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Optimal Control of a Two-Stage Stochastic Hybrid Manufacturing System with Poisson Arrivals and Exponential Service Times

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Abstract— Extending earlier work on single-stage stochastic hybrid system models, we consider a two-stage stochastic hybrid system where the job arrivals are represented through a Poisson process, and the service times required to attain a desired physical state are exponentially distributed dependent on the controllable process rates. For the case where the costs associated with the process rates and the inventory levels are non-decreasing convex, and the process rates take values from finite sets, we show that there exist threshold policies on both inventory levels for selecting the optimal process rates at each station.

Index Terms— Stochastic, Hybrid Systems, Two-Stage, Optimal Rate Control

I. INTRODUCTION

The term "hybrid" is used to characterize systems that include time-driven and event-driven dynamics. The former are represented by differential (difference) equations, while the latter may be described through various frameworks used for Discrete Event Systems (DES), such as timed automata, max-plus equations, queueing networks, or Petri nets (see [1]). Broadly speaking, two categories of modeling frameworks have been proposed to study hybrid systems: Those that extend event-driven models to include time-driven dynamics; and those that extend the traditional time-driven models to include event-driven dynamics (for an overview, see [2], [3], [4], [5])

The hybrid system modeling framework used in this paper falls into the first category above and is motivated by the structure of many manufacturing systems. In these systems, discrete entities (referred to as jobs) move through a network of work-centers which process the jobs so as to change their physical characteristics according to certain specifications. Associated with each job are a physical state and a temporal state. The physical state $z_{i,j}$ evolves according to time-driven dynamics modeled through differential equations

$$\dot{z}_{i,j} = f_j(z_{i,j}, u_{i,j}) \quad \text{for } i = 1, 2, \ldots \quad (1)$$
$$z_{i,j}(\tau_{i,j}) = \zeta_{i,j}^0 \quad z_{i,j}(\tau_{i,j} + s_{i,j}) = \zeta_{i,j}^d \quad (2)$$

which, depending on the particular problem being studied, describe changes in such quantities as the temperature, size, weight, chemical composition, bacteria level, or some other measure of the "quality" of the job. The temporal state of a job evolves according to event-driven dynamics, e.g., by the Lindley Equation (see in [1])

$$x_{i,1} = \max(a_i, x_{i-1,1}) + s_{i,1} \quad x_{0,1} = -\infty \quad (3)$$
$$x_{i,2} = \max(x_{i,1}, x_{i-1,2}) + s_{i,2} \quad x_{0,2} = -\infty \quad (4)$$

and includes information such as the arrival time a_i, departure times $x_{i,j}$, and service times $s_{i,j}$ of job i at work-center j dependent on the control input $u_{i,j}$ applied on job i to bring it to a desired final state $\zeta_{i,j}^d$. The interaction of time-driven with event-driven dynamics leads to a natural trade-off between temporal requirements on job completion times and physical requirements on the quality of the completed jobs. For example, while the physical state of a job can be made arbitrarily close to a desired "quality target," this usually comes at the expense of long processing times resulting in excessive inventory costs or violation of constraints on job completion deadlines. Our objective, therefore, is to formulate and solve optimal control problems associated with such trade-offs.

In [6], [7], [8], and [9], the hybrid system framework is adopted to analyze a single-stage manufacturing process assuming a deterministic setting, i.e., a known job arrival schedule and controllable service times for all jobs. An efficient algorithm to determine the optimal service times for a class of single-stage systems is presented in [8]. In [10], however, a stochastic model of a single-stage manufacturing system is studied, where the job arrivals are represented through a Poisson process with the control variable being the exponential service's process rate. Adopting an M/M/1 queueing model to describe the event driven dynamics, it is shown that when the inventory-level-dependent service process rates take values from a finite set, and the costs associated with the process rate and the inventory level are non-decreasing convex, there exists a threshold policy on the inventory level for selecting the optimal process rate. In this paper, we extend the model in [10] to a two-stage hybrid system model and show that similar threshold policies exist on inventory levels for each station.

II. PROBLEM FORMULATION

Consider the two stage serial manufacturing system depicted in Figure 1. Jobs are arriving to the system according to a Poisson process with rate λ, while servers process one job

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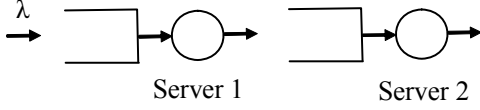
Fig. 1. Two-stage serial manufacturing system

at a time on a first-come first-served non-preemptive basis (i.e. a job in service can not be interrupted until its service completion). Service times for both servers are exponentially distributed and server l ($l = 1, 2$) operates with rate $u^l \in U_l$ where U_l 's are finite sets such that

$$U_l = \{u_1^l, u_2^l, ..., u_{m_l}^l\} \tag{5}$$

The rates u_i^l are indexed so that $u_i^l < u_{i+1}^l$ for $i = 1, ..., m_l-1$.

