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On the Minimal Robust Positively Invariant Set for Linear Difference Inclusions

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Abstract— **This paper provides a new and efficient method for the computation of an arbitrarily close outer robust positively invariant (RPI) approximation to the minimal robust positively invariant (mRPI) set for linear difference inclusions. It is assumed that the linear difference inclusion is absolutely asymptotically stable (AAS) in the absence of an additive state disturbance, which is the case for parametrically uncertain or switching linear discrete-time systems controlled by a stabilizing linear state feedback controller.**

I. Introduction

One of the fundamental tools employed in robust control of constrained dynamical systems is set invariance theory [1]. Set invariance is used in the design of reference governors [2] and predictive controllers [3]–[5] to guarantee constraint satisfaction, stability and convergence properties. One technique for robust control of constrained discrete-time systems is robust time-optimal control [6]–[9] that is based on the computation of a sequence of robust control (positively) invariant sets when the target set is also a robust control (positively) invariant set. A suitable target set in robust time-optimal control is the minimal robust positively invariant set [10]. The relevance of the minimal (in an appropriate sense) robust control (positively) invariant set is demonstrated by the novel robust predictive controllers, recently proposed in [11]–[13].

Computational issues and algorithmic procedures for the calculation of the robust control (positively) invariant sets and application of these sets in robust control for constrained systems are also discussed by a number of researchers [10], [14]–[22]. One of the outstanding problems for autonomous linear discrete-time systems is exact characterization of the minimal robustly positively invariant set [1], [10], [23]. Several authors have developed procedures for the computation of the so–called outer, RPI ε-approximation, of the minimal robust positively invariant (mRPI) set; see, for instance, a procedure proposed in [24] and an alternative, simpler and improved, procedure in [25]. However, these approximate techniques are developed for autonomous linear discrete-time invariant systems.

In this paper we address the issue of the computation of the outer, RPI ε-approximation, of the mRPI set for linear difference inclusions. Linear Difference Inclusions (LDI) are used for modeling linear systems with parametric uncertainty; see, for example, [26], [27], and linear systems with switching dynamics [28]. This paper extends the results reported in [18], [20], [21], [25].

This paper is organized as follows. Section II is concerned with preliminaries. Section III establishes existence of the mRPI set for linear difference inclusions and provides a characterization of a family of robust positively invariant sets for linear difference inclusions that are outer approximations of the mRPI set. Section IV considers the limiting behaviour of these RPI approximations and provides a condition for characterization of a family of outer, RPI ε-approximations of the mRPI set for linear difference inclusions. Section V presents efficient computational procedures when the disturbance set is a polytope. Section VI gives an illustrative example. Finally, Section VII presents conclusions. All proofs for the results stated in this paper can be found in [29].

Basic Notation: Let $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$, $\mathbb{N}^+ \triangleq \{1, 2, \ldots\}$; for $q \in \mathbb{N}^+$ let $\mathbb{N}_q \triangleq \{0, \ldots, q\}$ and $\mathbb{N}_q^+ \triangleq \{1, \ldots, q\}$. Let $\mathbb{B}_p^n(\gamma) \triangleq \{x \in \mathbb{R}^n \mid \|x\|_p \le \gamma\}$, where $\|\cdot\|_p$ denotes the vector p-norm. Given an integer $s \in \mathbb{N}^+$ and the sets $\Omega_i \subset \mathbb{R}^n$, $i \in \mathbb{N}_s^+$, the Minkowski set addition is defined as $\bigoplus_{i=1}^s \Omega_i \triangleq \Omega_1 \oplus \Omega_2 \oplus \ldots \oplus \Omega_s = \{\sum_{i=1}^s \omega_i \mid \omega_i \in \Omega_i\}$. Given the set $\Omega \subset \mathbb{R}^n$, $\text{interior}(\Omega)$ denotes its interior, $\text{closure}(\Omega)$ its closure and $\text{co}(\Omega)$ its convex hull. $\mathbb{R}_+^n$ denotes the set of non–negative vectors in $\mathbb{R}^n$, i.e. $\mathbb{R}_+^n \triangleq \{x \in \mathbb{R}^n \mid x \ge 0\}$. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces, a *polytope* is a closed and bounded polyhedron. A set $\Omega \subset \mathbb{R}^n$ is a C-set if it is a convex, compact, set containing the origin in its non-empty interior.

II. Preliminaries

We consider the following linear difference inclusion:

$$\begin{aligned} x^+ &\in \mathcal{D}(x, \mathbb{A}, \mathbb{W}) \\ \mathcal{D}(x, \mathbb{A}, \mathbb{W}) &\triangleq \{Ax + w \mid A \in \text{co}(\mathbb{A}),\ w \in \mathbb{W}\} \\ \mathbb{A} &\triangleq \{A_i \in \mathbb{R}^{n \times n} \mid i \in \mathbb{N}_q^+\} \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, x^+ is the successor state, $w \in \mathbb{R}^n$ is an unknown disturbance and $q \in \mathbb{N}^+$ is a finite integer. The system is subject to an external additive state disturbance w that is contained in a C-set $\mathbb{W} \subset \mathbb{R}^n$. The system transition matrix A is uncertain and is known only to the extent that it belongs to the convex hull of a finite set $\mathbb{A}$ of known matrices A_i; furthermore, A is in principle time-varying and different A from the set $\text{co}(\mathbb{A})$ can occur

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at different times. Let $\lambda \triangleq (\lambda_1, \lambda_2, \ldots, \lambda_q)$ and

$$\Lambda \triangleq \{\lambda \in \mathbb{R}^q_+ \mid \sum_{i=1}^{q} \lambda_i \leq 1\} \tag{2}$$

The system transition matrix A can take any arbitrary value in the set $\text{co}(\mathbb{A})$ so that:

$$A = \sum_{i=1}^{q} \lambda_i A_i,\ \lambda \in \Lambda \tag{3}$$

where λ can vary with time. In this paper we adapt the following standing assumption:

Assumption 1: (i) The set $\mathbb{W}$ is a C-set in $\mathbb{R}^n$ and (ii) The matrix A at any point in time is given by (3) for some (possibly time-varying) $\lambda = (\lambda_1, \ldots, \lambda_q) \in \Lambda$.

