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Port-based Modelling and Control of the Mindlin Plate

Alessandro Macchelli, Claudio Melchiorri and Luca Bassi

Abstract— **The purpose of this paper is to show how the Mindlin model of a plate can be fruitfully described within the framework of distributed port Hamiltonian systems (dpH systems) so that rather simple and elegant considerations can be drawn regarding both the modeling and control of this mechanical system. Once the distributed port Hamiltonian (dpH) model of the plate is introduced, a simple boundary or distributed control methodology based on damping injection is discussed.**

I. Introduction

The port Hamiltonian representation of a finite dimensional system [1], [2] has been recently generalized to the distributed parameter case [3], [4], [5], [6] by introducing the notion of Dirac structure on an infinite dimensional space of power variables. The port Hamiltonian approach overcomes the limitations in dealing with non-zero boundary conditions of *classical* Hamiltonian formulations [7], [8] by introducing the notion of infinite dimensional interconnection structure.

From a *physical* point of view, the dynamics of an infinite dimensional system with spatial domain $\mathcal{Z}$ and boundary $\partial\mathcal{Z}$ can be considered as the result of the interaction among (at least) two energy domains within $\mathcal{Z}$ and/or between the system and its environment through $\partial\mathcal{Z}$. This interaction is mathematically described by a generalization of the Dirac structure to the distributed parameter case. Since this new class of power conserving interconnection deeply relies on the Stokes Theorem, we speak about Stokes–Dirac structure [4]. This class of infinite dimensional systems is quite general, thus allowing the description of several physical phenomena, such as heat conduction [5], [6], electromagnetism, fluid dynamics [3], piezoelectricity [9] and elasticity, [5], [10], and the development of control schemes based on energy considerations, generalization of results valid in finite dimensions, [10], [11], [12], [13].

In this paper, the advantages of the proposed modelling and control techniques are illustrated by formulating the Mindlin model of a flexible plate (e.g. see [14], [15]) within this new framework. It is shown how boundary conditions can be explicitly taken into account and how they can *naturally* define a boundary *port* through which the distributed parameter system can interact, i.e. exchange power, with the environment. Then, by properly interconnecting a dissipative element at the boundary and/or along the spatial domain, it is shown how it is possible to stabilize the system around the *zero* configuration corresponding to the undeformed state.

This paper is organized as follows. Firstly, a short introduction about distributed parameter systems in port Hamiltonian form is given in Sect. II. In particular, the definition and some basic properties of the Dirac structures are presented in Sect. II-A, while the Stokes–Dirac structures defined by skew-adjoint differential operators are introduced in Sect. II-B and the corresponding class of distributed parameter port Hamiltonian systems in Sect. II-C. The Mindlin model of a flexible plate is illustrated in Sect. III in its original formulation, while its port Hamiltonian interpretation is discussed in Sect. IV. Then, simple control techniques based on damping injection through the boundary and along the spatial domain are illustrated in Sect. V-A and Sect. V-B respectively. Conclusions and final remarks are given in Sect. VI.

II. Port Hamiltonian formulation of distributed parameter systems

A. Basics on Dirac structures

Denote by $\mathcal{F} \times \mathcal{E}$ the space of power variables, with $\mathcal{F}$ an n-dimensional linear space, the space of flows (e.g. velocities and currents) and $\mathcal{E} \equiv \mathcal{F}^*$ its dual, the space of efforts (e.g. forces and voltages), and by $\langle e, f\rangle$ the power associated to the *port* $(f, e) \in \mathcal{F} \times \mathcal{E}$, with $\langle \cdot, \cdot \rangle$ the dual product between f and e. Based on the dual product, the following symmetric bilinear form (+pairing operator) is well-defined.

$$\ll (f_1, e_1), (f_2, e_2) \gg \;:= \langle e_1, f_2\rangle + \langle e_2, f_1\rangle \qquad (1)$$

Consider a linear subspace $\mathbb{S} \subset \mathcal{F} \times \mathcal{E}$ of dimension m and denote by $\mathbb{S}^\perp$ its orthogonal complement with respect to the +pairing operator (1), which is again a linear subspace of $\mathcal{F} \times \mathcal{E}$ with dimension $2n - m$ since (1) is a non-degenerate form. Based on the +pairing operator (1), it is possible to give the fundamental definition of Dirac structure, that is the basic mathematical tool that is used to describe the interconnection structure between physical systems.