Defining a rate $\gamma = \lambda + u_{m_1}^1 + u_{m_2}^2$, this system can be modeled as a discrete-time Markov Chain (DTMC). The state space for this DTMC can be defined as

$$S = \{(i, j) : i, j \in \mathbb{Z}_0^+\}$$

where the state (i, j) describes the system with i jobs in the first server and j jobs in the second server. The transition probabilities for this DTMC are

$$P_{(0,0),(k,l)} = \begin{cases} \frac{\lambda}{\gamma} & (k,l) = (1,0) \\ 1 - \frac{\lambda}{\gamma} & (k,l) = (0,0) \\ 0 & \text{otherwise} \end{cases}$$

$$P_{(i,0),(k,l)} = \begin{cases} \frac{\lambda}{\gamma} & (k,l) = (i+1,0) \\ \frac{u^1(i,0)}{\gamma} & (k,l) = (i-1,1) \\ 1 - \frac{\lambda+u^1(i,0)}{\gamma} & (k,l) = (i,0) \\ 0 & \text{otherwise} \end{cases}$$

$$P_{(0,j),(k,l)} = \begin{cases} \frac{\lambda}{\gamma} & (k,l) = (1,j) \\ \frac{u^2(0,j)}{\gamma} & (k,l) = (0,j-1) \\ 1 - \frac{\lambda+u^2(0,j)}{\gamma} & (k,l) = (0,j) \\ 0 & \text{otherwise} \end{cases}$$

$$P_{(i,j),(k,l)} = \begin{cases} \frac{\lambda}{\gamma} & (k,l) = (i+1,j) \\ \frac{u^1(i,j)}{\gamma} & (k,l) = (i-1,j+1) \\ \frac{u^2(i,j)}{\gamma} & (k,l) = (i,j-1) \\ 1 - \frac{\lambda+u^1(i,j)+u^2(i,j)}{\gamma} & (k,l) = (i,j) \\ 0 & \text{otherwise} \end{cases}$$

Assuming that the servers start with system sizes q_0^l, the infinite horizon problem we consider is to determine the stationary state-dependent rate setting policy π such that the discounted cost

$$V_\pi(q_0^1, q_0^2) = E_\pi \left[\sum_{k=0}^{\infty} \alpha^k C(q_k^1, q_k^2, u_k^1, u_k^2) \right] \tag{6}$$

is minimized. Note that in (6), α is the discount factor, q_k^l denotes the system size for server l, and the pair

$$(u_k^1, u_k^2) = \pi(q_k^1, q_k^2)$$

denotes the corresponding policy-determined process rates for the servers. The one step cost $C(q_k^1, q_k^2, u_k^1, u_k^2)$ in (6) is assumed to be separable, i.e.,

$$C(q_k^1, q_k^2, u_k^1, u_k^2) = b_1(q_k^1) + b_2(q_k^2) + c_1(u_k^1) + c_2(u_k^2)$$

where the inventory costs $b_l(.)$ and the service costs $c_l(.)$ are non-decreasing convex functions.

III. OPTIMAL RATE CONTROL POLICY

Applying Dynamic Programming (DP), in e.g. [11], to solve the optimal control problem (6), the discounted cost-to-go DP equation for state (i, j) becomes

$$V_{n+1}(i,j) = \min_{\substack{u^1(i,j)\in U_1 \\ u^2(i,j)\in U_2}} \left\{ \begin{array}{c} b_1(i) + b_2(j) \\ +c_1(u^1(i,j)) + c_2(u^2(i,j)) \\ +\alpha\frac{\lambda}{\gamma}V_n(i+1,j) \\ +\alpha\frac{u^1(i,j)}{\gamma}V_n(i-1,j+1) \\ +\alpha\frac{u^2(i,j)}{\gamma}V_n(i,j-1) \\ +\alpha V_n(i,j) \\ -\alpha\frac{\lambda+u^1(i,j)+u^2(i,j)}{\gamma}V_n(i,j) \end{array} \right\} \tag{7}$$