We refer to $\mathcal{D}(x, \mathbb{A}, \{0\})$ (i.e. when $\mathbb{W} = \{0\}$) as the nominal part of the linear difference inclusion (1).

The main motivation for considering linear difference inclusions of the form (1) lies in the fact that a broad class of systems can be modeled by this form. For example, consider the following uncertain, linear discrete-time system:

$$\begin{aligned} & x^+ = Fx + Gu + w\ ,\ (F, G) \in \text{co}(\mathbb{C}) \\ & \mathbb{C} \triangleq \{(F_i, G_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \mid i \in \mathbb{N}^+_q\} \end{aligned} \tag{4}$$

where, $w \in \mathbb{W}$. It is well known that the system (4) with $\mathbb{W} = \{0\}$ can be stabilized if there exists a solution to the following, possibly conservative, linear matrix inequality problem [27]:

$$(F_i+G_iK)^T P(F_i+G_iK)-P < 0,\ P = P^T > 0,\ \forall i \in \mathbb{N}^+_q \tag{5}$$

When $\mathbb{W} = \{0\}$ and (5) has a solution, the stabilizing state feedback controller takes the form $u = Kx$, the closed loop system is then of the form (1) with $A_i = F_i + G_iK$ and the corresponding Lyapunov function is $V(x) \triangleq ||x||^2_P$.

In view of (5) we assume, without loss of generality, that:

Assumption 2: There exists a pair $(P, \psi) \in \mathbb{R}^{n\times n} \times (0, 1)$ such that $P = P^T > 0$ and

$$A_i^T P A_i - P \leq -\psi P,\ \forall i \in \mathbb{N}^+_q \tag{6}$$

Recalling a set of relevant results in [28, Section 3] on stability of linear difference inclusions when $\mathbb{W} = \{0\}$, it follows that if (6) holds then the linear difference inclusion (1), when $\mathbb{W} = \{0\}$, is *Absolutely Asymptotically Stable (AAS)* [28] so that $\lim_{k\to\infty} x(k) \to 0$, where $x(k)$ is any particular solution, at time k, of the nominal part of the linear difference inclusion ($x^+ \in \mathcal{D}(x, \mathbb{A}, \{0\})$) given the initial condition $x(0)$ and a particular realization of the matrix sequence $\{A(0), A(1), \ldots\}$ with $A(i) \in \mathbb{A}$. It is also known that, if (6) holds then the nominal part of the linear difference inclusions (1) (with $\mathbb{W} = \{0\}$) is AAS for all $A \in \text{co}(\mathbb{A})$ [27, Section 5.1.3].

Given a non-empty set $X \subset \mathbb{R}^n$, a finite set of matrices $\mathbb{A}$, and a set $\mathbb{W}$ we use, as is standard, the following notation for the one step forward reachable set for the difference inclusion (1):

$$\mathcal{D}(X, \mathbb{A}, \mathbb{W}) \triangleq \{Ax+w \mid x \in X,\ A \in \text{co}(\mathbb{A}),\ w \in \mathbb{W}\} \tag{7}$$

The following two definitions are standard definitions in set invariance theory (see [1, Section 2] and [10, Section 4]):

Definition 1: A set Ω is a robust positively invariant (RPI) set of the difference inclusion (1) if $\mathcal{D}(\Omega, \mathbb{A}, \mathbb{W}) \subseteq \Omega$.

Definition 2: A set D_∞ is the minimal robust positively invariant (mRPI) set for the difference inclusion (1), if D_∞ is an RPI set and D_∞ is contained in every closed RPI set for the difference inclusion (1).

In this paper, following the abstract results developed in [30] and used in [10], [18], [20]–[22], we establish that the set D_∞ exists and is unique over the class of closed RPI sets for the difference inclusion (1), provided that Assumption 2 holds. Moreover, we provide an appropriate characterization of a family of RPI sets for the difference inclusion (1) and provide a method for the computation in finite time of the so called outer, RPI ε-approximation [25, Section II] of the mRPI set D_∞ – this is an RPI set D that satisfies $D_\infty \subseteq D \subseteq D_\infty \oplus \mathbb{B}^n_p(\varepsilon)$.

In order to discuss the convergence of the set sequences (taken in the Hausdorff metric sense) and to clarify our use of the term outer, RPI ε-approximation of the sets we recall the following two definitions:

Definition 3: If Ω and Φ are two non-empty, compact sets in $\mathbb{R}^n$, then the Hausdorff metric is defined as

$$\delta(\Omega, \Phi) \triangleq \max\{\sup_{\omega\in\Phi} d(\omega, \Omega), \sup_{\phi\in\Omega} d(\phi, \Phi)\} \tag{8}$$

where $d(z, \mathcal{Z}) \triangleq \inf_{y\in\mathcal{Z}} ||z - y||_p$.

Definition 4: Given a scalar $\varepsilon > 0$ and a non-empty set $\Omega \subset \mathbb{R}^n$, the set $\Phi \subset \mathbb{R}^n$ is an *outer ε-approximation* of Ω if $\Omega \subseteq \Phi \subseteq \Omega \oplus \mathbb{B}^n_p(\varepsilon)$ and an inner ε-approximation of Ω if $\Phi \subseteq \Omega \subseteq \Phi \oplus \mathbb{B}^n_p(\varepsilon)$.

We also need the following definition in Section V, where computational issues for the approximation of D_∞ are discussed:

Definition 5: The support function $h_{\mathcal{X}}(\cdot)$ of a set $\mathcal{X} \subset \mathbb{R}^n$, evaluated at a vector $\eta \in \mathbb{R}^n$, is defined by:

$$h_{\mathcal{X}}(\eta) \triangleq \sup_x \{\eta^T x \mid x \in \mathcal{X}\}.$$

Note that if $\mathcal{X}$ is a polytope then the *supremum* in Definition 5 is in fact *maximum*; furthermore, the evaluation of $h_{\mathcal{X}}(\eta)$ is a linear programming problem.