Definition 2.1 (Dirac structure): Consider the space of power variables $\mathcal{F} \times \mathcal{E}$ and the symmetric bilinear form (1). A (constant) Dirac structure on $\mathcal{F}$ is a linear subspace $\mathbb{D} \subset \mathcal{F} \times \mathcal{E}$ such that

$$\mathbb{D} = \mathbb{D}^\perp$$

Note 2.1: Suppose that $(f, e) \in \mathbb{D}$; from (1), we have that $\langle e, f\rangle = 0$, i.e. a Dirac structure on $\mathcal{F}$ defines a power-conserving relation between power variables $(f, e) \in \mathcal{F} \times \mathcal{E}$.

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Alessandro Macchelli, Claudio Melchiorri and Luca Bassi are with CASY–DEIS, University of Bologna, viale Risorgimento 2, 40136 Bologna, Italy {amacchelli,cmelchiorri}@deis.unibo.it, lucabassi@libero.it.

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B. Constant Stokes–Dirac structures

Denote by $\mathcal{Z}$ a compact subset of $\mathbb{R}^d$, the spatial domain of the distributed parameter system, and by $\mathcal{U}$ and $\mathcal{V}$ a pair of *smooth* functions from $\mathcal{Z}$ to $\mathbb{R}^{q_u}$ and $\mathbb{R}^{q_v}$ respectively.

Definition 2.2: A constant matrix differential operator of order N is a map L from $\mathcal{U}$ to $\mathcal{V}$ such that, given $u = (u^1, \dots, u^{q_u}) \in \mathcal{U}$ and $v = (v^1, \dots, v^{q_v}) \in \mathcal{V}$

$$v = Lu \iff v^b := \sum_{\#\alpha=0}^{N} P^\alpha_{a,b} D^\alpha u^a \tag{2}$$

where $\alpha := \{\alpha_1, \dots, \alpha_d\}$ is a multi-index of order $\#\alpha := \sum_{i=1}^d \alpha_i$, P^α are a set of constant $q_u \times q_v$ matrices and $D^\alpha := \partial^{\alpha_1}_{z_1} \cdots \partial^{\alpha_d}_{z_d}$ is an operator resulting from a combination of spatial derivatives. Note that, in (2), the sum is intended over all the possible multi-indexes α with order 0 to N and, implicitly, on a from 1 to q.

Definition 2.3: Consider the constant matrix differential operator (2). Its formal adjoint is the map L^* from $\mathcal{V}$ to $\mathcal{U}$ such that

$$u = L^* v \iff u^b := \sum_{\#\alpha=0}^{N} (-1)^{\#\alpha} P^\alpha_{b,a} D^\alpha v^a$$

Definition 2.4: Denote by J a constant matrix differential operator. Then, J is *skew-adjoint* if and only if $J = -J^*$.

An important relation satisfied by skew-adjoint differential operators is expressed by the following lemma, which generalizes an analogous result presented in [16] to the multi variable case. It is an extension of the integration by part formula and it turns out to be fundamental in the definition of the Stokes–Dirac structure associated to this class of differential operators.

Lemma 2.1: Consider a skew-adjoint matrix differential operator J. Then, for every functions $u \in \mathcal{U}$ and $v \in \mathcal{V}$ with $q_u = q_v$, we have that

$$\int_{\mathcal{Z}} \left[v^{\mathrm{T}} J u + u^{\mathrm{T}} J v \right] \mathrm{dV} = \int_{\partial\mathcal{Z}} B_J(u, v) \cdot \mathrm{dA} \tag{3}$$

where B_J is a symmetric differential operator on $\partial\mathcal{Z}$ determined by J.

