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On a Parameter Adaptive Extremum Controller

Bo Egardt and Stefan Larsson

***Abstract*— An extremum controller based on parameter adaptive control is investigated in this contribution. The objective function to be minimized is parameterized using a simple nonlinear model, and the control is based on real-time estimation of the parameters of this model. Asymptotic analysis of the extremum controller gives insight concerning the compromise between two contradicting goals, namely accurate control and accurate estimation. This is similar in spirit to stochastic dual control. One motivation of this study is an application to automotive engine control, which is briefly described.**

I. INTRODUCTION

In many applications it is of interest to minimize (or maximize) a measurable quantity by manipulating a control input. This gives rise to a somewhat different control problem than is usually considered, and the term *extremum control* has been used for this class of control problems for many years. The extremum control problem does not lend itself to a straightforward application of ordinary feedback control — trying e.g. to convert it to a standard regulator problem will immediately reveal a fundamental difficulty, namely the change of sign of the process gain around the optimum.

Classical approaches to extremum control go back to the 1950s and 1960s. A couple of useful references from this early development are [1] and [2]. A later overview with an adaptive control perspective is [3]. The recent book [4] contains many references and in addition presents stability results for some of the classical perturbation based techniques.

This paper is motivated by an application to combustion engine control. In an ordinary spark ignited engine, the engine efficiency (or net work) depends on the timing of ignition. This is illustrated in Fig. 1, which is based on experimental data from an engine rig. The figure shows the net work from the engine as a function of TR_{50}, a measure of the combustion phasing, measured in crank angle degrees (TR stands for Torque Ratio; TR_{50} as defined in [5] is calculated from a torque sensor signal). The optimal combustion phasing depends on external variables like engine load and angular speed. The state-of-the-art solution is to store this dependence in the form of maps in the engine control system, based on exhaustive calibration. In an on-going project, the goal is to obtain a more accurate control by measuring the crankshaft torque and using an on-line extremum controller. The structure of the control system is depicted in Fig. 2. The extremum controller adjusts the setpoint for TR_{50}, which in turn is controlled by an inner control loop, manipulating the spark advance. See [5] for further details on this application.

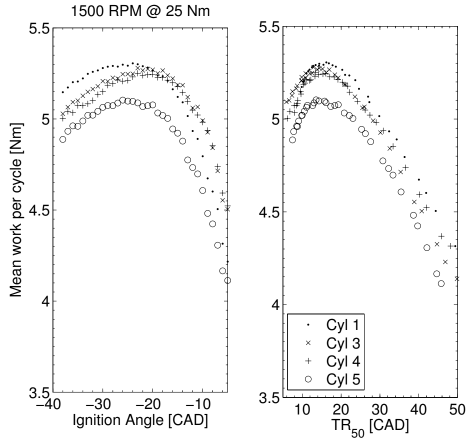

Fig. 1. Net work of an engine as a function of ignition angle (left) and combustion phasing, measured as TR_{50} (right, see text). CAD means Crank Angle Degrees.

The extremum controller applied to the application described above is investigated in this contribution. The approach taken is to apply a very simple, static polynomial model to the extremum control problem,

$$y(t) = ax^2(t) + bx(t) + c, \tag{1}$$

where $y(t)$ is the measured quantity to be minimized, $x(t)$ is the manipulated control variable, and a, b, and c are model parameters. If we have confidence in the model, it would be natural to apply the control input

$$x(t) = -\frac{b}{2a} \tag{2}$$

which is obtained by minimizing $y(t)$ with respect to $x(t)$. Using the idea of *certainty equivalence* (see e.g. [6]), it is

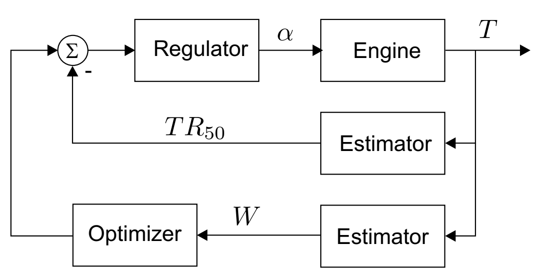

Fig. 2. The control system investigated (W is estimated work, T is measured torque, and α is the spark advance).