Note that for all $i, j, n \in \mathbb{Z}_0^+$, we have assumed

$$V_n(-1, j+1) = V_n(0, j) \tag{8}$$

and

$$V_n(i, -1) = V_n(i, 0) \tag{9}$$

Let us define

$$\begin{aligned} \Delta V_n^{(1)}(i,j) &= V_n(i,j) - V_n(i-1,j+1) \\ \Delta V_n^{(2)}(i,j) &= V_n(i,j) - V_n(i,j-1) \end{aligned}$$

By (8) and (9), for all $i, j, n \in \mathbb{Z}_0^+$

$$\Delta V_n^{(1)}(0,j) = \Delta V_n^{(2)}(i,0) = 0$$

The following theorem establishes the optimal control policy:

Theorem 1: The optimal controls for the $(n+1)$th step are

$$u_{n+1}^1(i,j) = \arg\min_{u^1 \in U_1} \left\{ c_1(u^1) - \alpha\frac{u^1}{\gamma}\Delta V_n^{(1)}(i,j) \right\}$$

$$u_{n+1}^2(i,j) = \arg\min_{u^2 \in U_2} \left\{ c_2(u^2) - \alpha\frac{u^2}{\gamma}\Delta V_n^{(2)}(i,j) \right\}$$

Proof: The cost-to-go equation in (7) can be written as

$$\begin{aligned} V_{n+1}(i,j) = {} & b_1(i) + b_2(j) + \alpha V_n(i,j) \\ & + \alpha\frac{\lambda}{\gamma}\left[V_n(i+1,j) - V_n(i,j)\right] \\ & + \min_{u^1(i,j)\in U_1} \left\{ \begin{array}{c} c_1(u^1(i,j)) \\ -\alpha\frac{u^1(i,j)}{\gamma}\Delta V_n^{(1)}(i,j) \end{array} \right\} \\ & + \min_{u^2(i,j)\in U_2} \left\{ \begin{array}{c} c_2(u^2(i,j)) \\ -\alpha\frac{u^2(i,j)}{\gamma}\Delta V_n^{(2)}(i,j) \end{array} \right\} \end{aligned}$$

Hence, the result follows. ∎

Corollary 1: The optimal rates for empty servers are the lowest cost rates for those servers, i.e.,

$$
\begin{aligned}
u^1(0,j) &= \arg\min_{u^1 \in U_1} \{c_1(u^1)\} = u_1^1 \\
u^2(i,0) &= \arg\min_{u^2 \in U_2} \{c_2(u^2)\} = u_1^2
\end{aligned}
$$

Proof: Since $\Delta V_n^{(1)}(0,j)$ and $\Delta V_n^{(2)}(i,0)$ are zero for all $i,j \in \mathbb{Z}_0^+$, the result follows from Theorem 1. ∎

Since it is computationally impossible to solve for $\Delta V_n^{(1)}(i,j)$ and $\Delta V_n^{(2)}(i,j)$ over all i, j and n, determining the optimal process rates for all states (i,j) analytically is not feasible. Instead, we will exploit monotonicity properties of $\Delta V_n^{(1)}(i,j)$ and $\Delta V_n^{(2)}(i,j)$ to establish inventory level thresholds for the optimal process rates.

Using notation $c_k^l = c_l(u_k^l)$, we can define the thresholds β_k^l as

$$
\beta_k^l = \begin{cases} 0 & k = 0 \\ \frac{\gamma}{\alpha}\left(\frac{c_{k+1}^l - c_k^l}{u_{k+1}^l - u_k^l}\right) & 0 < k < m_l \\ \infty & k = m_l \end{cases} \tag{10}
$$

Given that $c_l(.)$ is a non-decreasing convex function, β_k^l is also non-decreasing in k for $l = 1, 2$, which allows us to establish the following optimality condition:

Lemma 1: The optimal process rate $u^l(i,j) = u_k^l$ if and only if $\beta_{k-1}^l \le \Delta V_n^{(l)}(i,j) \le \beta_k^l$

Proof: ($\Leftarrow$) Let us assume that

$$
\beta_{k-1}^l \le \Delta V_n^{(l)}(i,j) \le \beta_k^l
$$

and

$$
u^l(i,j) = u_t^l \neq u_k^l
$$

Then,

$$
c_t^l - \alpha\frac{u_t^l}{\gamma}\Delta V_n^{(l)}(i,j) < c_k^l - \alpha\frac{u_k^l}{\gamma}\Delta V_n^{(l)}(i,j)
$$

If $t > k$ then

$$
\beta_k^l \le \frac{\gamma}{\alpha}\frac{c_t^l - c_k^l}{u_t^l - u_k^l} < \Delta V_n^{(l)}(i,j)
$$

If $t < k$ then

$$
\beta_{k-1}^l \ge \frac{\gamma}{\alpha}\frac{c_k^l - c_t^l}{u_k^l - u_t^l} > \Delta V_n^{(l)}(i,j)
$$

Both cases contradict our assumption, so the optimal process rate $u^l(i,j) = u_k^l$.

