mathbf{B}(k)(\mathbf{v}(k)-\tilde{\mathbf{v}}(k))\\ &\in \mathcal{W}(k),\ \forall \mathbf{z}(k) \in \mathcal{Z}(k), \mathbf{v}(k) \in \mathcal{V}(k) \qquad (25)\end{aligned}$$

These sets $\mathcal{W}(k)$ should be used for the constraint tightening in (7). This completes the derivation of the representative LTV system parameters $\mathbf{A}(k), \mathbf{B}(k), \mathbf{C}(k), \mathbf{D}(k), \mathcal{Y}(k)$ and $\mathcal{W}(k)$. The method in Section III can be applied to construct the MPC optimization, which is employed in the following algorithm.

Algorithm 2 (MPC for Nonlinear System)

1) Find deviation from reference $\mathbf{x}(k) = \mathbf{z}(k) - \tilde{\mathbf{z}}(k)$
2) Solve linearized problem $\mathsf{P}(k)$
3) Apply control $\mathbf{v}(k) = \mathbf{u}^*(k|k) + \tilde{\mathbf{v}}(k)$
4) Increment k. Go to Step 1

Theorem 2 (Robust Feasibility for Nonlinear System): If $\mathsf{P}(0)$ has a feasible solution then the system (18) controlled by Algorithm 2 satisfies the constraints (19) and all subsequent optimizations $\mathsf{P}(k)$ are feasible for all $k > 0$,

Proof: Begin by assuming that the constraints (19) are always satisfied. This assumption must be verified as part of the proof. Now find the dynamics of the state perturbation $\mathbf{x}$

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{z}(k+1) - \tilde{\mathbf{z}}(k+1)\\ &= \mathbf{f}(\mathbf{z}(k),\mathbf{v}(k),k) - \mathbf{f}(\tilde{\mathbf{z}}(k),\tilde{\mathbf{v}}(k),k)\end{aligned}$$

Then substituting from (24) gives an LTV system of the same form as (1), including the affine disturbance $\mathbf{w}(k)$. Under the assumption made at the start of the proof, the bound on $\mathbf{w}(k)$ from (25) holds. Therefore, the result of Theorem 1 can be applied, guaranteeing that all optimizations are feasible and the constraints (6) on the LTV system are satisfied. Using the constraint construction in (23), this implies $\mathbf{x}(k) \in \mathcal{X}(k)$ and $\mathbf{u}(k) \in \mathcal{U}(k)$. Then using the construction of these sets (22) and the property (9) of the Pontryagin difference, this implies satisfaction of the constraints on the nonlinear system (19). Finally, this verifies the assumption made at the start of the proof, completing the result. ■

V. Examples

A. Linear System with Time-Varying Constraints

This section demonstrates the method from Section III applied to the control of a system with constant dynamics but time-varying constraints. The system is a simple point mass model

$$\mathbf{A}(k) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ \mathbf{B}(k) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \ \forall k$$

subjected to a norm-bounded disturbance $\mathcal{W} = \{\mathbf{w} \in \Re^2 : \|\mathbf{w}\|_\infty \le 0.2\}$. The control is constrained to have unit magnitude $|u| \le 1$ and the position has a sinusoidal limit $|x_1| \le 0.8 + 0.2\sin(0.2k)$, hence

$$\mathbf{C}(k) = \begin{bmatrix} \frac{1}{(0.8+0.2\sin 0.2k)} & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{D}(k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ \forall k$$

and the output set $\mathcal{Y}(k) = \{\mathbf{y} \in \Re^2 : \|\mathbf{y}\|_\infty \le 1\} \ \forall k$. The cost function was weighted towards control $\ell(\mathbf{u}, \mathbf{x}, k) = 1000u^2 + \mathbf{x}^T\mathbf{x}$. The problem used a horizon of $N = 15$ steps and a candidate controller with $M = 2$ steps.

Figure 1(a) shows the results of 100 simulations of the system, controlled using Algorithm 1 from Section III, each run with different randomly-generated disturbances. The plot shows the position x_1, constrained to remain inside the shaded area. It is clear that the constraints were satisfied throughout. Also notice that the position frequently gets close to the constraint boundaries. Since the cost function penalized control effort more heavily than state deviation, this shows that the controller was using all the available position "space" in order to conserve control.