Let, for any $k \in \mathbb{N}$, $\mathbf{i}_k \triangleq \{i_0, i_1, \ldots, i_k\}$ denote a sequence of integer variables such that $i_j \in \mathbb{N}^+_q$ for each $j \in \mathbb{N}_k$ and $\mathbf{i}_0 \triangleq i_0 \in \mathbb{N}^+_q$. We denote the set of all integer sequences $\mathbf{i}_k$ by $\mathcal{I}_k \triangleq \{\mathbf{i}_k \mid i_j \in \mathbb{N}^+_q,\ j \in \mathbb{N}_k\}, \forall k \in \mathbb{N}$. We define the matrices $\mathcal{A}_{\mathbf{i}_k} \triangleq A_{i_k} \ldots A_{i_1} A_{i_0}$ for each $\mathbf{i}_k \in \mathcal{I}_k$ and $\mathcal{A}_{\mathbf{i}_0} \triangleq I$ where I is the identity matrix and $A_{i_j} \in \mathbb{A}$.

It is easily shown that the set sequence $\{D_k\}$ defined by:

$$D_{k+1} \triangleq \mathcal{D}(D_k, \mathbb{A}, \mathbb{W}),\ k \in \mathbb{N}^+,\ D_0 = \{0\} \tag{9}$$

is the set sequence describing the forward reachable tube [30]–[32] starting from the origin for the difference inclusion (1). An alternative form for the set sequence $\{D_k\}$

is given by:

$$D_{k+1} = \bigoplus_{j=0}^{k} \text{co}\left(\bigcup_{\mathbf{i}_j \in \mathcal{I}_j} \mathcal{A}_{\mathbf{i}_j}\mathbb{W}\right),\ k \in \mathbb{N}^+, D_0 = \{0\} \quad (10)$$

It follows from (9) and (10) that, for any finite integer $k \in \mathbb{N}$, the set D_k is a convex and compact set, since it is the Minkowski addition of a finite number of convex sets, each of which is the convex hull of a finite union of compact sets. Moreover, since $0 \in \text{interior}(\mathbb{W})$ it follows that $0 \in \text{interior}(D_k)$ for all $k \in \mathbb{N}^+$.

In the following section we show that there exists a compact RPI set D_∞ satisfying $\delta(D_\infty, D_k) \to 0$ as $k \to \infty$, where $\{D_k\}$ is the set sequence defined in (9) ((10)). Moreover, we characterize a family of RPI sets $D(\alpha, s)$ for the difference inclusion (1) that are outer, RPI approximations of D_∞ for a given pair $(\alpha, s) \in (0, 1) \times \mathbb{N}$, i.e the sets $D(\alpha, s)$ such that they are RPI and $D_\infty \subseteq D(\alpha, s)$. We also show that this family of sets contains another family of the sets $D(\alpha, s)$ that are outer, RPI ε-approximation of the mRPI set D_∞, i.e. the sets satisfy $D_\infty \subseteq D(\alpha, s) \subseteq D_\infty \oplus \mathbb{B}_p^n(\varepsilon)$, for $\varepsilon > 0$. Additionally we establish an appropriate condition allowing the computation of a suitable pair $(\alpha, s) \in (0, 1) \times \mathbb{N}$ such that $D(\alpha, s)$ is an RPI set and satisfies $D_\infty \subseteq D(\alpha, s) \subseteq D_\infty \oplus \mathbb{B}_p^n(\varepsilon)$ for an *a–priori* given value of $\varepsilon > 0$.

III. The mRPI set D_∞ for the linear difference inclusion $\mathcal{D}(x, \mathbb{A}, \mathbb{W})$

In this section we discuss the existence of the mRPI set D_∞ and a characterization of a collection of the RPI sets that are RPI approximations of the mRPI set D_∞.

A. Existence of a compact RPI set D_∞ satisfying $\delta(D_\infty, D_k) \to 0$ as $k \to \infty$

A collection of non-empty compact sets in $\mathbb{R}^n$, equipped with the Hausdorff Metric, form a complete metric space [33]. Hence, every convergent or Cauchy sequence (whose elements belong to this collection) converges to an element of the space. It can be shown by exploiting ideas from [10], [18], [20]–[22] that since each set D_k is compact, the set sequence $\{D_k\}$ is a Cauchy sequence.

It follows from (10) that each D_k, $k \in \mathbb{N}^+$, can be expressed as:

$$\begin{aligned} D_{k+1} &= \bigoplus_{j=0}^{k} \text{co}\left(\bigcup_{\mathbf{i}_j \in \mathcal{I}_j} \mathcal{A}_{\mathbf{i}_j}\mathbb{W}\right) \\ &= \text{co}\left(\bigcup_{\mathbf{i}_k \in \mathcal{I}_k} \mathcal{A}_{\mathbf{i}_k}\mathbb{W}\right) \oplus \bigoplus_{j=0}^{k-1} \text{co}\left(\bigcup_{\mathbf{i}_j \in \mathcal{I}_j} \mathcal{A}_{\mathbf{i}_j}\mathbb{W}\right) \\ &= R_k \oplus D_k \end{aligned} \quad (11)$$

where the sets R_k are defined by:

$$R_k \triangleq \text{co}\left(\bigcup_{\mathbf{i}_k \in \mathcal{I}_k} \mathcal{A}_{\mathbf{i}_k}\mathbb{W}\right),\ k \in \mathbb{N}^+,\ R_0 \triangleq \mathbb{W} \quad (12)$$

In going from the first line of (11) to the second line of (11) we have used a result from [34, Theorem 1.1.2], which states that $\text{co}(\mathcal{X} \oplus \mathcal{Y}) = \text{co}(\mathcal{X}) \oplus \text{co}(\mathcal{Y})$ for sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$.

