Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Gain Scheduling Controller Synthesis with Spline-Type Parameter-Dependent Quadratic Forms via Dilated Linear Matrix Inequalities

Yoshiro Hamada, Isao Yamaguchi and Takashi Kida

Abstract— **This paper proposes a new synthesis of a gain scheduling controller based on the approximation of Lyapunov matrices by spline functions. The synthesis condition is described as dilated linear matrix inequalities which can be solved numerically. While in a previous study the derived feedback gains always have the same knots as the approximated Lyapunov matrices, our condition enables the feedback gains to have fewer knots without conservatism. Scheduled gains which are piecewise-linear on a parameter can also be obtained by adding constraints to the proposed synthesis condition.**

I. INTRODUCTION

Gain scheduling control techniques have been considered applicable to and useful for linear parameter varying (LPV) systems, and have been studied for the past two decades. The basic idea of gain scheduling control analysis and synthesis is to search for adequate Lyapunov functions which ensure the stability and performance of the closed-loop system. In previous studies, gain scheduling techniques for polytopic or other LPV systems have been developed by restricting the Lyapunov functions to be fixed functions (for example, [1], [2], [3]), although several studies have shown that parameter-dependent Lyapunov functions give less conservative performance than fixed functions [4].

Recently, parameter-dependent linear matrix inequality (LMI) problems have been studied, by which gain scheduling techniques using parameter-dependent Lyapunov functions are characterized as shown in [5]. While those problems are inherently equivalent to an infinite number of LMIs corresponding to each value of the parameter and are hard to solve numerically, sufficient conditions to parameter-dependent LMIs have been derived as finite sets of LMIs by some relaxation techniques[6],[7],[8],[9]. Spline-type functions were introduced in [10] and [11] to approximate the solution to a class of parameter-dependent convex differential inequalities which includes parameter-dependent LMIs, and a sufficient condition with a finite number of LMIs was derived by dividing the region of the parameter's values. Since the number of "knots" of the spline-type function coincides with the number of elements in the division of the parameter region, a finer division gives a more accurate approximation of the solution. The condition was also proved to be necessary with a sufficiently fine division of the region.

Y. Hamada and I. Yamaguchi are with Institute of Space Technology and Aeronautics, Japan Aerospace Exploration Agency, 7-44-1 Jindaiji Higashi-machi, Chofu-shi, Tokyo 181-8522, JAPAN hamada.yoshiro@jaxa.jp, yamaguchi.isao@jaxa.jp

T. Kida is with the Graduate School of Information Systems, the University of Electro-Communications, 1-5-1 Chofugaoka, Chofu-shi, Tokyo 182-8585, JAPAN kida@mce.uec.ac.jp

A gain scheduling controller synthesis for LPV systems by this approach was also shown in [12]. Although a feedback gain which achieves L_2 gain performance can be derived using the synthesis, the derived gain has always the same knots as the approximated Lyapunov matrix, even if there exists a feedback gain with fewer knots which achieves required performance. That is, when the region of the parameter's value is divided more finely in order to achieve better performance, the number of knots of the scheduled gain always increase. Since a greater number of knots of the scheduled gain requires a large control program, it is not favorable for implementation.

We propose a new synthesis of a gain scheduling controller which overcomes the above-mentioned deficiency; that is, a design method for a scheduled gain which achieves an L_2 gain performance with fewer knots (if such a scheduled gain exists) than those of an approximated Lyapunov matrix. It has been shown that parameter-dependent Lyapunov functions can be derived for polytopic LPV systems synthesis via so-called dilated LMIs[13] (or extended LMIs[14]). The advantage of the dilated LMI approach is "separation[14]" between the Lyapunov matrix and some of the dynamic system matrices in the derived conditions, and this property is utilized to obtain a feedback gain with fewer knots than the Lyapunov matrix. We also show that scheduled gains that are piecewise-linear on a parameter, which do not require complex calculations for scheduling but simply interpolation, can be obtained using our synthesis by imposing some constraints. Although such scheduled gains can also be derived from a former result [1] in which the Lyapunov matrix is fixed (i.e. parameter-independent), our approach is less conservative because the Lyapunov matrix in our synthesis can be parameter-dependent. The resulting gains are presented as an example.

This paper is organized as follows. An LPV system that will be considered here is presented in Section 2, and the results of previous studies[5],[12] are reviewed. The parameter-dependent condition which assures L_2 gain performance is rewritten in a dilated LMI form and reduced to a finite set of dilated LMIs in Section 3. Section 4 shows examples of derived feedback gains using the proposed synthesis.