B. Response Shape Constraints

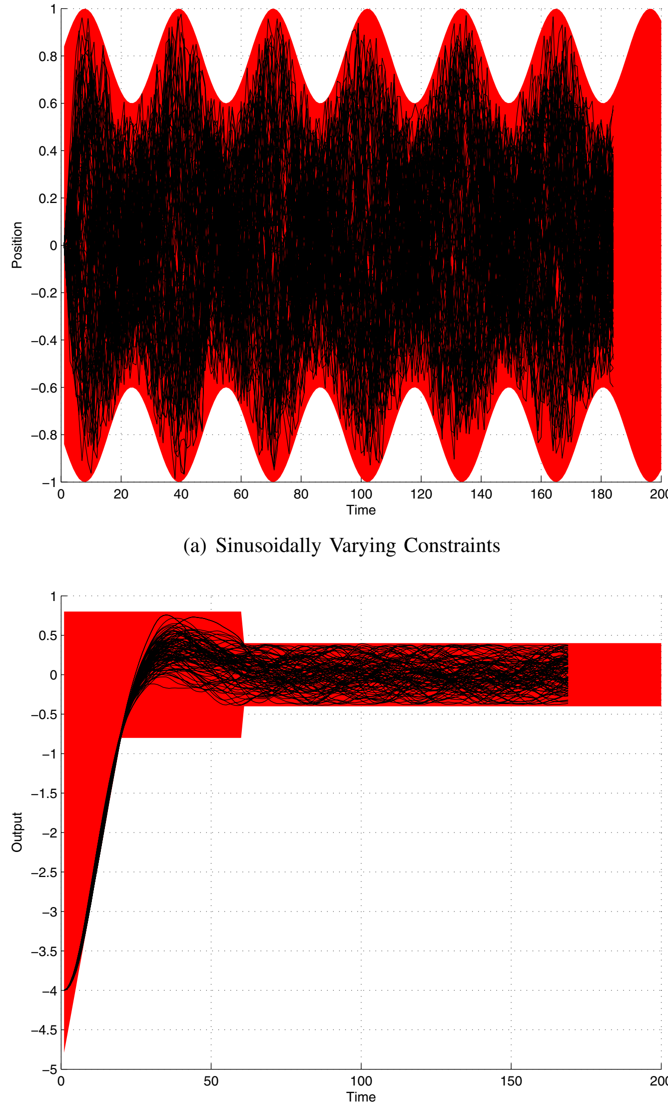
Figure 1(b) shows 100 simulation results for an example in which the constraints define a convergence envelope. The envelope is defined in terms of common step response characteristics: rise time, settling time, and peak overshoot.

The system used in this example is the same as in the previous section, except for the disturbance level and constraints. The new disturbance set is $\mathcal{W} = \{\mathbf{w} \in \Re^2 : \|\mathbf{w}\|_\infty \le 0.02\}$. The modified constraints enforce a reduced control magnitude and the output limits shown by the shaded region in Figure 1(b). The output matrices are

$$\mathbf{C}(k) = \begin{bmatrix} \frac{1}{y_{\max}(k)} & 0 \\ \frac{-1}{y_{\min}(k)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{D}(k) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \ \forall k$$

and the constraint set is $\mathcal{Y}(k) = \{\mathbf{y} \le \mathbf{1}\} \ \forall k$ where the inequality holds for all elements. The convergence limits $y_{\max}(k)$ and $y_{\min}(k)$ are defined as

$$y_{\max}(k) = \begin{cases} e_{\mathrm{p}}, & 0 \le k \le k_{\mathrm{s}} \\ e_{\mathrm{s}}, & k_{\mathrm{s}} < k \end{cases}$$



(a) Sinusoidally Varying Constraints

(b) Convergence Constraints

Fig. 1. Plots of Position vs Time from 100 Simulations of Linear System with Time-Varying Constraints. The constraints require the position to remain within the shaded region.

$$y_{\min}(k) = \begin{cases} \left(\frac{k_{\mathrm{r}}-k}{k_{\mathrm{r}}}\right) e_{\mathrm{m}} + \left(\frac{k}{k_{\mathrm{r}}}\right) e_{\mathrm{p}}, & 0 \le k \le k_{\mathrm{r}} \\ e_{\mathrm{p}}, & k_{\mathrm{r}} < k \le k_{\mathrm{s}} \\ e_{\mathrm{s}}, & k_{\mathrm{s}} < k \end{cases}$$

where k_{r} is the rise time, k_{s} is the settling time, e_{m} is the maximum step size, e_{p} is the peak overshoot limit and e_{s} is the settling limit.