($\Rightarrow$)Conversely, let us assume that $u^l(i,j) = u_k^l$ and consider the following cases: If

$$
\Delta V_n^{(l)}(i,j) > \beta_k^l = \frac{\gamma}{\alpha}\left(\frac{c_{k+1}^l - c_k^l}{u_{k+1}^l - u_k^l}\right)
$$

then

$$
c_k^l - \alpha\frac{u_k^l}{\gamma}\Delta V_n^{(l)}(i,j) > c_{k+1}^l - \alpha\frac{u_{k+1}^l}{\gamma}\Delta V_n^{(l)}(i,j)
$$

which contradicts the $u^l(i,j) = u_k^l$ assumption. Similarly, if

$$
\Delta V_n^{(l)}(i,j) < \beta_{k-1}^l = \frac{\gamma}{\alpha}\left(\frac{c_k^l - c_{k-1}^l}{u_k^l - u_{k-1}^l}\right)
$$

then

$$
c_{k-1}^l - \alpha\frac{u_{k-1}^l}{\gamma}\Delta V_n^{(l)}(i,j) < c_k^l - \alpha\frac{u_k^l}{\gamma}\Delta V_n^{(l)}(i,j)
$$

which contradicts the $u^l(i,j) = u_k^l$ assumption. ∎

An immediate corollary of Lemma 1 is the following: If $\Delta V_n^{(l)}(i_1,j_1) \le \Delta V_n^{(l)}(i_2,j_2)$ then the optimal process rates satisfy $u^l(i_1,j_1) \le u^l(i_2,j_2)$. Depending on the $[\beta_{k-1}^l, \beta_k^l]$ interval that the $\Delta V_n^{(l)}(i,j)$ values fall in, an optimal process rate u_k^l is selected.

The following theorem establishes monotonicity properties of the $\Delta V_n^{(l)}(i,j)$ and implies the existence of a threshold policy.

Theorem 2: Given $b(.)$ and $c(.)$ as non-decreasing convex functions, for all $i,j,n \in \mathbb{Z}_0^+$.

i) $\Delta V_n^{(1)}(i,j)$ is non-decreasing in i and non-increasing in j, i.e.,

$$
\begin{aligned}
\Delta V_n^{(1)}(i,j) &\le \Delta V_n^{(1)}(i+1,j) \\
\Delta V_n^{(1)}(i,j) &\ge \Delta V_n^{(1)}(i,j+1)
\end{aligned}
$$

ii) $\Delta V_n^{(2)}(i,j)$ is non-decreasing in both i and j, i.e.,

$$
\begin{aligned}
\Delta V_n^{(2)}(i,j) &\le \Delta V_n^{(2)}(i+1,j) \\
\Delta V_n^{(2)}(i,j) &\le \Delta V_n^{(2)}(i,j+1)
\end{aligned}
$$

Proof: (By Induction)
Let us define

$$
\begin{aligned}
\delta V_{k,i,j}^l(1) &= \Delta V_k^{(l)}(i+1,j) - \Delta V_k^{(l)}(i,j) \\
\delta V_{k,i,j}^l(2) &= \Delta V_k^{(l)}(i,j+1) - \Delta V_k^{(l)}(i,j)
\end{aligned}
$$

We need to show that for all k

$$
\begin{aligned}
\delta V_{k,i,j}^1(1) &\ge 0, & \delta V_{k,i,j}^1(2) &\le 0 \\
\delta V_{k,i,j}^2(1) &\ge 0, & \delta V_{k,i,j}^2(2) &\ge 0
\end{aligned}
$$

For $k = 0$, since $V_0(i,j) = 0$ for all i,j,

$$
\begin{aligned}
\delta V_{0,i,j}^1(1) &= 0, & \delta V_{0,i,j}^1(2) &= 0 \\
\delta V_{0,i,j}^2(1) &= 0, & \delta V_{0,i,j}^2(2) &= 0
\end{aligned}
$$

Next, assume that for $k = n$, for all i,j

$$
\begin{aligned}
\delta V_{n,i,j}^1(1) &\ge 0, & \delta V_{n,i,j}^1(2) &\le 0 \\
\delta V_{n,i,j}^2(1) &\ge 0, & \delta V_{n,i,j}^2(2) &\ge 0
\end{aligned}
$$