Note 2.2: Given $u \in \mathcal{U}$ and $v \in \mathcal{V}$, from the Stokes' Theorem, it is well known that relation (3) can be equivalently written as $v^{\mathrm{T}} J u + u^{\mathrm{T}} J v = \operatorname{div} B_J(u, v)$, i.e. $v^{\mathrm{T}} J u + u^{\mathrm{T}} J v$ can be expressed in divergence form. Moreover, it is important to note that B_J is a constant differential operator, that is a constant linear combination, of the functions u and v, restricted on $\partial\mathcal{Z}$, together with their spatial derivatives up to a certain order. Equivalently, it can be interpreted as a constant linear combination of $B_{\mathcal{Z}}(u)$ and $B_{\mathcal{Z}}(v)$, where $B_{\mathcal{Z}}$ is an operator providing the boundary terms (conditions), i.e. all the spatial derivatives required in (3).

As in finite dimensions, the definition of a power conserving interconnection structure is possible once the notion of power is properly introduced. Denote by $\mathcal{F}$ a space of *smooth* functions from $\mathcal{Z} \subset \mathbb{R}^d$ to $\mathbb{R}^q$ (space of flows) and, as far as concerns the space of efforts $\mathcal{E}$, assume that $\mathcal{E} \equiv \mathcal{F}$. Then, given $f = (f^1, \dots, f^q) \in \mathcal{F}$ and $e = (e^1, \dots, e^q) \in \mathcal{E}$, define the dual product as follows:

$$\langle e, f \rangle := \int_{\mathcal{Z}} \sum_{i=1}^{q} e^i f^i \, \mathrm{dV} = \int_{\mathcal{Z}} e^{\mathrm{T}} f \, \mathrm{dV}$$

A wide class of constant Stokes–Dirac structures is provided by the following proposition, [6]. The Stokes–Dirac structure associated with the Mindlin model of an elastic plate belongs to this class.

Proposition 2.2: Denote by $\mathcal{Z} \subset \mathbb{R}^d$ a compact set and by $\mathcal{F}$ a space of vector values smooth functions on $\mathcal{Z}$, the space of flows. For simplicity, suppose that $\mathcal{E} \equiv \mathcal{F}$ is the space of efforts. Moreover, assume that J is a skew-adjoint matrix differential operator. Then,

$$\mathbb{D} = \Big\{ (f, e, w) \in \mathcal{F} \times \mathcal{E} \times \mathcal{W} \mid f = -Je, w = B_{\mathcal{Z}}(e) \Big\} \tag{4}$$

is a Stokes–Dirac structure with respect to the pairing

$$\begin{aligned} &\ll (f_1, e_1, w_1), (f_2, e_2, w_2) \gg_J = \\ &= \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1 \right] \mathrm{dV} + \int_{\partial\mathcal{Z}} B_J(w_1, w_2) \cdot \mathrm{dA} \end{aligned} \tag{5}$$

where $B_{\mathcal{Z}}$ is the analogous of the boundary operator of Note 2.2 and $B_J(\cdot, \cdot)$ is the boundary differential operator induced by J.

Note 2.3: Suppose that $(f, e, w) \in \mathbb{D}$. From (5), we have:

$$-\int_{\mathcal{Z}} e^{\mathrm{T}} f \, \mathrm{dV} = \frac{1}{2} \int_{\partial\mathcal{Z}} B_J(w, w) \cdot \mathrm{dA}$$

This relation is a direct consequence of the definition of Dirac structure and expresses the property that the variation of internal energy is equal to the power provided to the system through the boundary $\partial\mathcal{Z}$. This means that no dissipative (or diffusive) effect is present in the distributed parameter system. As discussed in [6], this property follows from the fact that J is a skew-adjoint differential operator as in finite dimension when the dynamics of the system is defined on a Poisson manifold.

C. Infinite dimensional port Hamiltonian systems

As in finite dimensions, the dynamics of a distributed parameter system can be obtained from its Stokes–Dirac structure once the power ports are terminated on the corresponding elements, that is the input/output behavior of the *components* are specified.

Denote by $\mathcal{X}$ the space of smooth real valued functions on $[0, +\infty) \times \mathcal{Z}$ representing the space of energy *configuration*. The total energy is a functional $\mathcal{H} : \mathcal{X} \to \mathbb{R}$ such that

$$\mathcal{H}(x) = \int_{\mathcal{Z}} H(z, x) \, \mathrm{dV}$$

where H is the energy density. As proposed in [4], the port behavior of the energy storing element is given by

$$f_s = -\frac{\partial x}{\partial t} \qquad e_s = \delta_x \mathcal{H} \tag{6}$$

where $\delta_x \mathcal{H}$ is the variational derivative of the Hamiltonian with respect to the energy configuration. Consequently, taking into account (4) and (6), the following definition makes sense.