0-7803-9568-9/05/$20.00 ©2005 IEEE

II. Problem Formulation

We focus our attention only on the state feedback case, and consider the following LPV plant:

$$\Sigma : \begin{cases} \dot{x} = A(\theta)x + B(\theta)w + B_u(\theta)u, \\ z = C(\theta)x + D(\theta)w + D_u(\theta)u, \end{cases} \tag{1}$$

where $x \in \boldsymbol{R}^n$ is the plant state, $w \in \boldsymbol{R}^m$ is the exogenous input, $u \in \boldsymbol{R}^{m_u}$ is the control input and $z \in \boldsymbol{R}^p$ is the controlled output. The scheduling parameter $\theta(t)$ is a differentiable function whose values are supposed to lie in the region

$$\begin{array}{l} \theta(t) \in \Theta_{\text{val}} := [\underline{\theta}, \overline{\theta}], \\ \dot{\theta}(t) = \omega(t) \in \Omega_{\text{val}} := [\underline{\omega}, \overline{\omega}]. \end{array} \tag{2}$$

In this discussion, the six matrices in (1) are assumed to be piecewise-linear functions of θ; for example, $A(\theta)$ is expressed as:

$$\begin{array}{rcl} A(\theta) & = & A_i^{\Sigma} + \dfrac{\theta - \theta_i^{\Sigma}}{\theta_{i+1}^{\Sigma} - \theta_i^{\Sigma}}(A_{i+1}^{\Sigma} - A_i^{\Sigma}), \\ \theta & \in & [\theta_i^{\Sigma}, \theta_{i+1}^{\Sigma}], i = 0, 1, \ldots, N_{\Sigma}, \end{array} \tag{3}$$

where θ_i^{Σ} are knots of $A(\theta)$, $\underline{\theta} = \theta_0^{\Sigma} < \theta_1^{\Sigma} < \ldots < \theta_{N_\Sigma}^{\Sigma} < \theta_{N_\Sigma+1}^{\Sigma} = \overline{\theta}$. We denote this division of Θ_{val} by $D^{\Sigma} = \{\underline{\theta} = \theta_0^{\Sigma}, \theta_1^{\Sigma}, \ldots, \theta_{N_\Sigma}^{\Sigma}, \theta_{N_\Sigma+1}^{\Sigma} = \overline{\theta}\}$. The other matrices are also assumed to be piecewise-linear functions of the same type with the common division D^{Σ}.

Consider the scheduled static state feedback controller:

$$\Gamma : u = F(\theta)x. \tag{4}$$

We denote the closed-loop system composed of Σ and Γ as $\Sigma\Gamma$. An LMI condition for stability and L_2 gain performance of an LPV system has been derived in a previous study, and can be applied to the closed-loop system:

Lemma 1: [5] The closed-loop system $\Sigma\Gamma$ is stable and its L_2 gain is less than γ if there exists a differentiable matrix $P(\theta)$ which satisfies the following LMI condition:

$$P(\theta) \gg 0, \tag{5}$$

$$\begin{bmatrix} Q_{cl}(\theta,\omega) & P(\theta)B(\theta) & C_{cl}^T(\theta) \\ B^T(\theta)P(\theta) & -\gamma I & D^T(\theta) \\ C_{cl}(\theta) & D(\theta) & -\gamma I \end{bmatrix} \ll 0, \tag{6}$$

where,

$$\begin{array}{l} Q_{cl}(\theta,\omega) = \omega \dfrac{\partial P(\theta)}{\partial \theta} + A_{cl}^T(\theta)P(\theta) + P(\theta)A_{cl}(\theta), \\ A_{cl}(\theta) = A(\theta) + B_u(\theta)F(\theta), \\ C_{cl} = C(\theta) + D_u(\theta)F(\theta) \end{array}$$

for any $(\theta, \omega) \in (\Theta_{\text{val}} \times \Omega_{\text{val}})$[1].
If such a $P(\theta)$ exists, the parameter-dependent quadratic form $x^T P(\theta) x$ acts as a Lyapunov function proving stability and L_2 gain performance. □

[1]Inequality $P(\theta) \gg 0$ ($P(\theta) \ll 0$) means that $P(\theta) \geq \alpha$ ($P(\theta) \leq -\alpha$) holds for some positive number α and any $\theta \in \Theta_{\text{val}}$.

The condition in Lemma 1 is a parameter-dependent LMI and involves an infinite number of LMIs corresponding to each pair of $(\theta, \omega) \in (\Theta_{\text{val}} \times \Omega_{\text{val}})$. The condition is reduced to a finite set of LMIs as in the following lemma, which gives a feedback gain matrix $F(\theta)$ and a spline-type solution $P(\theta)$ to (5) and (6).