We need to show that for $k = n+1$ and for all i,j

$$
\begin{aligned}
\delta V_{n+1,i,j}^1(1) &\ge 0, & \delta V_{n+1,i,j}^1(2) &\le 0 \\
\delta V_{n+1,i,j}^2(1) &\ge 0, & \delta V_{n+1,i,j}^2(2) &\ge 0
\end{aligned}
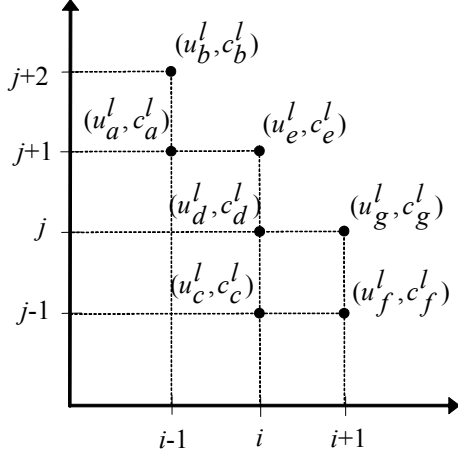$$

inequalities are satisfied.

Fig. 2. Optimal process rates and their costs for step $(n+1)$

Let us assume that optimal process rates from Theorem 1 and their costs are calculated (given in Figure 2.) Then, $\Delta V_{n+1}^{(1)}(i,j)$ and $\Delta V_{n+1}^{(2)}(i,j)$ can be given as

$$
\begin{aligned}
\Delta V_{n+1}^{(1)}(i,j) &= b_1(i) - b_1(i-1) + b_2(j) - b_2(j+1) \\
&\quad + \alpha \Delta V_n^{(1)}(i,j) + \alpha \frac{\lambda}{\gamma} \delta V_{n,i,j}^1(1) \\
&\quad + c_d^1 - \alpha \frac{u_d^1}{\gamma} \Delta V_n^{(1)}(i,j) \\
&\quad - c_a^1 + \alpha \frac{u_a^1}{\gamma} \Delta V_n^{(1)}(i-1,j+1) \\
&\quad + c_d^2 - \alpha \frac{u_d^2}{\gamma} \Delta V_n^{(2)}(i,j) \\
&\quad - c_a^2 + \alpha \frac{u_a^2}{\gamma} \Delta V_n^{(2)}(i-1,j+1)
\end{aligned}
$$

$$
\begin{aligned}
\Delta V_{n+1}^{(2)}(i,j) &= b_2(j) - b_2(j-1) + \alpha \Delta V_n^{(2)}(i,j) \\
&\quad + \alpha \frac{\lambda}{\gamma} \delta V_{n,i,j}^2(1) + c_d^1 - \alpha \frac{u_d^1}{\gamma} \Delta V_n^{(1)}(i,j) \\
&\quad - c_c^1 + \alpha \frac{u_c^1}{\gamma} \Delta V_n^{(1)}(i,j-1) \\
&\quad + c_d^2 - \alpha \frac{u_d^2}{\gamma} \Delta V_n^{(2)}(i,j) \\
&\quad - c_c^2 + \alpha \frac{u_c^2}{\gamma} \Delta V_n^{(2)}(i,j-1)
\end{aligned}
$$

Note that by the induction hypothesis for all i,j

$$
\begin{aligned}
&\Delta V_n^{(2)}(i,j+1) - \Delta V_n^{(2)}(i+1,j) \quad (11) \\
&= [V_n(i,j+1) - V_n(i,j)] \\
&\quad - [V_n(i+1,j) - V_n(i+1,j-1)] \\
&= [V_n(i+1,j-1) - V_n(i,j)] \\
&\quad - [V_n(i+1,j) - V_n(i,j+1)] \\
&= -\delta V_{n,i+1,j-1}^1(2) \geq 0
\end{aligned}
$$

hence by Lemma 1 $u_{n+1}^2(i+1,j) \leq u_{n+1}^2(i,j+1)$ holds for all i,j. This enables us to claim the following inequalities:

$$
\begin{aligned}
u_b^1 &\leq u_a^1 \leq u_e^1 \leq u_d^1 \leq u_g^1 \leq u_f^1 \\
u_d^1 &\leq u_c^1 \leq u_f^1 \\
u_c^2 &\leq u_f^2 \leq u_d^2 \leq u_g^2 \leq u_e^2 \leq u_b^2 \\
u_d^2 &\leq u_a^2 \leq u_e^2
\end{aligned}
$$