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The authors are with Department of Signals and Systems, Chalmers University of Technology, SE-41296 Göteborg, Sweden egardt,last@s2.chalmers.se

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then straightforward to replace the unknown model parameters by their estimates, obtained from e.g. a recursive least squares estimator.

The simple description above hides several difficult and important issues. In practice, the extremum controller needs to deal with modelling errors and disturbances. In addition, the parameters of the model often vary in time (implying e.g. that the optimal control $x(t)$ varies), and the process may have dynamics combined with the nonlinear static characteristics. This gives rise to questions such as stability, parameter identifiability and convergence, noise rejection and tracking properties. Some of these issues will be considered in this paper.

A method, similar to the one described above, has been used in [7] for exactly the same application as considered here. The algorithm investigated in [7] differs from the one used here in two ways. Firstly, a model in incremental form is used, i.e. the data used is $\Delta y(t) = y(t) - y(t-1)$ and $\Delta x(t) = x(t) - x(t-1)$. Secondly, the algorithm used and analyzed estimates the curvature of the model nonlinearity, a procedure we have found less suitable, for reasons we will explain later on.

The next section gives a more detailed description of the extremum controller. An asymptotic analysis is carried out in section III, and conclusions are drawn concerning the convergence properties. The role of probing in the control law is analyzed in section IV. Simulations illustrating the performance of the extremum controller are provided in section V, and conclusions end the paper.

II. AN EXTREMUM CONTROLLER

As outlined in the introduction, our extremum controller is based on the static model

$$y(t) = ax^2(t) + bx(t) + c, \tag{3}$$

where we assume from now on that $a > 0$, i.e. we are dealing with a minimization problem. The reason for choosing this model is primarily its simplicity; in addition, it can be argued that many objective functions can, at least locally, be approximated by a second order polynomial. The parameter adaptive approach we are adopting is based on a combination of parameter estimation and control, using the certainty equivalence principle. We will treat these two parts in the following sections.

A. Real-time Parameter Estimation

It is straightforward to formulate a linear regression problem based on the model (3), and to estimate the three parameters a, b, and c. However, it turns out that this gives a control signal with an extremely erratic behavior. One way to explain this is that with standard assumptions, the parameter estimates are described by Gaussian distributions. The quotient of two such estimates – as suggested by the control law (2) – has a Cauchy distribution, which does not have a finite variance. Hence, such a control signal is not suitable. As has been described in [7] for a similar problem, this can to some degree be circumvented by introducing limitations of the parameter estimates, e.g. avoiding estimates of a becoming too close to zero; however, we will choose another route here.

As a remedy of the problem mentioned, the parameter a in the model is fixed to an a priori estimate of the curvature of the true system's objective function. Thus, the resulting model becomes

$$y(t) = ax^2(t) + bx(t) + c = ax^2(t) + \theta^T\varphi(t), \tag{4}$$

where $\theta^T = [b \quad c]$ contains the parameters to be estimated and $\varphi^T(t) = [x(t) \quad 1]$ contains the regressors.

The recursive least-squares (RLS) algorithm is used to update the parameter estimates $\hat{\theta}$:

$$\begin{aligned} \hat{\theta}(t) &= \hat{\theta}(t-1) + P(t)\varphi(t)\varepsilon(t) \\ \varepsilon(t) &= y(t) - ax^2(t) - \hat{\theta}^T(t-1)\varphi(t) \\ P(t) &= \frac{1}{\lambda}(P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{\lambda + \varphi^T(t)P(t-1)\varphi(t)}) \end{aligned} \tag{5}$$

Here, $0 < \lambda \leq 1$ is the forgetting factor, and ε is the prediction error. The matrix $P(t)$ can, in the case $\lambda = 1$, be interpreted as the estimated covariance matrix of the parameter estimates.