Lemma 2: [12] The following two statements are equivalent.
(i)There exist a differentiable matrix $P(\theta)$ and a feedback gain matrix $F(\theta)$ which satisfy (5) and (6).
(ii) There exist a subdivision D of D^{Σ} ($D = \{\underline{\theta} = \theta_0, \theta_1, \ldots, \theta_N, \theta_{N+1} = \overline{\theta}\}$), matrices W_k and symmetric matrices $X_k (k = 0, 1, \ldots, N+1)$ which satisfy the following LMIs for $\omega = \underline{\omega}, \overline{\omega}$:

$$X_k > 0, \quad k = 0, 1, ..., N+1, \tag{7}$$

$$\begin{bmatrix} Q_{cl(k)} & B_k & S_k^T \\ B_k^T & -\gamma I & D_k^T \\ S_k & D_k & -\gamma I \end{bmatrix} := J_{cl(k)} < 0, \quad k = 0, 1, ..., N, \tag{8}$$

$$\begin{bmatrix} Q_{cl(k)}^- & B_k & S_k^T \\ B_k^T & -\gamma I & D_k^T \\ S_k & D_k & -\gamma I \end{bmatrix} < 0, \quad k = 1, 2, ..., N+1, \tag{9}$$

$$J_{cl(k)} + \frac{1}{2}(L_{cl(k)} + L_{cl(k)}^T) < 0, \quad k = 0, 1, ..., N, \tag{10}$$

$$\begin{array}{lrcl} \text{where} & Q_{cl(k)} & := & A_k X_k + X_k A_k^T - \dfrac{\omega}{\Delta\theta_k}\Delta X_k \\ & & & +B_{u(k)}W_k + W_k^T B_{u(k)}^T, \\ & Q_{cl(k)}^- & := & A_k X_k + X_k A_k^T - \dfrac{\omega}{\Delta\theta_{k-1}}\Delta X_{k-1} \\ & & & +B_{u(k)}W_k + W_k^T B_{u(k)}^T, \\ & S_k & := & C_k X_k + D_{u(k)} W_k, \end{array}$$

$$\begin{array}{rcl} L_{cl(k)} & := & \begin{bmatrix} L_{cl(k)}^{11} & \Delta B_k & 0 \\ 0 & 0 & 0 \\ L_{cl(k)}^{31} & \Delta D_k & 0 \end{bmatrix}, \\ L_{cl(k)}^{11} & := & (\Delta A_k X_k + A_k \Delta X_k) \\ & & +(\Delta B_{u(k)} W_k + B_{u(k)} \Delta W_k), \\ L_{cl(k)}^{31} & := & (\Delta C_k X_k + C_k \Delta X_k) \\ & & +(\Delta D_{u(k)} W_k + D_{u(k)} \Delta W_k), \end{array}$$

and Δ denotes difference; $\Delta\theta_k := \theta_{k+1} - \theta_k, \Delta X := X_{k+1} - X_k, \Delta A := A(\theta_{k+1}) - A(\theta_k)$ and so on.
If (ii) holds, one of the solutions $P(\theta)$ and $F(\theta)$ is given as:

$$\begin{array}{rcl} P(\theta) & = & \left\{ \dfrac{1}{l} \displaystyle\int_{\theta - \frac{l}{2}}^{\theta + \frac{l}{2}} X_S(h) dh \right\}^{-1}, \\ F(\theta) & = & W_S(\theta) X_S^{-1}(\theta), \end{array} \tag{11}$$

where l is some small positive constant and

$$X_S(\theta) = X_k + \frac{\theta - \theta_k}{\theta_{k+1} - \theta_k}(X_{k+1} - X_k),$$
$$W_S(\theta) = W_k + \frac{\theta - \theta_k}{\theta_{k+1} - \theta_k}(W_{k+1} - W_k),$$

for $\theta \in [\theta_k, \theta_{k+1}]$. □

Since the spline-type solution (11) with sufficiently large N can represent almost any continuous matrix function, even non-convex solutions with respect to θ can be derived from this lemma. It thus gives less conservative solutions than the earlier results in which $P(\theta)$ was supposed to be a fixed matrix P.

Note that the resulting knots of $F(\theta)$ are the same as those of $X_S(\theta)$ because $F(\theta)$ is given as $W_S(\theta)X_S^{-1}(\theta)$. As a result, the finer the region $\Theta_{\rm val}$ is divided to obtain a non-conservative Lyapunov function, the more complicated the feedback gain $F(\theta)$ becomes.

III. A New Gain Scheduling Controller Synthesis

A. Stability Condition via Dilated LMIs

There has been much research published on dilated LMIs, and it has been shown that dilated LMI conditions are suitable for some control problems. For example, in H_2 synthesis of discrete-time[14] and continuous-time[15],[13],[16] polytopic LPV systems, non-common Lyapunov functions for all vertices of the polytope can be derived via a dilated LMI approach. For multiobjective synthesis problems[13],[16], non-common Lyapunov functions for each specification can be obtained using a dilated LMI condition in which there is no product between the Lyapunov-related matrix X and the feedback gain F. As a result, less conservative performance is realized.

An equivalent condition of Lemma 1 can be derived via a dilated LMI approach. In the following theorem, parameter-dependent matrices are expressed by calligraphic characters and the argument (θ) is omitted for simplicity ($\mathcal{A}$ as $A(\theta)$, for example).