Step 1: (Proof of $\delta V_{n+1,i,j}^1(1) \geq 0$)
We have

$$
\begin{aligned}
\delta V_{n+1,i,j}^1(1) &= [b_1(i+1) - b_1(i)] - [b_1(i) - b_1(i-1)] \\
&\quad + \alpha \frac{\lambda}{\gamma} \delta V_{n,i+1,j}^1(1) + T_1^{(1)} + T_2^{(1)}
\end{aligned}
$$

where

$$
\begin{aligned}
T_1^{(1)} &= \alpha \frac{u_{m_1}^1 - u_g^1}{\gamma} \delta V_{n,i,j}^1(1) + \alpha \frac{u_a^1}{\gamma} \delta V_{n,i-1,j+1}^1(1) \\
&\quad + \left[(c_g^1 - c_d^1) - \alpha \frac{u_g^1 - u_d^1}{\gamma} \Delta V_n^{(1)}(i,j) \right] \\
&\quad - \left[(c_e^1 - c_a^1) - \alpha \frac{u_e^1 - u_a^1}{\gamma} \Delta V_n^{(1)}(i,j+1) \right]
\end{aligned}
$$

$$
\begin{aligned}
T_2^{(1)} &= \alpha \frac{u_{m_2}^2 - u_g^2}{\gamma} \delta V_{n,i,j}^1(1) + \alpha \frac{u_g^2}{\gamma} \delta V_{n,i,j-1}^1(1) \quad (12) \\
&\quad + \alpha \frac{u_a^2 - u_g^2}{\gamma} \delta V_{n,i-1,j+1}^2(1) \\
&\quad + \left[(c_g^2 - c_d^2) - \alpha \frac{u_g^2 - u_d^2}{\gamma} \Delta V_n^{(2)}(i,j) \right] \\
&\quad - \left[(c_e^2 - c_a^2) - \alpha \frac{u_e^2 - u_a^2}{\gamma} \Delta V_n^{(2)}(i,j+1) \right]
\end{aligned}
$$

Note that we can manipulate (12) to derive an equivalent definition for $T_2^{(1)}$:

$$
\begin{aligned}
T_2^{(1)} &= \alpha \frac{u_{m_2}^2 - u_a^2}{\gamma} \delta V_{n,i,j}^1(1) + \frac{\alpha}{\gamma} u_a^2 \delta V_{n,i,j-1}^1(1) \quad (13) \\
&\quad - \alpha \frac{u_g^2 - u_a^2}{\gamma} \delta V_{n,i+1,j-1}^1(2) \\
&\quad + \left[(c_a^2 - c_d^2) - \frac{\alpha}{\gamma} (u_a^2 - u_d^2) \Delta V_n^{(2)}(i,j) \right] \\
&\quad - \left[(c_e^2 - c_g^2) - \frac{\alpha}{\gamma} (u_e^2 - u_g^2) \Delta V_n^{(2)}(i,j+1) \right]
\end{aligned}
$$

Observe that
1. Since $b_1(.)$ is non-decreasing convex in i,

$$[b_1(i+1) - b_1(i)] - [b_1(i) - b_1(i-1)] \geq 0$$

2. By the induction hypothesis,

$$\alpha \frac{\lambda}{\gamma} \delta V_{n,i+1,j}^1(1) \geq 0$$

3. By the induction hypothesis, and since $u_{m_1}^1 \geq u_g^1$,

$$\alpha \frac{u_{m_1}^1 - u_g^1}{\gamma} \delta V_{n,i,j}^1(1) + \alpha \frac{u_a^1}{\gamma} \delta V_{n,i-1,j+1}^1(1) \geq 0$$

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4. a. If $u_g^1 = u_d^1$, then

$$\left(c_g^1 - c_d^1\right) - \alpha \frac{u_g^1 - u_d^1}{\gamma} \Delta V_n^{(1)}(i,j) = 0$$

4. b. If $u_g^1 > u_d^1$, then

$$\Delta V_n^{(1)}(i,j) \leq \beta_d^1 \leq \frac{\gamma}{\alpha} \frac{(c_g^1 - c_d^1)}{(u_g^1 - u_d^1)}$$

Hence,

$$\left(c_g^1 - c_d^1\right) - \alpha \frac{u_g^1 - u_d^1}{\gamma} \Delta V_n^{(1)}(i,j) \geq 0$$

5. a. If $u_e^1 = u_a^1$, then

$$\left(c_e^1 - c_a^1\right) - \alpha \frac{u_e^1 - u_a^1}{\gamma} \Delta V_n^{(1)}(i,j+1) = 0$$