Figure 1(b) shows 100 simulation results with randomly-generated disturbances. Each example had an initial condition $\mathbf{x}(0) = [-4 \ 0]^T$. The step envelope settings were $k_{\mathrm{r}} = 20$, $k_{\mathrm{s}} = 60$, $e_{\mathrm{m}} = 5$, $e_{\mathrm{p}} = 0.8$ and $e_{\mathrm{s}} = 0.4$. The corresponding response envelope is shown by the shaded region in Figure 1(b) and all 100 responses satisfy those constraints.

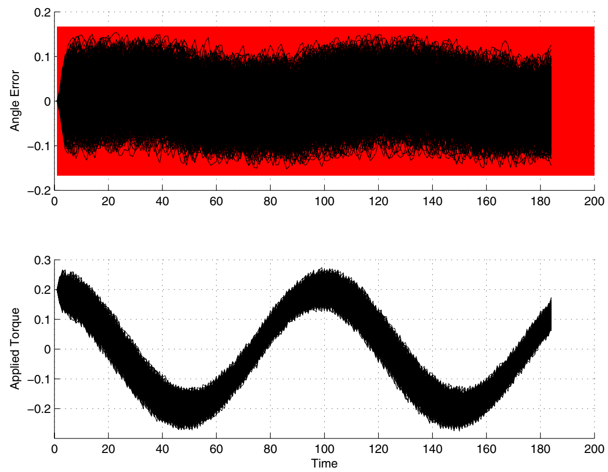


Fig. 2. Plots of angle error (top) and applied torque (bottom) from 500 simulations of nonlinear system. The constraints require the angle error to remain within the shaded region.

C. Nonlinear System

This section demonstrates the application of the method in Section IV to the control of a nonlinear system. A simple pendulum is required to rotate about a horizontal axis at a steady rate $\Omega = \frac{1}{50}\pi$. The pendulum is acted upon by gravity and a driving torque at the pivot. The dynamics are nonlinear, $\ddot{\theta} + \lambda \cos\theta = \tau + \tau_D$, where θ is the angle of displacement from vertically down, $\lambda = 0.2$ is a constant combining the mass, length and gravitational acceleration, τ is the driving torque and $\tau_D \in [-0.05, 0.05]$ is a random disturbance torque. Linearizing this system about the nominal trajectory gives a time-varying linear system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\lambda \sin\Omega t & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

where $\mathbf{x}(t) = (\theta(t) - \Omega t, \dot{\theta}(t) - \Omega)^T$ and $u(t) = \tau(t) - \lambda\cos\Omega t$. This system was discretized with a time step of 1.0 s to generate a system of the form (1). The constraints were $|x_1| \leq \frac{1}{6}$, $|u| \leq 1$. The horizon $N = 15$ and cost function were identical to that in the previous example. The uncertainty sets $\mathcal{W}(k)$ were identified by experimental simulation, similar to the adaptive method in Ref. [15].

Figure 2 shows the angle tracking error x_1 and the applied control torque τ from 500 simulations of the pendulum system controlled by Algorithm 2 from Section IV. The simulations used the nonlinear system model, controlled using the linearized and discretized model within MPC. The angle error always remains inside the shaded region, indicating satisfaction of the constraints, but deviates within that region, showing again that the control allows movement within the constraint limits to reduce control effort.

VI. Conclusion

A formulation for Model Predictive Control has been presented for linear time-varying systems subject to persistent disturbances. Under the assumption that the disturbance is bounded, all optimizations are guaranteed to be feasible and the constraints will be satisfied at all times. The new controller is applicable to robust trajectory tracking for nonlinear systems, by linearizing those systems about a reference trajectory. The uncertainty model within the MPC can incorporate the effect of linearization error. Simulation results have demonstrated the controller applied to examples with time-varying constraints and another example with time-varying dynamics.

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