Theorem 1: The closed-loop system $\Sigma\Gamma$ is stable and its L_2 gain is less than γ if there exist a positive constant ϵ, a differentiable matrix $\mathcal{X}$ and a square matrix $\mathcal{G}$ which satisfy the following condition:

$$\mathcal{X} \gg 0, \quad (12)$$

$$\begin{bmatrix} -\omega\frac{\partial\mathcal{X}}{\partial\theta} + \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T & \mathcal{B} & \mathcal{X}\mathcal{C}^T & -\mathcal{X} \\ \mathcal{B}^T & -\gamma I & \mathcal{D}^T & 0 \\ \mathcal{C}\mathcal{X} & \mathcal{D} & -\gamma I & 0 \\ -\mathcal{X} & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{B}_u\mathcal{F} \\ 0 \\ \mathcal{D}_u\mathcal{F} \\ I \end{bmatrix} \mathcal{G} \begin{bmatrix} I & 0 & 0 & -\epsilon I \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \\ -\epsilon I \end{bmatrix} \mathcal{G}^T \begin{bmatrix} \mathcal{F}^T\mathcal{B}_u^T & 0 & \mathcal{F}^T\mathcal{D}_u^T & I \end{bmatrix} \ll 0, \quad (13)$$

for any $(\theta, \omega) \in (\Theta_{\rm val} \times \Omega_{\rm val})$. □

Proof: This theorem can be proved in a similar way to [13] and [16].

Suppose that (12) and (13) hold. By multiplying

$$\begin{bmatrix} \mathcal{B}_u\mathcal{F} \\ 0 \\ \mathcal{D}_u\mathcal{F} \\ I \end{bmatrix}^{\perp} = \begin{bmatrix} I & 0 & 0 & -\mathcal{B}_u\mathcal{F} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -\mathcal{D}_u\mathcal{F} \end{bmatrix}$$

and its transpose from the left and right sides respectively, the inequality (13) leads to

$$\begin{bmatrix} \star & \mathcal{B} & \mathcal{X}\mathcal{C}^T \\ \mathcal{B}^T & -\gamma I & \mathcal{D}^T \\ \mathcal{C}\mathcal{X} & \mathcal{D} & -\gamma I \end{bmatrix} \ll 0, \quad (14)$$

$$(\star := -\omega\frac{\partial\mathcal{X}}{\partial\theta} + \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T + \mathcal{B}_u\mathcal{F}\mathcal{X} + \mathcal{X}\mathcal{F}^T\mathcal{B}_u^T).$$

Then the inequality (6) is obtained by letting $\mathcal{X} = \mathcal{P}^{-1}$. It is apparent that (5) holds.

On the other hand, suppose that (5) and (6) hold. Note that (6) can be rewritten as (14) by defining $\mathcal{X} := \mathcal{P}^{-1}$. There exists a small positive constant ϵ such that

$$\begin{bmatrix} \epsilon\mathcal{B}_u\mathcal{F}\mathcal{X} \\ 0 \\ \epsilon\mathcal{D}_u\mathcal{F}\mathcal{X} \end{bmatrix} (2\epsilon\mathcal{X})^{-1} \begin{bmatrix} \epsilon\mathcal{X}\mathcal{F}^T\mathcal{B}_u^T & 0 & \epsilon\mathcal{X}\mathcal{F}^T\mathcal{D}_u^T \end{bmatrix} \geq 0, \quad (15)$$

since $\mathcal{X} > 0$.

If ϵ is sufficiently small, then

$$[\text{left side of (14)}] + [\text{left side of (15)}] \ll 0$$

holds. Applying the Schur complement to this inequality leads to

$$\begin{bmatrix} \star & \mathcal{B} & \mathcal{X}\mathcal{C}^T + \mathcal{X}\mathcal{F}^T\mathcal{D}_u^T & -\epsilon\mathcal{B}_u\mathcal{F}\mathcal{X} \\ \mathcal{B}^T & -\gamma I & \mathcal{D}^T & 0 \\ \mathcal{C}\mathcal{X} + \mathcal{D}_u\mathcal{F}\mathcal{X} & \mathcal{D} & -\gamma I & -\epsilon\mathcal{D}_u\mathcal{F}\mathcal{X} \\ -\epsilon\mathcal{X}\mathcal{F}^T\mathcal{B}_u^T & 0 & -\epsilon\mathcal{X}\mathcal{F}^T\mathcal{D}_u^T & -2\epsilon\mathcal{X} \end{bmatrix} \ll 0,$$

$$(\star := -\omega\frac{\partial\mathcal{X}}{\partial\theta} + \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T + \mathcal{B}_u\mathcal{F}\mathcal{X} + \mathcal{X}\mathcal{F}^T\mathcal{B}_u^T).$$

This inequality shows that, by choosing $\mathcal{G} = \mathcal{G}^T = \mathcal{X}$, the differentiable matrices $\mathcal{X}$ and $\mathcal{G}$ satisfy (12) and (13). ■

The advantage of (13) is that it has no product between $\mathcal{X}$ and $\mathcal{F}$ in the matrix elements. This property is utilized in the next theorem.