The set R_k is the set of states that can be reached at time k, *with respect to the nominal part of the difference inclusion* (1)*, i.e.* $\mathcal{D}(x, \mathbb{A}, \{0\})$, starting from an initial condition that belongs to the set $\mathbb{W}$. The set sequence $\{R_k\}$ has an alternative expression given by the following set recursion:

$$R_k \triangleq \mathcal{D}(R_{k-1}, \mathbb{A}, \{0\}),\ k \in \mathbb{N}^+,\ R_0 \triangleq \mathbb{W} \quad (13)$$

where $\mathcal{D}(X, \mathbb{W}, \{0\})$ is defined by (7). It follows from (11)–(12) that for all $k \in \mathbb{N}$ we have:

$$D_k \subseteq D_{k+1} = D_k \oplus R_k \quad (14)$$

The properties of the sequence $\{D_k\}$ are summarized in the following theorem:

Theorem 1: Suppose Assumptions 1 and 2 hold. Then the set sequence $\{D_k\}$ defined by (9) ((10)) satisfies :
(i) $D_k \subseteq D_{k+1} \subseteq \mathcal{D}_k \oplus \theta^k \mathbb{B}_p^n(\mu)$ for all $k \in \mathbb{N}$ with $\theta \in (0, 1)$ and $\mu < \infty$,
(ii) there exists a compact set D_∞ such that $\delta(D_\infty, D_k) \to 0$ as $k \to \infty$.
Since $\{D_k\}$ is a Cauchy sequence of compact sets, D_∞ is the limit of this sequence and is given by:

$$D_\infty = \text{closure}\left(\bigoplus_{j=0}^{\infty} \text{co}\left(\bigcup_{\mathbf{i}_j \in \mathcal{I}_j} \mathcal{A}_{\mathbf{i}_j}\mathbb{W}\right)\right) \quad (15)$$

Robust positive invariance of the set D_∞ in (15) is obvious. It is difficult to obtain an explicit characterization of the set D_∞ even for the simple case when $q = 1$ (so that $\mathbb{A}$ is singleton) and the linear difference inclusion (1) is simply a linear time-invariant system, except maybe in some restrictive cases. We proceed to exploit linearity of the difference inclusion (1), Assumptions 1 and 2 and basic properties of Minkowski addition in order to characterize a set $D(\alpha, s)$ that is an RPI approximation of the mRPI set D_∞.

B. An RPI approximation of the set D_∞ – the set $D(\alpha, s)$

The discussion in the previous subsection motivates further investigation regarding robust positive invariance of a collection of sets for the difference inclusion (1). The following result, established in [18], [21], allows one to compute an RPI outer approximation of the mRPI set for the difference inclusion (1):

Theorem 2: Suppose Assumptions 1 and 2 hold, then there exists a finite integer $s \in \mathbb{N}^+$ and a scalar $\alpha \in [0, 1)$ such that

$$R_s \subseteq \alpha\mathbb{W} \quad (16)$$

where R_s is defined in (12) ((13)). Moreover, given any pair $(\alpha, s) \in [0, 1) \times \mathbb{N}^+$ such that (16) is true, the set $D(\alpha, s)$ defined by

$$D(\alpha, s) \triangleq (1 - \alpha)^{-1} D_s \quad (17)$$

is a convex, compact RPI set for the difference inclusion (1) such that $D_\infty \subseteq D(\alpha, s)$.

Theorem 2 can be used to develop and implement an algorithm for the approximation of D_∞. Clearly, from Theorem 2, the set $D(\alpha, s)$ is an outer RPI approximation of D_∞. However, the former can be a poor approximation of the latter, hence we proceed to present an extension of the results for the LTI systems case, reported in [25], in order to provide a way to obtain a set $D(\alpha, s)$ such that $D_\infty \subseteq D(\alpha, s) \subseteq D_\infty \oplus \mathbb{B}_p^n(\varepsilon)$ for an *a-priori* given $\varepsilon > 0$.

IV. Limiting Behavior of the RPI set $D(\alpha, s)$

In order to be able to evaluate how "well" $D(\alpha, s)$ approximates D_∞, we have to study the limiting behaviour of $D(\alpha, s)$ as $s \to \infty$ and $\alpha \searrow 0$. Given any $\alpha \in (0, 1)$, the smallest value of s such that (16) holds is:

$$s^0(\alpha) \triangleq \inf_s \{s \in \mathbb{N}^+ \mid R_s \subseteq \alpha \mathbb{W}\} \tag{18}$$

The smallest α such that (16) holds for a given $s \in \mathbb{N}^+$ is:

$$\alpha^0(s) \triangleq \inf_\alpha \{\alpha \in \mathbb{R}_+ \mid R_s \subseteq \alpha \mathbb{W}\} \tag{19}$$

Note that, for any $\alpha \in (0, 1)$ the $s^0(\alpha)$ in (18) is finite and that $\alpha^0(s)$ satisfies $\alpha^0(s) \in [0, 1)$ if and only if s is sufficiently large.

The following two theorems extend the results established in [20], [22] for switching systems and in [25] for linear systems to the class of linear difference inclusions (1).

The first theorem addresses the issue of the limiting behaviour of $D(\alpha, s)$:

Theorem 3: Suppose Assumptions 1 and 2 hold, then

i) $D(\alpha^0(s), s) \to D_\infty$ as $s \to \infty$

ii) $D(\alpha, s^0(\alpha)) \to D_\infty$ as $\alpha \searrow 0$

Theorem 3 implies that $D(\alpha, s)$ converges to D_∞ as $s \to \infty$ or $\alpha \searrow 0$. Thus, by increasing s and calculating α from (19), or by decreasing α and calculating s from (18), one can obtain a better approximation of D_∞. However, given a pre-specified accuracy, it is not clear yet how to obtain a pair (α, s) such that $D(\alpha, s)$ efficiently approximates D_∞ with the given accuracy.