5. b. If $u_e^1 > u_a^1$, then

$$\Delta V_n^{(1)}(i,j+1) \geq \beta_{e-1}^1 \geq \frac{\gamma}{\alpha} \frac{(c_e^1 - c_a^1)}{(u_e^1 - u_a^1)}$$

Hence,

$$\left(c_e^1 - c_a^1\right) - \alpha \frac{u_e^1 - u_a^1}{\gamma} \Delta V_n^{(1)}(i,j+1) \leq 0$$

From 3, 4, and 5, we establish that $T_1^{(1)} \geq 0$. In order to show that $\delta V_{n+1,i,j}^1(1) \geq 0$, we will also show $T_2^{(1)} \geq 0$. For this purpose we will consider the $u_g^2 \leq u_a^2$ case with (12) and the $u_g^2 > u_a^2$ case with (13).
Case 1: $u_g^2 \leq u_a^2$
6. By the induction hypothesis, and that $u_g^2 \leq u_a^2 \leq u_{m_2}^2$,

$$\alpha \frac{u_{m_2}^2 - u_g^2}{\gamma} \delta V_{n,i,j}^1(1) + \alpha \frac{u_g^2}{\gamma} \delta V_{n,i,j-1}^1(1) \geq 0$$
$$\alpha \frac{u_a^2 - u_g^2}{\gamma} \delta V_{n,i-1,j+1}^2(1) \geq 0$$

7. a. If $u_g^2 = u_d^2$, then

$$\left(c_g^2 - c_d^2\right) - \alpha \frac{u_g^2 - u_d^2}{\gamma} \Delta V_n^{(2)}(i,j) = 0$$

7. b. If $u_g^2 > u_d^2$, then

$$\Delta V_n^{(2)}(i,j) \leq \beta_d^2 \leq \frac{\gamma}{\alpha} \frac{(c_g^2 - c_d^2)}{(u_g^2 - u_d^2)}$$

Hence,

$$\left(c_g^2 - c_d^2\right) - \alpha \frac{u_g^2 - u_d^2}{\gamma} \Delta V_n^{(2)}(i,j) \geq 0$$

8. a. If $u_e^2 = u_a^2$, then

$$\left(c_e^2 - c_a^2\right) - \alpha \frac{u_e^2 - u_a^2}{\gamma} \Delta V_n^{(2)}(i,j+1) = 0$$

8. b. If $u_e^2 > u_a^2$, then

$$\Delta V_n^{(2)}(i,j+1) \geq \beta_{e-1}^2 \geq \frac{\gamma}{\alpha} \frac{(c_e^2 - c_a^2)}{(u_e^2 - u_a^2)}$$

Hence,

$$\left(c_e^2 - c_a^2\right) - \alpha \frac{u_e^2 - u_a^2}{\gamma} \Delta V_n^{(2)}(i,j+1) \leq 0$$

Case 2: $u_g^2 > u_a^2$
9. By the induction hypothesis, by $u_a^2 < u_g^2 \leq u_{m_2}^2$, and by the argument in (11)

$$\alpha \frac{u_{m_2}^2 - u_a^2}{\gamma} \delta V_{n,i,j}^1(1) + \frac{\alpha}{\gamma} u_a^2 \delta V_{n,i,j-1}^1(1) \geq 0$$
$$-\alpha \frac{u_g^2 - u_a^2}{\gamma} \delta V_{n,i+1,j-1}^1(2) \geq 0$$

10. a. If $u_a^2 = u_d^2$, then

$$\left(c_a^2 - c_d^2\right) - \frac{\alpha}{\gamma}(u_a^2 - u_d^2)\Delta V_n^{(2)}(i,j) = 0$$

10. b. If $u_a^2 > u_d^2$

$$\Delta V_n^{(2)}(i,j) \leq \beta_d^2 \leq \frac{\gamma}{\alpha} \frac{(c_a^2 - c_d^2)}{(u_a^2 - u_d^2)}$$

Hence,

$$\left(c_a^2 - c_d^2\right) - \frac{\alpha}{\gamma}(u_a^2 - u_d^2)\Delta V_n^{(2)}(i,j) \geq 0$$

11. a. If $u_e^2 = u_g^2$, then

$$\left(c_e^2 - c_g^2\right) - \frac{\alpha}{\gamma}(u_e^2 - u_g^2)\Delta V_n^{(2)}(i,j+1) = 0$$

11. b. If $u_e^2 > u_g^2$, then

$$\Delta V_n^{(2)}(i,j+1) \geq \beta_{e-1}^2 \geq \frac{\gamma}{\alpha} \frac{(c_e^2 - c_g^2)}{(u_e^2 - u_g^2)}$$

Hence,

$$\left(c_e^2 - c_g^2\right) - \alpha \frac{u_e^2 - u_g^2}{\gamma} \Delta V_n^{(2)}(i,j+1) \leq 0$$

For both cases we showed that $T_2^{(1)} \geq 0$ completing this step, proving that $\delta V_{n+1,i,j}^1(1) \geq 0$.