B. Controller Synthesis via Dilated LMIs

Before moving on to the theorem, let us consider a division D^Γ of $\Theta_{\rm val}$, of which D is a subdivision:

$$D^\Gamma = \{\underline{\theta} = \theta_0^\Gamma, \theta_1^\Gamma, \ldots, \theta_{N_\Gamma}^\Gamma, \theta_{N_\Gamma+1}^\Gamma = \overline{\theta}\},$$
$$(N_\Gamma \leq N_\Sigma \leq N). \quad (16)$$

Note that D^Γ has the same number or fewer knots than D (that is, the knots of $X_S(\theta)$).

Example 1: Consider the case $\Theta_{\mathrm{val}} = [0,6]$, $D^\Sigma = \{0,6\}$ and $D = \{0,2,4,6\}$, $(N_\Sigma = 0, N = 2)$. Then the candidates of D^Γ are

$$D^\Gamma = \begin{cases} \{0,6\}, & (N_\Gamma = 0), \\ \{0,2,6\},\{0,4,6\}, & (N_\Gamma = 1), \\ \{0,2,4,6\}, & (N_\Gamma = 2). \end{cases}$$

□

The following theorem gives a feedback controller $F(\theta)$ whose knots are in D^Γ based on Theorem 1.

Theorem 2: The following two statements are equivalent.
(i) There exist a positive constant ϵ, a differentiable matrix $\mathcal{X}$, a square matrix $\mathcal{G}$ and a feedback gain matrix $\mathcal{F}$ which satisfy (12) and (13).
(ii) There exist a positive constant ϵ, a subdivision D of $D^\Sigma (D = \{\theta_0, \theta_1, \ldots, \theta_N, \theta_{N+1}\})$, a division D^Γ as in (16), symmetric matrices $X_k (k = \{0,1,\ldots,N+1\})$, matrices W_j^Γ and $G_j^\Gamma (j = \{0,1,\ldots,N_\Gamma+1\})$ which satisfy the following conditions for $\omega = \underline{\omega}, \overline{\omega}$:

$$X_k > 0, \quad k = 0,1,...,N+1, \tag{17}$$

$$\begin{bmatrix} Q_{cl(k)} & B_k & S_k^T & R_k \\ B_k^T & -\gamma I & D_k^T & 0 \\ S_k & D_k & -\gamma I & -\epsilon D_{u(k)} W_k \\ R_k^T & 0 & -\epsilon W_k^T D_{u(k)}^T & -\epsilon(G_k + G_k^T) \end{bmatrix} := J_{cl(k)} < 0, \quad k = 0,1,...,N, \tag{18}$$

$$\begin{bmatrix} Q_{cl(k)}^- & B_k & S_k^T & R_k \\ B_k^T & -\gamma I & D_k^T & 0 \\ S_k & D_k & -\gamma I & -\epsilon D_{u(k)} W_k \\ R_k^T & 0 & -\epsilon W_k^T D_{u(k)}^T & -\epsilon(G_k + G_k^T) \end{bmatrix} < 0, \quad k = 1,2,...,N+1, \tag{19}$$

$$J_{cl(k)} + \frac{1}{2}(L_{cl(k)} + L_{cl(k)}^T) < 0, \quad k = 0,1,...,N, \tag{20}$$

where

$$\begin{aligned}
G_k &:= G_j^\Gamma + \frac{\theta_k - \theta_j^\Gamma}{\theta_{j+1}^\Gamma - \theta_j^\Gamma}(G_{j+1}^\Gamma - G_j^\Gamma), \\
W_k &:= W_j^\Gamma + \frac{\theta_k - \theta_j^\Gamma}{\theta_{j+1}^\Gamma - \theta_j^\Gamma}(W_{j+1}^\Gamma - W_j^\Gamma), \\
& \quad (\text{for} \quad k \quad \text{s.t.} \quad \theta_j^\Gamma \le \theta_k \le \theta_{j+1}^\Gamma), \\
Q_{cl(k)} &:= A_k X_k + X_k A_k^T - \frac{\omega}{\Delta\theta_k}\Delta X_k \\
& \quad + B_{u(k)} W_k + W_k^T B_{u(k)}^T, \\
Q_{cl(k)}^- &:= A_k X_k + X_k A_k^T - \frac{\omega}{\Delta\theta_{k-1}}\Delta X_{k-1} \\
& \quad + B_{u(k)} W_k + W_k^T B_{u(k)}^T, \\
R_k &:= G_k - X_k - \epsilon B_{u(k)} W_k, \\
S_k &:= C_k X_k + D_{u(k)} W_k,
\end{aligned}$$