B. Control Law

Assuming that the vector of parameter estimates $\hat{\theta}^T(t) = [\hat{b}(t) \quad \hat{c}(t)]$ are obtained by the RLS algorithm, the following control law is used:

$$x(t+1) = -\frac{\hat{b}(t)}{2a} + f(t+1) \tag{6}$$

This is basically the certainty equivalence control law corresponding to (2), i.e. the unknown parameter b is replaced by its most recent estimate $\hat{b}(t)$. We note that there is a delay of one sampling interval in the controller, since the plant model (3) has no delay; note also that the estimate $\hat{c}(t)$ does not affect the control law. The additional term $f(t)$ is an extra probing signal. It is motivated by the risk of losing identifiability when the excitation becomes weak (as would be the case for an almost constant estimate $\hat{b}$). The argument will be substantiated in the analysis to follow, and guidelines will be provided for the choice of $f(t)$. Let it suffice here to say that it is assumed that $f(t)$ is periodic with period T_f and that the following relations hold:

$$\sum_{t=1}^{T_f-1} f(t) = \sum_{t=1}^{T_f-1} f^3(t) = 0 \tag{7}$$

$$\frac{1}{T_f}\sum_{t=1}^{T_f-1} f^2(t) = \rho_f^2 \tag{8}$$

The conditions (7) are fulfilled if $f(t)$ is symmetric around zero, as is the case for e.g. a sinusoid or a square wave. The parameter ρ_f can be seen as the RMS value of f.

C. The Extremum Control Algorithm

The final extremum control algorithm is obtained by combining, in the classical way, identification and control. The algorithm is as follows:

1) Record the latest control input $x(t)$ and system output $y(t)$.
2) Update the parameter estimates according to (5), resulting in estimates $\hat{\theta}^T(t) = [\hat{b}(t) \quad \hat{c}(t)]$.
3) Apply the control signal (6) to the process.
4) Update the time index $t \to t+1$ and go to 1.

The state ("memory") for this algorithm is comprised of $(\hat{\theta}(t), P(t), x(t))$.

III. ASYMPTOTIC ANALYSIS

A. Method of Analysis

The asymptotic properties of the extremum controller described in the previous section will now be analyzed using Ljung's ODE analysis, similarly to the analysis in [7]. In [8] it is shown that the asymptotic properties of many recursive, stochastic algorithms, like the one given by (5) and (6), can be analyzed in terms of an associated ordinary differential equation (ODE). In our case, the ODE is given by

$$\frac{d}{d\tau}\theta(\tau) = R^{-1}(\tau)\overline{E}\{\overline{\varphi}(t,\theta)\overline{\varepsilon}(t,\theta)\} \tag{9}$$

$$\frac{d}{d\tau}R(\tau) = \overline{E}\{\overline{\varphi}(t,\theta)\overline{\varphi}^T(t,\theta)\} - R(\tau) \tag{10}$$

The operator $\overline{E}$ is defined by

$$\overline{E}(\cdot) = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}E(\cdot), \tag{11}$$

where E denotes expectation with respect to the stochastic disturbance $e(t)$. The variables $\overline{\varphi}(t,\theta)$ and $\overline{\varepsilon}(t,\theta)$ are the stationary processes, obtained by "freezing" the parameter estimates at a constant value θ in the relations defining φ and ε, respectively. The matrix variable R corresponds to P^{-1}, normalized by time. The significance of this ODE is that its solutions predict the asymptotic behavior of the solution to the original recursive algorithm, assuming no forgetting of old data, i.e. $\lambda = 1$. In practice, the ODE gives useful indications of algorithm properties even when $\lambda < 1$.

In [8], it is shown that under quite general assumptions, the ODE can be used to conclude the following:

- Possible convergence points of the original algorithm are stable stationary points of the ODE (9), (10).
- If the variables of the original algorithm belong to the domain of attraction of a stable stationary point (θ^*, R^*) of the ODE infinitely often w.p.1, then $(\hat{\theta}(t), \frac{1}{t}P^{-1}(t))$ will converge w.p.1 to this stationary point as t tends to infinity.