This issue is dealt in the next theorem, which provides conditions that the pair (α, s) has to satisfy in order to guarantee that the set $D(\alpha, s)$ is an outer RPI ε-approximation of the mRPI set D_∞.

Theorem 4: Suppose Assumptions 1 and 2 hold, then for all $\varepsilon > 0$ there exists a pair $(\alpha, s) \in [0, 1) \times \mathbb{N}^+$ such that (16) and

$$\alpha(1-\alpha)^{-1} D_s \subseteq \mathbb{B}_p^n(\varepsilon) \tag{20}$$

hold. Moreover, for any pair $(\alpha, s) \in [0, 1) \times \mathbb{N}^+$ such that (16) and (20) hold, the set $D(\alpha, s)$ is an outer RPI ε-*approximation* of D_∞.

Theorem 4 clearly states that given an *a priori* $\varepsilon > 0$, a collection of (α, s) can be found to satisfy (16) and (20). Following this, any set $D(\alpha, s)$ is an outer RPI ε-approximation of D_∞, i.e. $D_\infty \subseteq D(\alpha, s) \subseteq D_\infty \oplus \mathbb{B}_p^n(\varepsilon)$.

Let $M(s) \triangleq \sup_z \{\|z\|_p \mid z \in D_s\}$ and $M_\infty \triangleq \sup_z \{\|z\|_p \mid z \in D_\infty\}$. Since $D_s \subseteq D_\infty$ it follows that $M(s) \leq M_\infty$ and

$$\alpha \leq \varepsilon(\varepsilon + M_\infty)^{-1} \leq \varepsilon(\varepsilon + M(s))^{-1} \tag{21}$$

Hence, an upper bound for α can be obtained by using (21). Note also that (16) gives a lower bound for α such that $D(\alpha, s)$ is a RPI set that contains D_∞.

V. Computational Issues

The first computational issue is checking the set inclusion in (16). Given an $\alpha \in [0, 1)$ we proceed as follows:

$$\begin{aligned} &R_s \subseteq \alpha \mathbb{W} \;\Leftrightarrow\; \mathrm{co}\left(\cup_{\mathbf{i}_s \in \mathcal{I}_s} \mathcal{A}_{\mathbf{i}_s} \mathbb{W}\right) \subseteq \alpha \mathbb{W} \;\Leftrightarrow \\ &\cup_{\mathbf{i}_s \in \mathcal{I}_s} \mathcal{A}_{\mathbf{i}_s} \mathbb{W} \subseteq \alpha \mathbb{W} \;\Leftrightarrow\; \mathcal{A}_{\mathbf{i}_s} \mathbb{W} \subseteq \alpha \mathbb{W},\; \forall \mathbf{i}_s \in \mathcal{I}_s \end{aligned} \tag{22}$$

where we have used Assumption 1(i) and the fact that $\mathbb{W}$ is a convex set. The equivalence between the second and the third term holds since for any compact set Ω and a convex set $\mathbb{W}$, the inclusion $\mathrm{co}(\Omega) \subseteq \mathbb{W}$ holds if and only if $\Omega \subseteq \mathbb{W}$. In order to ensure satisfaction of the set inclusion (16) we need to choose a sufficiently large s such that a finite set of simple and convex inclusions hold.

An algorithm for the computation of an RPI set $D(\alpha, s)$ satisfying $D_\infty \subseteq D(\alpha, s) \subseteq D_\infty \oplus \mathbb{B}_p^n(\varepsilon)$ for a given $\varepsilon > 0$ can be formulated from Theorem 4 by observing that the lower and upper bounds imposed on α are specified by (16) and (21) respectively. The computation of $M(s)$ depends on the calculation of $\mathcal{A}_{\mathbf{i}_k} \mathbb{W}$ for all $\mathbf{i}_k \in \mathcal{I}_k$ with $k \in \mathbb{N}_s$. When $\mathbb{W}$ is a polytope, the pair (α, s) and $M(s)$ can be calculated *without having to explicitly compute* any of the afore-mentioned sets D_k and R_k.

Suppose that $\mathbb{W} \triangleq \{w \in \mathbb{R}^n | f_j^T w \leq g_j, j \in \mathbb{N}_l\}$, where $l \in \mathbb{N}_+$. The fact that $0 \in \mathrm{interior}(\mathbb{W})$ implies that $(f_j, g_j) \in \mathbb{R}^n \times (0, \infty), \forall j \in \mathbb{N}_l$. By definition 5 and by basic properties of the support function it can be shown that (22) is satisfied if and only if

$$f_j^T \mathcal{A}_{\mathbf{i}_s} w \leq \alpha g_j,\; \forall w \in \mathbb{W} \Leftrightarrow h_{\mathbb{W}}(\mathcal{A}_{\mathbf{i}_s}^T f_j) \leq \alpha g_j \tag{23}$$

for all $\mathbf{i}_s \in \mathcal{I}_s$ and $j \in \mathbb{N}_l$. Furthermore,

$$\begin{aligned} &h_{\mathbb{W}}(\mathcal{A}_{\mathbf{i}_s}^T f_j) \leq \alpha g_j,\; \forall \mathbf{i}_s \in \mathcal{I}_s,\; \forall j \in \mathbb{N}_l && \Leftrightarrow \\ &\max_{w \in \mathbb{W}} f_j^T \mathcal{A}_{\mathbf{i}_s} w \leq \alpha g_j,\; \forall \mathbf{i}_s \in \mathcal{I}_s,\; \forall j \in \mathbb{N}_l && \Leftrightarrow \\ &\max_{\mathbf{i}_s \in \mathcal{I}_s} \max_{w \in \mathbb{W}} f_j^T \mathcal{A}_{\mathbf{i}_s} w \leq \alpha g_j,\; \forall j \in \mathbb{N}_l && \Leftrightarrow \\ &\max_{j \in \mathbb{N}_l} \frac{\max_{\mathbf{i}_s \in \mathcal{I}_s} \max_{w \in \mathbb{W}} f_j^T \mathcal{A}_{\mathbf{i}_s} w}{g_j} \leq \alpha && \end{aligned} \tag{24}$$

Then, equation (24) yields the simple observation that, given an $s \in \mathbb{N}^+$,

$$\alpha^o(s) = \max_{j \in \mathbb{N}_l} \frac{\max_{\mathbf{i}_s \in \mathcal{I}_s} \max_{w \in \mathbb{W}} f_j^T \mathcal{A}_{\mathbf{i}_s} w}{g_j} \tag{25}$$

Equation (25) allows us to calculate $\alpha^o(s)$ for a given s without having to explicitly compute the set R_s. Of course, (16) is satisfied if and only if $\alpha^o(s) \in [0, 1)$.