$$\begin{aligned}
L_{cl(k)} &:= \begin{bmatrix} L_{cl(k)}^{11} & \Delta B_k & 0 & L_{cl(k)}^{14} \\ 0 & 0 & 0 & 0 \\ L_{cl(k)}^{31} & \Delta D_k & 0 & L_{cl(k)}^{34} \\ 0 & 0 & 0 & -2\epsilon\Delta G_k \end{bmatrix}, \\
L_{cl(k)}^{11} &:= (\Delta A_k X_k + A_k \Delta X_k) \\
& \quad + (\Delta B_{u(k)} W_k + B_{u(k)} \Delta W_k), \\
L_{cl(k)}^{31} &:= (\Delta C_k X_k + C_k \Delta X_k) \\
& \quad + (\Delta D_{u(k)} W_k + D_{u(k)} \Delta W_k), \\
L_{cl(k)}^{14} &:= \Delta G_k - \Delta X_k - \epsilon(\Delta B_{u(k)} W_k + B_{u(k)} \Delta W_k), \\
L_{cl(k)}^{34} &:= -\epsilon(\Delta D_{u(k)} W_k + D_{u(k)} \Delta W_k),
\end{aligned}$$

and Δ denotes difference; $\Delta\theta_k := \theta_{k+1} - \theta_k, \Delta X := X_{k+1} - X_k, \Delta A := A(\theta_{k+1}) - A(\theta_k)$ and so on.
If (ii) holds, one of the solutions $P(\theta)$ and $F(\theta)$ is given as:

$$\begin{aligned}
P(\theta) &= \left\{ \frac{1}{l} \int_{\theta - \frac{l}{2}}^{\theta + \frac{l}{2}} X_S(h) dh \right\}^{-1}, \\
F(\theta) &= W_S^\Gamma(\theta) G_S^{\Gamma^{-1}}(\theta),
\end{aligned}$$

where l is some small positive constant,

$$X_S(\theta) = X_k + \frac{\theta - \theta_k}{\theta_{k+1} - \theta_k}(X_{k+1} - X_k),$$

for $\theta \in [\theta_k, \theta_{k+1}]$ and

$$\begin{aligned}
W_S^\Gamma(\theta) &= W_j^\Gamma + \frac{\theta - \theta_j^\Gamma}{\theta_{j+1}^\Gamma - \theta_j^\Gamma}(W_{j+1}^\Gamma - W_j^\Gamma), \\
G_S^\Gamma(\theta) &= G_j^\Gamma + \frac{\theta - \theta_j^\Gamma}{\theta_{j+1}^\Gamma - \theta_j^\Gamma}(G_{j+1}^\Gamma - G_j^\Gamma),
\end{aligned}$$

for $\theta \in [\theta_j^\Gamma, \theta_{j+1}^\Gamma]$. □

Proof:
(ii)⇒(i): This can be proved in the same way as in [11].
(i)⇒(ii): This is also proven by choosing $D^\Gamma = D (N_\Gamma = N)$. ■

Note 1: The inequalities (17)–(20) are not LMIs since the variable ϵ is involved in some products with the matrix variables W_k and G_k. However, by letting ϵ be a line-search parameter, these inequalities can be solved using standard LMI solvers[17][16].

The main point of the theorem is that the feedback gain $F(\theta)$ is not related to $X_S(\theta)$ and consequently the number of knots of $F(\theta)$ does not always increase even if the number of knots of $X_S(\theta)$ $(= N+2)$ increases in order to obtain a less conservative Lyapunov matrix. Fig. 1 illustrates the difference between Lemma 2 and Theorem 2 with the same $N(=5)$. A feedback gain $F(\theta)$ derived from Lemma 2 always has the same number of knots as $X_S(\theta)$ (Fig. 1 left). On the other hand, a feedback gain with fewer knots can be derived using Theorem2 (Fig. 1 right) by letting $N_\Gamma < N$ ($N_\Gamma = 0$ in Fig. 1). This is preferable for implementation in a controller since a scheduled gain with fewer knots requires a smaller control program.

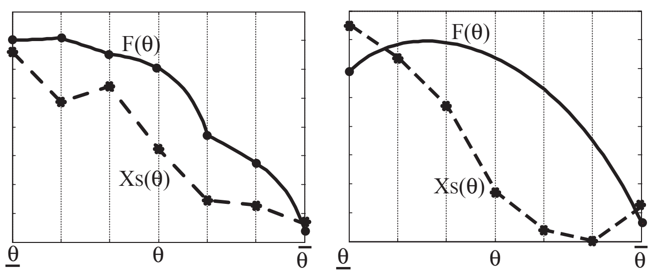

Fig. 1. Examples of function $X_S(\theta)$, which is related to Lyapunov solution, and feedback gain $F(\theta)$ derived from Lemma 1 (left hand) and Theorem 2 (right hand).

Furthermore, if we set $G_* = G$ (a common matrix) in Theorem 2 and a solution to (17)–(20) is obtained, the feedback gain $F(\theta)$ that results is simply a piecewise-linear function of θ. While scheduled gains derived from Lemma 2 need a matrix inverse calculation ($W_S X_S^{-1}$) for scheduling at each value of θ, the piecewise-linear scheduled gain requires only simple interpolation, which reduces the processing requirements for embedded control computers and is therefore favorable from the standpoint of implementability. These derived gains are shown in the example that follows.