B. Assumptions on the True System

In order to apply the results of the previous section, we need to make certain assumptions about the actual system and the data it generates. We assume that the true system admits a description, which is compatible with the model (4). The system output is thus given by

$$\begin{aligned} y(t) &= a_0x^2(t) + b_0x(t) + c_0 + e(t) \\ &= a_0x^2(t) + \theta_0^T\varphi(t) + e(t), \end{aligned} \tag{12}$$

where $\{e(t)\}$ is a sequence of independent, identically distributed random variables with zero mean and variance σ^2. We will assume that the model curvature a in (4) and the system's curvature a_0 have the same sign; without loss of generality it is assumed to be positive in the sequel.

C. Stationary Points and Local Stability Analysis

Going back to the analysis of the adaptive optimizer using the ODE approach, the quantities used in (9) and (10) have to be computed. For easy reference, define

$$g(\theta) = \overline{E}\{\overline{\varphi}(t,\theta)\overline{\varepsilon}(t,\theta)\} \tag{13}$$

$$G(\theta) = \overline{E}\{\overline{\varphi}(t,\theta)\overline{\varphi}^T(t,\theta)\} \tag{14}$$

From the definition of $\varphi(t)$ in (4) we get

$$G(\theta) = \overline{E}\begin{bmatrix} x^2(t) & x(t) \\ x(t) & 1 \end{bmatrix} = \begin{bmatrix} (\frac{b}{2a})^2 + \rho_f^2 & -\frac{b}{2a} \\ -\frac{b}{2a} & 1 \end{bmatrix} \tag{15}$$

where the definition (6) of $x(t)$ and the properties (7) and (8) of $f(t)$ have been used. Using (5) and the system description (12), we obtain

$$\begin{aligned} g(\theta) &= \overline{E}\,\overline{\varphi}(t,\theta)[(a_0-a)x^2(t) + (\theta_0-\theta)^T\overline{\varphi}(t,\theta) + e(t)] \\ &= (a_0-a)\begin{bmatrix} \overline{E}x^3(t) \\ \overline{E}x^2(t) \end{bmatrix} + G(\theta)(\theta_0-\theta) \\ &= (a_0-a)\begin{bmatrix} -(\frac{b}{2a})^3 - \frac{3b}{2a}\rho_f^2 \\ (\frac{b}{2a})^2 + \rho_f^2 \end{bmatrix} + G(\theta)(\theta_0-\theta) \end{aligned} \tag{16}$$

The immediate impression from (16) is that correct estimates of θ_0 can be expected only when the a priori estimate of the curvature is correct, i.e. $a = a_0$. However, a further look at the condition for a stationary point of (9), i.e. $g(\theta) = 0$, reveals a more encouraging result — setting $g(\theta) = 0$ in (16) and solving for $(\theta - \theta_0)$ gives:

$$\begin{aligned} \theta - \theta_0 &= (a_0-a)G^{-1}(\theta)\begin{bmatrix} \overline{E}x^3(t) \\ \overline{E}x^2(t) \end{bmatrix} \\ &= \frac{a_0-a}{\rho_f^2}\begin{bmatrix} 1 & -\overline{E}x(t) \\ -\overline{E}x(t) & \overline{E}x^2(t) \end{bmatrix}\begin{bmatrix} \overline{E}x^3(t) \\ \overline{E}x^2(t) \end{bmatrix} \\ &= (a_0-a)\begin{bmatrix} -\frac{b}{a} \\ \rho_f^2 - (\frac{b}{2a})^2 \end{bmatrix}, \end{aligned} \tag{17}$$

which finally gives the following parameters at the stationary point:

$$\begin{cases} b^* = \dfrac{a}{a_0}b_0 \\ c^* = c_0 + (a_0-a)(\rho_f^2 - (\dfrac{b_0}{2a_0})^2) \end{cases} \tag{18}$$

Note that $b^*/a = b_0/a_0$ at the stationary point, so that the latter corresponds to a correct estimation of the position of the optimum, *in spite of a possibly erroneous a priori estimation of the curvature* a_0. The computations carried out above implicitly assume that the probing signal is non-vanishing, i.e. $\rho_f^2 > 0$. Without the probing signal, $G(\theta)$ would become singular and the set of stationary points becomes a one-dimensional manifold in the parameter space.