The second issue is the calculation of $M(s)$ without having to calculate D_s. Since $\mathbb{W}$ (and D_s) are polytopes, it is appropriate to use the infinity norm for the calculation of $M(s)$. Then:

$$M(s) = \sup_{z \in D_s} \|z\|_\infty = \min_\gamma \{\gamma \mid D_s \subseteq \mathbb{B}_\infty^n(\gamma)\}. \tag{26}$$

which is the minimal value of γ for which $D_s \subseteq \mathbb{B}^n_\infty(\gamma)$ holds. The corresponding value of γ, and hence of $M(s)$, can be computed without having to explicitly compute D_s, as shown next.

By recalling the definition of the set sequences $\{R_k\}$ and $\{D_k\}$ (see equations (10) and (12), respectively) it follows that:

$$D_s = \bigoplus_{k=0}^{s-1} R_k \tag{27}$$

It is easily shown that:

$$D_s \subseteq \mathbb{B}^n_\infty(\gamma) \Leftrightarrow \bigoplus_{k=0}^{s-1} R_k \subseteq \mathbb{B}^n_\infty(\gamma) \Leftrightarrow \bigoplus_{k=0}^{s-1} \mathcal{A}_{\mathbf{i_k}}\mathbb{W} \subseteq \mathbb{B}^n_\infty(\gamma)$$
$$\forall \mathbf{i}_k \in \mathcal{I}_k,\ \forall k \in \mathbb{N}_{s-1} \tag{28}$$

The last inclusion is satisfied if and only if the following inequalities hold:

$$\sum_{k=0}^{s-1} e_j^T \mathcal{A}_{\mathbf{i}_k} w \leq \gamma,\ \sum_{k=0}^{s-1} (-e_j^T)\mathcal{A}_{\mathbf{i}_k} w \leq \gamma$$
$$\forall w \in \mathbb{W},\ \forall \mathbf{i}_k \in \mathcal{I}_k,\ \forall k \in \mathbb{N}_{s-1},\ \forall j \in \mathbb{N}_n^+ \tag{29}$$

where e_j is the j^{th} standard basis vector in $\mathbb{R}^n$. The smallest value for γ can be computed by calculating the maximum of the terms $\sum_{k=0}^{s-1} e_j^T \mathcal{A}_{\mathbf{i}_k} w$ and $\sum_{k=0}^{s-1}(-e_j^T)\mathcal{A}_{\mathbf{i}_k} w$ for all $w \in \mathbb{W}$, $\mathbf{i}_k \in \mathcal{I}_k$, $k \in \mathbb{N}_{s-1}$ and $j \in \{1,\ldots,n\}$. Then

$$M(s) = \max_{j\in\{1,\ldots,n\}} \{\sum_{k=0}^{s-1} \max_{\mathbf{i}_k\in\mathcal{I}_k} \max_{w\in\mathbb{W}} e_j^T \mathcal{A}_{\mathbf{i}_k} w,$$
$$\sum_{k=0}^{s-1} \max_{\mathbf{i}_k\in\mathcal{I}_k} \max_{w\in\mathbb{W}} (-e_j^T)\mathcal{A}_{\mathbf{i}_k} w\} \tag{30}$$

The values for $\alpha^o(s)$ and $M(s)$ can be computed from (25) and (30). The results of the above analysis can now be used to formulate Algorithm 1 for the calculation of $D(\alpha, s)$.

Algorithm 1 Computation of an RPI outer ε-approximation of the mRPI set D_∞

Require: A, W and $\varepsilon > 0$

1: Choose any $s \in \mathbb{N}$ (ideally, set $s \leftarrow 0$).
2: **repeat**
3: Increment s by one.
4: Compute $\alpha^o(s)$ using (25) and set $\alpha \leftarrow \alpha^o(s)$.
5: Compute $M(s)$ using (30).
6: **until** $\alpha \leq \varepsilon/(\varepsilon + M(s))$
7: Compute D_s as the Minkowski sum (11) and scale it to give $D(\alpha, s) \triangleq (1-\alpha)^{-1} D_s$.

In order to reduce the computational effort for the calculation of $M(s)$ we observe that it is not necessary to calculate $\sum_{k=0}^{s-2} \max_{\mathbf{i}_k\in\mathcal{I}_k} \max_{w\in\mathbb{W}} e_j^T \mathcal{A}_{\mathbf{i}_k} w$ and $\sum_{k=0}^{s-2} \max_{\mathbf{i}_k\in\mathcal{I}_k} \max_{w\in\mathbb{W}}(-e_j^T)\mathcal{A}_{\mathbf{i}_k} w$ at each iteration of Algorithm 1. These sums would have been calculated at a previous iteration; they can be stored and then been updated in the next iteration by simply adding $\max_{\mathbf{i}_{s-1}\in\mathcal{I}_{s-1}} \max_{w\in\mathbb{W}} e_j^T \mathcal{A}_{\mathbf{i}_{s-1}} w$ and $\max_{\mathbf{i}_{s-1}\in\mathcal{I}_{s-1}} \max_{w\in\mathbb{W}}(-e_j^T)\mathcal{A}_{\mathbf{i}_{s-1}} w$ respectively.