IV. Numerical Example

Consider the LPV system Σ:

$$\begin{aligned} A(\theta) &= \begin{bmatrix} -4.1 - 3.0\theta & 1 \\ -2.0\theta & 2.0 - 3.2\theta \end{bmatrix}, \\ B(\theta) &= \begin{bmatrix} -0.03 - 0.3\theta \\ -0.47 + 0.9\theta \end{bmatrix}, \\ B_u(\theta) &= \begin{bmatrix} 3.0 \\ 2.0 - 1.0\theta \end{bmatrix}, \; C(\theta) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \\ D(\theta) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; D_u(\theta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

In this example, we assume $\Theta_{\text{val}} = [0, 6], \Omega_{\text{val}} = [-10, 10]$ and $D^{\Sigma} = \{0, 6\}$. We also assume that L_2 gain performance γ is required to be less than 0.7.

Table 1 shows the results of minimizing γ. The columns A and B show the minimized values of γ under (7)–(10) (Lemma 2) and (17)–(20) (Theorem 2) respectively. The number of knots of $F(\theta)$ is supposed to be two ($N_\Gamma = 0$ and $D^\Gamma = \{0, 6\}$) in Theorem 2. In addition, we also minimize γ under (17)–(20) and $G_1 = G_2 = G$, with the results shown in column B'. In each case, D is given so that Θ is divided equally with $N = 0, \ldots, 5$.Using Lemma 2, N should be at least four in order to meet the requirement of L_2 gain performance. Consequently, the knots of the derived feedback gain should be six ($= N+2$) or more. On the other hand, a feedback gain with two knots can be obtained using Theorem 2 with $N = 4$ and $N_\Gamma = 0$. Moreover, by letting $N = 5, N_\Gamma = 0$ and imposing $G_1 = G_2$, a feedback gain is derived which is a linear function of θ.

These resulting gains are shown in Fig. 2. Gain (A) has six knots, whereas (B) and (B') have two. Gain (B') is linear on the parameter θ. Now let us suppose that these scheduled gains are to be implemented in a control system. It is apparent that gain (A) requires the most resources for implementation since six gain matrices are necessary to schedule it, three times more than those necessary for (B) and (B'). This may not seem to be a problem in this case since the size of each gain matrix is 1×2 and the difference in required storage would be only several tens of bytes. However, this difference would not be negligible when dealing with large-scale systems which have several tens or hundreds of states and inputs. Moreover, because matrix inverse calculation is required for scheduling (A) and (B), a controller would require greater computational resources than for scheduling (B'). Therefore, considering the fact that all these gains satisfy the performance requirement, gain (B') is preferable from the standpoint of implementability.

TABLE I

Minimized value of γ in each case.

	(A) Lemma 2	(B) Theorem 2 ($N_\Gamma = 0$)	(B') Theorem 2 ($N_\Gamma = 0; G_*$:const.)
$N = 0$	1.47	1.47	1.48
1	0.992	1.07	1.10
2	0.822	0.836	0.883
3	0.743	0.747	0.783
4	**0.690**	**0.698**	0.723
5	**0.657**	**0.668**	**0.692**

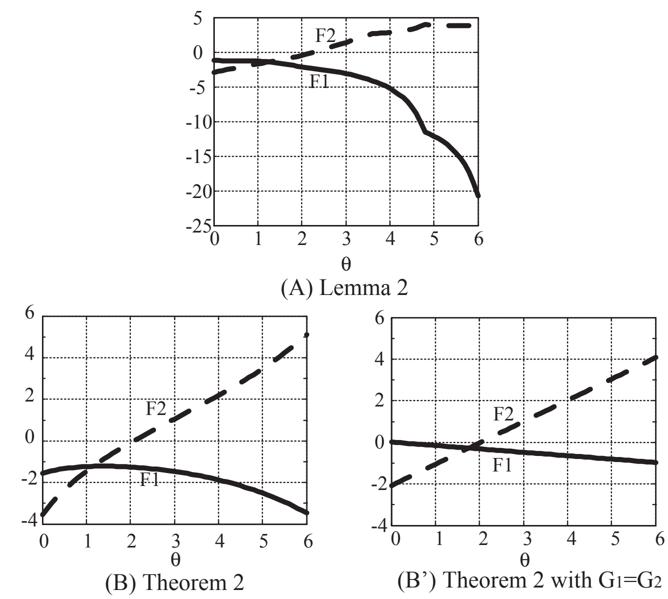

Fig. 2. The derived feedback gains. (A): Lemma2 with $N = 4$ ($\gamma = 0.690$). (B): Theorem 2 with $N = 4$ and $N_\Gamma = 0$ ($\gamma = 0.698$). (B'): Theorem 2 with $N = 5, N_\Gamma = 0$ and $G_1 = G_2$ ($\gamma = 0.692$).