The ODE analysis requires stationary points to be stable in order for these to be possible convergence points for the original algorithm. This condition is fulfilled for our algorithm; a linearization of (9) and (10) at the stationary point θ^*, defined by (18), results in a block triangular system. The eigenvalues associated with the R part are all -1 and the θ part is characterized by the "system matrix"

$$\begin{aligned} G^{-1}(\theta^*)\left.\frac{\partial g(\theta)}{\partial\theta}\right|_{\theta=\theta^*} &= \frac{1}{\rho_f^2}\begin{bmatrix} 1 & \frac{b_0}{2a_0} \\ \frac{b_0}{2a_0} & (\frac{b_0}{2a_0})^2+\rho_f^2 \end{bmatrix}\begin{bmatrix} -(\frac{b_0}{2a_0})^2-\frac{a_0}{a}\rho_f^2 & \frac{b_0}{2a_0} \\ \frac{b_0}{2a_0} & -1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{a_0}{a} & 0 \\ (1-\frac{a_0}{a})\frac{b_0}{2a_0} & -1 \end{bmatrix} \end{aligned} \tag{19}$$

This matrix is asymptotically stable as long as the sign of a is the same as the sign of a_0 — certainly a reasonable assumption. The analysis can now be summarized as

Result 1: The adaptive optimizer (5), (6) is applied to the system (12), assuming the a priori estimate a and the corresponding system parameter a_0 have equal sign. Then there is one possible convergence point (θ^*, R^*), defined by (18) and $R^* = G(\theta^*)$. The corresponding control signal, averaged over the period of the probing signal, is optimal.

This result is obtained using stochastic averaging by the ODE method. Alternatively, very similar stability results could be obtained in a deterministic setting using conventional averaging techniques along the same lines as in e.g. [4].

D. Global Stability Analysis

The convergence analysis made so far has only been local. As stated in the beginning of this section, further analysis of the associated ODE can give stronger results. The standard tool to deduce stability properties of the differential equation is then to use Lyapunov functions. However, the differential equation for our problem is not easy to analyze — it is strongly non-linear and of 5th order (taking into account the fact that P is symmetric). To give an idea of the character of the dynamics of the ODE, a few trajectories of $\theta(\tau)$ are shown in Fig. 3. In the same figure, level curves of the criterion to be minimized by the RLS algorithm, $\overline{V}(\theta) = \overline{E}\varepsilon^2(\theta)$ are depicted. From the figure, it can be seen how the trajectories converge to the optimum position, located at the bottom of a curved "valley".

Simulations of the ODE indicate that the stationary point has a large domain of attraction, and it may even be globally stable in many cases. However, so far we have not been able to prove this theoretically. It should also be pointed out that, in order to guarantee convergence of the original algorithm using the ODE method, a separate boundedness condition

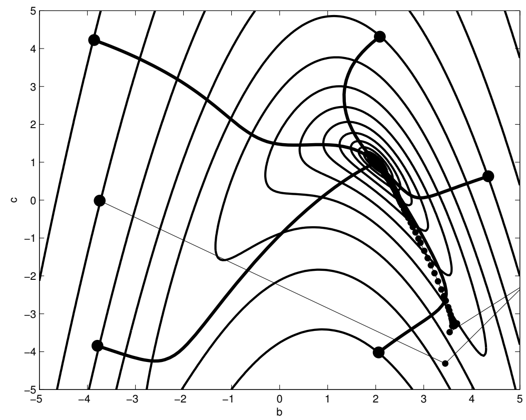

Fig. 3. Parameter trajectories ($\theta(\tau)$) of the associated ODE for different initial conditions. Level curves of the function $\overline{V}(\theta) = \overline{E}\varepsilon^2(t,\theta)$ are also shown. The trace with dots represents one run with the extremum controller.

needs to be fulfilled. It seems from simulations that this can not be expected for large enough initial parameter errors.