Algorithm 1 initially sets s to a fixed value (usually 0) and increases it at each step. The values of α and $M(s)$ are calculated in each iteration using (25) and (30). The algorithm stops when the inequality (21) is satisfied, in which case the *a–priori* specified accuracy $\varepsilon > 0$ has been obtained. The ε-approximation $D(\alpha, s)$ of D_∞ can then be computed as the Minkowski sum of a finite number of sets.

The complexity of Algorithm 1 may increase as the dimension of the linear difference inclusion and q increases. However, the algorithm involves the solution of a number of linear programming problems ((25) and (30)) that can be solved more efficiently than working with set calculations (as in (16) and (17)). It is also very useful to note that if $\mathbb{W} = \{Ew \mid \|w\|_\infty \leq 1\}$, where E is non-singular, then one can compute $\alpha^o(s)$ and $M(s)$ without having to resort to solving linear programs, since $\max_{w\in\mathbb{W}} e_j^T \mathcal{A}_{\mathbf{i}_k} w = \|E^T \mathcal{A}_{\mathbf{i}_k}^T e_j\|_1$.

VI. Illustrative Example

The proposed procedure is illustrated by considering an uncertain discrete-time (4) with

$$F_1 = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix},\ F_2 = \begin{bmatrix} 0.8 & 1 \\ 0 & 1 \end{bmatrix} \tag{31}$$

and $G = G_1 = G_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. The additive disturbance set is $\mathbb{W} \triangleq \{w \in \mathbb{R}^n \mid \|w\|_\infty \leq 0.1\}$. The nominal part of the uncertain system (31) can be quadratically stabilized by the state feedback controller $K = [-1.2 \quad -1]$. Assumption 2 is satisfied with $\psi = 0.33$ and

$$P = \begin{bmatrix} 2.9048 & 0 \\ 0 & 1 \end{bmatrix}. \tag{32}$$

The sets D_k, $k = 1, 2, 3, 4$, are shown in Figure 1 together with $D(3.07\cdot 10^{-2}, 6)$ and $D(2.0134\cdot 10^{-5}, 14)$. The approximations $D(3.07 \cdot 10^{-2}, 6)$ and $D(2.0134 \cdot 10^{-5}, 14)$ have

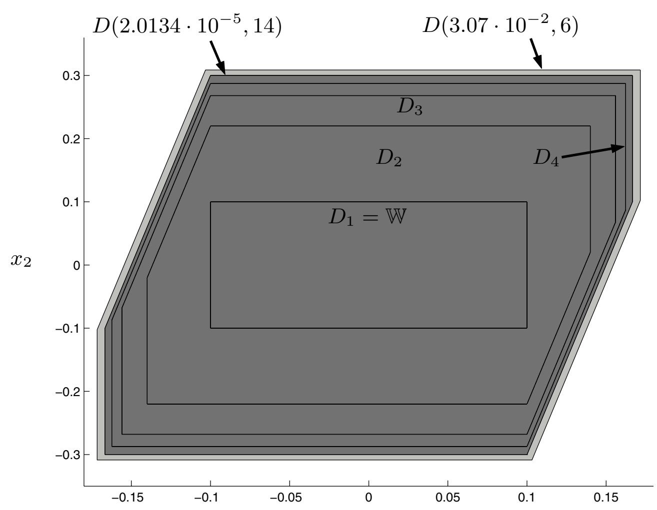

Fig. 1. Approximations $D(3.07 \cdot 10^{-2}, 6)$ and $D(2.0134 \cdot 10^{-5}, 14)$ of D_∞, and the sets D_k, $k = 1, 2, 3, 4$.

been computed for the given accuracies of $\varepsilon = 10^{-2}$ and 10^{-5} respectively.

The sequence $\{D_k\}$ is nondecreasing and $D(\alpha, s)$ decreases as α decreases or s increases. Hence both set sequences converge to D_∞. Moreover, for $\alpha = 2.0134 \cdot 10^{-5}$ and $s = 14$, we have $(1-\alpha)^{-1} \cong 1$ and hence, $D_{14} \cong D(2.0134 \cdot 10^{-5}, 14)$. Since $D_{14} \subseteq D_\infty \subseteq D(2.0134 \cdot 10^{-5}, 14)$ then $D(2.0134 \cdot 10^{-5}, 14) \cong D_\infty$.

VII. Conclusions

The novel results reported in this paper further extend the existing research for the computation and approximation of the mRPI set for autonomous linear discrete-time systems [25]. The results have been extended to address the more general and difficult case of *linear difference inclusions*. A relevant contribution is a method for the computation of the outer RPI ε-approximation, of the mRPI set for *linear difference inclusions*, for an *a priori* given $\varepsilon > 0$. The proposed method is efficient in that it involves the computation of a number of linear programming problems and simple algebraic calculations instead of less tractable calculations with sets. It is in principle possible to further improve computational aspects and this extension is a subject of current research.

The results presented in this paper can be exploited in robust control of linear difference inclusions subject to constraints and additive but bounded disturbances [5], [7], [11].

References

[1] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, pp. 1747–1767, 1999.

[2] E. G. Gilbert and I. Kolmanovsky, "Fast reference governors for systems with state and control constraints and disturbance inputs," *International Journal of Robust and Nonlinear Control*, vol. 9, no. 15, pp. 1117–41, December 1999.

[3] A. Bemporad and M. Morari, *Robustness in Identification and Control*, ser. Lecture Notes in Control and Information Sciences. Springer-Verlag, 1999, no. 245, ch. Robust model predictive control: A survey, pp. 207–226, survey paper.

[4] R. Findeisen, L. Imsland, F. Allgöwer, and B. A. Foss, "State and output feedback nonlinear model predictive control: An overview," *European Journal of Control*, vol. 9, no. 2–3, pp. 190–206, 2003, survey paper.

[5] D. Q. Mayne, "Control of constrained dynamic systems," *European Journal of Control*, vol. 7, pp. 87–99, 2001, survey paper.