V. Conclusion

We proposed a new gain scheduling controller synthesis with spline-type parameter-dependent quadratic forms. The condition is described as dilated LMIs and the synthesis can be performed using standard LMI solvers. Since scheduled gain matrices are derived independently of the Lyapunov matrix, they are simpler than those in previous studies. Moreover, scheduled gains which are piecewise-linear on a parameter can be obtained by imposing some equality constraints on matrix variables. Scheduled gains derived from the synthesis were illustrated, and we showed that these are desirable for practical implementation.

In the example, we supposed $D^\Gamma = \{0, 6\}(N_\Gamma = 0)$ and then designed the scheduled gains. It is, however, difficult to decide *a priori* a division D^Γ in general. Thus, an algorithm which sets a division D^Γ and simultaneously gives a feedback gain needs to be developed. In addition, output feedback controller synthesis remains a topic for further research. These studies are ongoing and the results will be shown in future presentations.

References

[1] G. Becker and A. Packard, "Gain-Scheduled State-Feedback with Quadratic Stability for Uncertain Systems", *Proceedings of the American Control Conference*, pp.2702-2703 (1991).

[2] A. Packard, "Gain Scheduling via Linear Fractional Transformation", *Systems & Control Letters*, vol.22, no.2, pp.79-92 (1994).

[3] P. Apkarian and P. Gahinet, "A Convex Characterization of Gain-Scheduling H_∞ Control", *IEEE Transactions on Automatic Control*, vol.40, no.7, pp.853-864 (1995).

[4] F. Wu, X. H. Yang, A. Packard and G. Becker, "Induced L_2-Norm Control for LPV Systems with Bounded Parameter-Variation Rates", *International Journal of Robust and Nonlinear Control*, vol.6, no.9, pp.983-998 (1996).

[5] R. Watanabe, K. Uchida, M. Fujita and E. Shimemura, "L_2 Gain and H_∞ Control of Linear Systems with Scheduling Parameter", *Proceedings of the 33rd IEEE Conference on Decision and Control*, pp.1412–1414 (1994).

[6] R. Watanabe, K. Uchida and M. Fujita, "A New LMI Approach to Analysis of Linear Systems with Scheduling Parameter – Reduction to Finite Number of LMI Conditions –", *Proceedings of the 35th IEEE Conference on Decision and Control*, pp.1663–1665 (1996).

[7] P. Gahinet, P. Apkarian and M. Chilali, "Affine Parameter-Dependent Lyapunov Functions and Real Parametric Uncertainty", *IEEE Transactions on Automatic Control*, vol.41, no.3, pp.436-442 (1996).

[8] H. D. Tuan and P. Apkarian, "Relaxations of Parameterized LMIs with Control Applications", *International Journal of Robust and Nonlinear Control*, vol.9, no.2, pp.59-84 (1999).

[9] F. Wu and S. Prajna, "A New Solution Approach to Polynomial LPV System Analysis and Synthesis", *Proceedings of the American Control Conference*, pp.1362-1367 (2004).

[10] I. Masubuchi, "On Numerical Solution to Parameter-Dependent Convex Differential Inequalities", *Proceedings of the 38th IEEE Conference on Decision and Control*, pp.305–309 (1999).

[11] I. Masubuchi, "Numerical Solution to a Class of Parameter-Dependent Convex Differential Inequalities", *Transaction of the Society of Instrument and Control Engineers*, vol.36, No.3, pp.248-254 (2000).

[12] I. Masubuchi, A. Kume and E. Shimemura, "Spline-Type Solution to Parameter Dependent LMIs", *Proceedings of the 37th IEEE Conference on Decision and Control*, pp.1753-1758 (1998).

[13] Y. Ebihara and T. Hagiwara, "New Dilated LMI Characterizations for Continuous-Time Control Design and Robust Multiobjective Control", *Proceedings of the American Control Conference*, pp.47-52 (2002).

[14] M. C. de Oliveira, J. C. Geromel and J. Bernussou, "Extended H_2 and H_∞ norm characterizations and controller parameterizations for discrete-time systems", *International Journal of Control*, vol.75, no.9, pp.666-679 (2002).

[15] P. Apkarian, H. D. Tuan and J. Bernussou, "Continuous-Time Analysis, Eigenstructure Assignment, and H_2 Synthesis With Enhanced Linear Matrix Inequalities (LMI) Characterizations", *IEEE Transactions on Automatic Control*, vol.46, no.12, 1941-1946 (2001).

[16] T. Shimomura, M. Takahashi and T. Fujii, "Extended-Space Control Design with Parameter-Dependent Lyapunov Functions", *Proceedings of the 40th IEEE Conference on Decision and Control*, pp.2157-2162 (2001).

[17] U. Shaked, "Improved LMI Representations for the Analysis and the Design of Continuous-Time Systems with Polytopic Type Uncertainty", *IEEE Transactions on Automatic Control*, vol.46, no.4, 652-656 (2001).