IV. THE TRADE-OFF BETWEEN ESTIMATION AND CONTROL

In the convergence analysis it was assumed that the probing signal $f(t)$ is non-vanishing, i.e. $\rho_f^2 > 0$. Even though this entered as a technical condition in the analysis, it is also intuitively clear that a probing signal is needed in order to avoid identifiability issues. However, it is equally clear that there is a penalty involved in making the probing signal too large — it will eventually drive the control signal too far away from the optimum position. This leads to the important question how the probing signal should be designed, a question which we will now address. An early contribution dealing with a similar problem can be found in [9].

A. Analysis of Variance

We assume in the subsequent analysis that the parameters $\hat{\theta}(t)$ converge asymptotically to the stationary point θ^*, given by (18), and we will analyze the stochastic variations around the limit. This is an idealized case, which e.g. assumes that $\lambda = 1$, but the results give useful information also about the case $\lambda < 1$.

We will base our analysis of variance on the result given in [10], Theorem 9.1, for a class of prediction error methods. There, it is shown that under a set of fairly weak conditions, the parameter estimates are asymptotically normal distributed with mean θ^* and covariance

$$\operatorname{cov}\hat{\theta}(t) = \frac{\sigma^2}{t}\overline{E}\{\psi(t,\theta^*)\psi^T(t,\theta^*)\}^{-1}, \tag{20}$$

where $\psi(t,\theta) = \frac{\partial}{\partial\theta}\varepsilon(t,\theta)$.

Remark 1: Care should be taken when applying this result to our problem. First, the derivation of this result uses the fact that the prediction errors $\varepsilon(t,\theta^*)$ corresponding to the "true" parameters are independent stochastic variables; in our case $\varepsilon(t,\theta^*) = (a_0-a)(f^2(t)-\rho_f^2)+e(t)$, so that this is no longer true if $a \neq a_0$. However, it turns out that the derivation is still valid with small modifications. The second issue to point out is that the data is generated in closed loop, where the control action is dependent on current parameter estimates. This makes it nontrivial to verify basic assumptions on the quasi-stationary behavior of input and output data. We will carry out the analysis disregarding this fact, and return to this question later on.

From the expression for ε as used in (16), the following is obtained:

$$\begin{aligned}\psi(t,\theta^*) &= \begin{bmatrix}\frac{1}{2a}(b - b_0 + \frac{a_0}{a}b) - \frac{a_0}{a}f(t) \\ -1\end{bmatrix}_{b=b^*} \\ &= \begin{bmatrix}\frac{b_0}{2a_0} - \frac{a_0}{a}f(t) \\ -1\end{bmatrix}\end{aligned} \tag{21}$$

from which we get

$$\overline{E}\{\psi(t,\theta^*)\psi^T(t,\theta^*)\} = \begin{bmatrix}(\frac{b_0}{2a_0})^2 + (\frac{a_0}{a})^2\rho_f^2 & -\frac{b_0}{2a_0} \\ -\frac{b_0}{2a_0} & 1\end{bmatrix} \tag{22}$$

Let us define the average output over one period of the probing signal as follows:

$$E_f\{y(t)\} = \frac{1}{T_f}\sum_{k=t}^{t+T_f-1} Ey(k) \tag{23}$$

Using the system description (12) and the control law (6), the following is obtained (neglecting the time argument for the estimates):

$$\begin{aligned}&E_f\{y(t)\} \\ &= E_f\Big\{a_0(f(t) - \frac{\hat{b}}{2a})^2 + b_0(f(t) - \frac{\hat{b}}{2a}) + c_0 + e(t)\Big\} \\ &= E_f\Big\{a_0(\frac{b_0}{2a_0} - \frac{\hat{b}}{2a})^2 - \frac{b_0^2}{4a_0} + (b_0 - \frac{a_0}{a}\hat{b})f(t) \\ &\qquad + a_0 f^2(t) + c_0 + e(t)\Big\} \\ &= y_{opt} + \frac{a_0}{4a^2}\operatorname{cov}\hat{b} + a_0\rho_f^2\end{aligned} \tag{24}$$

where y_{opt} is the output value at the optimum. From the last term in this expression, it can be seen that a large probing signal deteriorates the performance. However, since the covariance of the estimate $\hat{b}$ depends on ρ_f^2, there is a trade-off in the choice of probing signal power — using (20) and (22) in (24), it follows that

$$E_f\{y(t)\} - y_{opt} = \frac{1}{4a_0}\frac{\sigma^2}{t}\frac{1}{\rho_f^2} + a_0\rho_f^2 \tag{25}$$