[6] D. P. Bertsekas and I. B. Rhodes, "On the minimax reachability of target sets and target tubes," *Automatica*, vol. 7, pp. 233–247, 1971.

[7] F. Blanchini, "Minimum-time control for uncertain discrete-time linear systems," in *Proc. 31st IEEE Conference on Decision and Control*, vol. 3, Tuczon AZ, USA, December 1992, pp. 2629–34.

[8] D. Q. Mayne and W. R. Schroeder, "Robust time-optimal control of constrained linear systems," *Automatica*, vol. 33, no. 12, pp. 2103–2118, 1997.

[9] S. Raković, P. Grieder, M. Kvasnica, D. Q. Mayne, and M. Morari, "Computation of invariant sets for piecewise affine discrete time systems subject to bounded disturbances," in *Proc. 43rd IEEE Conference on Decision and Control*, Paradise Island, Bahamas, December 2004.

[10] I. Kolmanovsky and E. G. Gilbert, "Theory and computation of disturbance invariant sets for discrete time linear systems," *Mathematical Problems in Engineering: Theory, Methods and Applications*, vol. 4, pp. 317–367, 1998.

[11] W. Langson, I. Chryssochoos, S. V. Raković, and D. Q. Mayne, "Robust model predictive control using tubes," *Automatica*, vol. 40, pp. 125–133, 2004.

[12] D. Q. Mayne, M. Seron, and S. V. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, pp. 219–224, 2005.

[13] S. V. Raković and D. Q. Mayne, "A simple tube controller for efficient robust model predictive control of constrained linear discrete time systems subject to bounded disturbances," in *Proceedings of the 16th IFAC World Congress IFAC 2005*, Praha, Czech Republic, July 2005, Invited Session.

[14] E. De Santis, "On positively invariant sets for discrete-time linear systems with disturbance: an applications of maximal disturbance sets," *IEEE Trans. Automat. Contr.*, vol. 39, no. 1, pp. 245–249, January 1994.

[15] F. Blanchini, "Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions," *IEEE Trans. Automatic Control*, vol. 39, no. 2, pp. 428–433, 1994.

[16] E. De Santis, "On invariant sets for constrained discrete time linear systems with disturbances and parametric uncertainties," *Automatica*, vol. 33, no. 11, pp. 2033–2039, 1997.

[17] E. C. Kerrigan, "Robust constraint satisfaction: Invariant sets and predictive control," Ph.D. dissertation, University of Cambridge, 2000.

[18] K. Kouramas, "Control of linear systems with state and control constraints," Ph.D. dissertation, Imperial College London, 2003.

[19] E. De Santis, M. D. Di Benedetto, and L. Berardi, "Computation of maximal safe sets for switching systems," *IEEE Trans. Automatic Control*, vol. 49, no. 2, pp. 184–195, 2004.

[20] S. Raković and P. Grieder, "Approximations and properties of the disturbance response set of pwa systems," Automatic Control Lab, ETHZ, Switzerland, Tech. Rep. AUT04-02, 2004, `http://control.ethz.ch/`.

[21] K. I. Kouramas and J. C. Allwright, "Computation of minimal and maximal disturbance invariant sets for uncertain time-varying discrete-time linear systems," *IEE Proc. Control Theory and Applications*, submitted.

[22] S. V. Raković, "Robust control of constrained discrete-time systems: Characterization and implementation," Ph.D. dissertation, Imperial College London, 2005.

[23] J. E. Gayek, "A survey of techniques for approximating reachable and controllable set," in *Proc. 30th IEEE Conf. Decision and Control*, Brighton,U.K., 1991, pp. 1724–1729.

[24] K. Hirata and Y. Ohta, "ε–feasible approximation of the state reachable set for discrete time systems," in *Proc. 42nd IEEE Conference on Decision and Control*, Maui HI, USA, 2003, pp. 5520–5525.

[25] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne, "Invariant approximations of the minimal robustly positively invariant sets," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 406–410, 2005.

[26] B. R. Barmish and J. Sarkaran, "The propagation of parametric uncertainty via polytopes," *IEEE Trans. Automat. Contr.*, vol. 24, pp. 346–349, February 1979.

[27] E. F. S. Boyd, L. El Ghaoui and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994, vol. 15, of Studies in Applied Mathematics.

[28] L. Gurvits, "Stability of discrete linear inclusion," *Linear Algebra and its Applications*, vol. 231, pp. 47–85, 1995.

[29] K. I. Kouramas, S. V. Raković, E. C. Kerrigan, J. C. Allwright, and D. Q. Mayne, "On the Minimal Robust Positively Invariant Set for Linear Difference Inclusions," Imperial College London, Downloadable from http://www2.ee.ic.ac.uk/cap/cappp/projects/11/reports.htm, Tech. Rep. EEE/C&P/SVR/9-d/2005, 2005.

[30] J. P. Aubin, *Viability theory*, ser. Systems & Control: Foundations & Applications. Boston, Basel, Berlin: Birkhauser, 1991.

[31] A. B. Kurzhanski and T. F. Filippova, *On the Theory of Trajectory Tubes: A Mathematical Formalism for Uncertain Dynamics, Viability and Control*, ser. in: Advances in Nonlinear Dynamics and Control: A Report from Russia, A.B. Kurzhanski, ed., ser. PSCT 17. Boston, Basel, Berlin: Birkhauser, 1993.

[32] A. B. Kurzhanski, "Dynamic optimization for nonlinear target control synthesis," in *Proc. 6th IFAC Symposium – NOLCOS2004*, Stuttgart, Germany, September 2004.

[33] J. P. Aubin, *Applied Abstract Analysis*, ser. Pure & Applied Mathematics. Wiley-Interscience, 1977.

[34] R. Schneider, *Convex bodies: The Brunn-Minkowski theory*. Cambridge, England: Cambridge University Press, 1993, vol. 44, of Encyclopedia of Mathematics and its Applications.