Other steps of the proof are similar and omitted for space considerations. The complete proof is given in [12]. ∎

It follows from Lemma 1 and Theorem 2 that an inventory level threshold policy is optimal for determining the process rates. Such thresholds are depicted in the following numerical example.

Note that the three possible events may affect the process rates as follows: an arrival to the system has the potential of increasing the process rates for both servers, a departure from the first server has the potential of decreasing the process rate for the first server and increasing the process rate for the second server, and a departure from the second server has the potential of increasing the process rate for the first server and decreasing the process rate for the second server.

IV. Numerical Example

Let the arrival rate to the two-stage serial manufacturing system be $\lambda = 17$ while the service rate sets and respective costs of operation for both stages are

$$
\begin{aligned}
U_1 &= \{30, 50, 70\} \\
c_1(30) &= 4, \quad c_1(50) = 7, \quad c_1(70) = 12 \\
U_2 &= \{40, 60, 90\} \\
c_2(40) &= 2, \quad c_2(60) = 6, \quad c_2(90) = 15
\end{aligned}
$$

The costs of holding (i, j) inventory is

$$
b_1(i) = 3i, \quad b_2(j) = 5j
$$

The optimal process rates obtained by the Genetic Algorithm (see in [13]) are shown in Figure 3 for the first server and in Figure 4 for the second server:
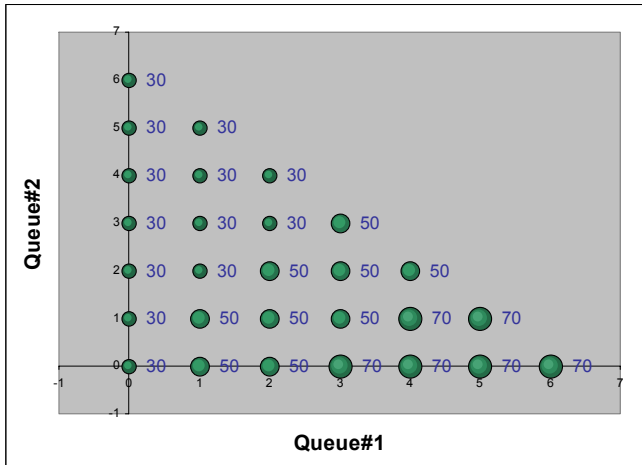

Fig. 3.   Optimal process rates for server 1

Fig. 4.   Optimal process rates for server 2

V. Conclusion

In this study, we modeled a stochastic two-stage manufacturing system with two M/M/1 queueing systems in series. The controllable process rates for both stations took values from finite sets. For the case where the single step costs associated with the process rates and the inventory levels are non-decreasing convex, we show that there exist threshold policies on both inventory levels for selecting the optimal process rates.

Extending these results for the N-machine series production line is the subject of ongoing research.

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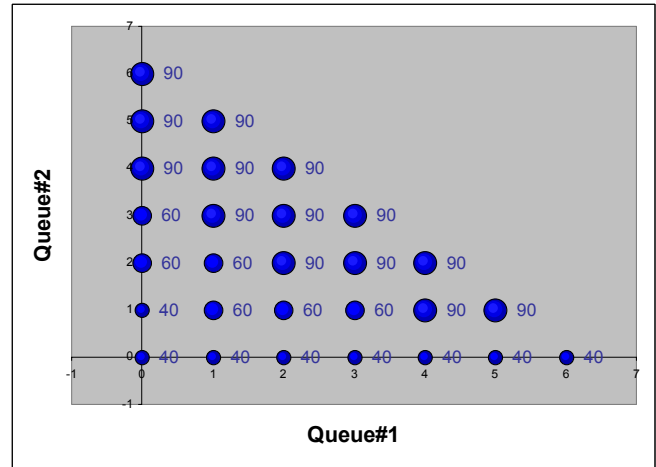[3] P. Antsaklis, W. Kohn, M. Lemmon, A. Nerode, and S. Sastry, eds., *Hybrid Systems*. Springer-Verlag, 1998.

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