The right hand side of (25) shows how much the performance deteriorates relative to the theoretical optimum. The first term is the result of parameter estimate variance, which of course decreases with increasing ρ_f^2, whereas the second term is the result of adding the probing component to the certainty equivalence control law — this term improves with a decreasing ρ_f^2. By differentiating the right hand side of (25) with respect to the design variable ρ_f^2, we get

$$\frac{\partial}{\partial\rho_f^2}[E_f\{y(t)\} - y_{opt}] = a_0(1 - \frac{\sigma^2/t}{4a_0^2(\rho_f^2)^2}) \tag{26}$$

and by equating this to zero, the optimal probing power and the corresponding output is given by

$$\rho_f^2 = \frac{\sigma/(2a_0)}{\sqrt{t}} \quad \Rightarrow \quad E_f\{y(t)\} - y_{opt} = \frac{\sigma}{\sqrt{t}} \tag{27}$$

The analysis has been carried out for the case $\lambda = 1$, i.e. without any discounting of old data in the estimation. In practice, we would like to use the algorithm with $\lambda < 1$. A rule of thumb often used is that the "effective" number of data (or the memory length) used for estimation when $\lambda < 1$ is roughly $1/(1-\lambda)$, see e.g. [10]. Based on this rule of thumb, we would expect the results above to hold, at least approximately. If so, the optimal probing power and the corresponding output would be given by

$$\rho_f^2 = \frac{\sigma\sqrt{1-\lambda}}{2a_0} \quad \Rightarrow \quad E_f\{y(t)\} - y_{opt} \approx \sigma\sqrt{1-\lambda} \tag{28}$$

Result 2: The probing signal $f(t)$ for the adaptive optimizer (5), (6), applied to the system (12) (assuming a and a_0 have equal sign), asymptotically affects the total system performance in two conflicting ways; estimation is improved and control is impaired, see (25). The optimal trade-off for the case with discounting of old data is given by (28).

This result resembles the classical trade-off between estimation and control in dual control problems, see e.g. [11], [6]. A distinction is that we have assumed a stationary probing signal and an asymptotic analysis, so that the only remaining task is to determine the power of the probing signal; other characteristics of the probing signal, e.g. the frequency, has no effect in the asymptotic analysis.

V. SIMULATION RESULTS

The proposed extremum controller has been applied to a simulated process, described by

$$y(t) = a_0x^2(t) + b_0x(t) + c_0 + e(t), \tag{29}$$

with $a_0 = 1$, $b_0 = 2$, $c_0 = 1$, and the noise variance $\sigma^2 = 0.1$. Thus, the optimal control input is $x_{opt} = -1$ and the corresponding output is $y_{opt} = 0$.

The extremum controller is applied to this system, using an a priori curvature estimate $a = 2$, i.e. a factor of 2 larger than the true value. The value of the forgetting factor $\lambda = 0.99$. A sinusoidal probing signal with frequency 0.1 rad/s and amplitude A is used (implying $\rho_f^2 = \frac{A^2}{2}$).

Fig. 4 gives an illustration of the role of the probing signal. The average value of the output is depicted as a function of the probing signal amplitude A. As can be seen from the figure, there is a close correspondence between the theoretical expression (24) and actual algorithm results for a range of different perturbation amplitudes from the optimal value and above. However, for smaller perturbation amplitudes the extremum controller performs better than

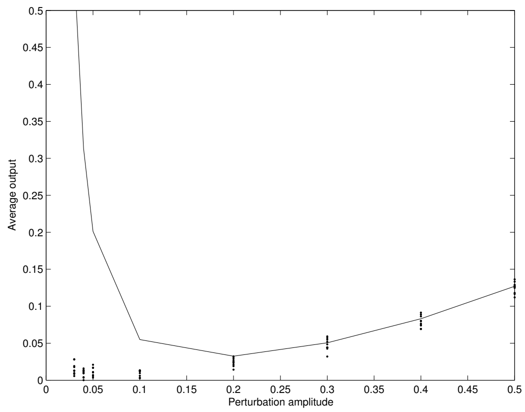

Fig. 4. The average cost as function of probing signal power. The solid curve indicates the theoretical expression (24), and the dots show algorithm outcomes.

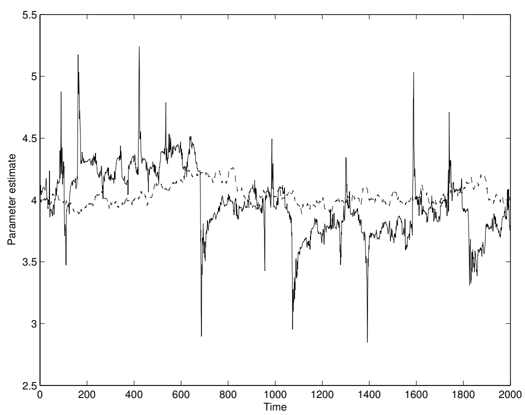

Fig. 5. Parameter estimate $\hat{b}(t)$ for two different perturbation amplitudes A. Solid, irregular curve: $A = 0.03$. Dashed curve: $A = 0.5$.

what theory predicts. The reason seems to be exactly what was discussed in the previous section — the very small perturbation levels give parameter estimates that behave very irregularly, i.e. far from the quasi-stationarity on which the analysis is based. This can clearly be seen in Fig. 5, where the parameter estimates $\hat{b}(t)$ for two different perturbation levels are shown. Despite the fact that the strongly varying parameter estimates give excitation enough in order for the controller to work quite satisfactorily, the recipe (28) for the probing level seems as a good design rule, since this leads to more regular and well behaved estimates.

The system input and output for the same example are depicted in Fig. 6. In this case, the perturbation level has been selected according to (28).

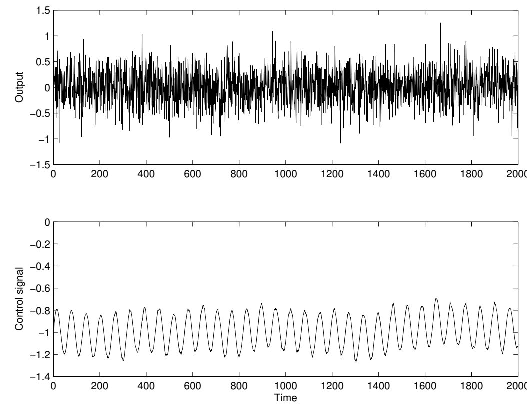

Fig. 6. System input and output for a medium perturbation level.

VI. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

Motivated by an automotive engine application, an extremum controller has been proposed in this contribution. The controller is based on real-time identification of a simple polynomial, static model, combined with a certainty equivalence controller. The asymptotic convergence properties deduced from a theoretical analysis are encouraging. An asymptotic analysis of variance of the parameter estimates has given guidelines for the design of a probing signal, which has to be added in order for the algorithm to be able to track a time-varying system.

B. Future Works

The extremum controller presented and analyzed in this contribution will be used in the near future in experiments on a 5-cylinder passenger car engine. The objective of these experiments is to optimize the net work of the engine by manipulating the setpoint of an inner feedback loop. Further details of this application can be found in [5].

There are several important theoretical issues to consider further. The stability and convergence analysis needs to be extended, and the algorithm's ability to handle dynamics should be investigated. It would also be of interest to compare the properties of the algorithm with other schemes suggested in the literature, e.g. the ones described and analyzed in [4].

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