Definition 2.5: Denote by $\mathcal{X}$ the space of vector valued smooth functions on $[0, +\infty) \times \mathcal{Z}$ (energy configurations) and by $\mathcal{W}$ the space of vector valued smooth functions on $\partial\mathcal{Z}$ representing the boundary terms. Moreover, denote by J a skew-adjoint differential operator and by $B_\mathcal{Z}$ the boundary operator introduced in Note 2.2. If $\mathcal{H} : \mathcal{X} \to \mathbb{R}$ is the Hamiltonian function,

$$\left\{ \begin{array}{rcl} \dfrac{\partial x}{\partial t} &=& J\, \delta_x \mathcal{H} \\ w &=& B_\mathcal{Z}(\delta_x \mathcal{H}) \end{array} \right. \tag{7}$$

is the multi-variable distributed port Hamiltonian system associated with the differential operator J.

Proposition 2.3: Consider the mdpH system (7). Then,

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{1}{2} \int_{\partial\mathcal{Z}} B_J(w, w) \cdot \mathrm{dA} \tag{8}$$

i.e. the variation of internal energy is equal to the power provided to the system through the boundary.

III. Mindlin model of a flexible plate

The Mindlin model of an elastic plate is the generalization to the 2D case of the Timoshenko model of a flexible beam, [10]. These models of simple flexible structures take into account the vertical deformation and the rotation of the cross section. The resulting mathematical descriptions are more accurate in predicting system response than the Euler-Bernoulli or the Kirchoff ones but more difficult to be utilized (e.g. in control applications) because of their complexity.

The *classical* formulation of the Mindlin plate with spatial domain $\mathcal{Z} \subset \mathbb{R}^2$ is

$$\begin{array}{rcl} \rho \dfrac{h^3}{12} \dfrac{\partial^2 \psi_x}{\partial t^2} &=& q_x - \dfrac{\partial m_x}{\partial x} - \dfrac{\partial m_{xy}}{\partial y} \\ \rho \dfrac{h^3}{12} \dfrac{\partial^2 \psi_y}{\partial t^2} &=& q_y - \dfrac{\partial m_{xy}}{\partial x} - \dfrac{\partial m_y}{\partial y} \\ \rho h \dfrac{\partial^2 w}{\partial t^2} &=& \dfrac{\partial q_x}{\partial x} + \dfrac{\partial q_y}{\partial y} \end{array} \tag{9}$$

where w is the vertical displacement, ψ_x and ψ_y are the deflection of the cross section in the x and y direction respectively, h is the thickness of the plate, while m_x, m_y, m_{xy}, q_x and q_y are given by

$$\begin{array}{rcl} m_x &=& -D\left(\dfrac{\partial \psi_x}{\partial x} + \nu \dfrac{\partial \psi_y}{\partial y}\right) \\ m_y &=& -D\left(\nu \dfrac{\partial \psi_x}{\partial x} + \dfrac{\partial \psi_y}{\partial y}\right) \\ m_{xy} &=& -D \dfrac{1-\nu}{2} \left(\dfrac{\partial \psi_x}{\partial x} + \dfrac{\partial \psi_y}{\partial y}\right) \\ q_x &=& kGh\left(\dfrac{\partial w}{\partial x} - \psi_x\right) \\ q_y &=& kGh\left(\dfrac{\partial w}{\partial y} - \psi_y\right) \end{array}$$

with ν the Poisson ratio, D the plate module, G the stiffness module and k a corrective factor, which is equal to $\pi^2/12$, [14], [15]. These quantities represent the stress of the plate, which is a function of the vertical deformation and of the deflection of the plate itself.

The total energy $\mathcal{H}$ of the system is

$$\mathcal{H} = \mathcal{K} + \mathcal{W} \tag{10}$$

where $\mathcal{K}$ is the kinetic and $\mathcal{W}$ the potential energy, with area densities K and W respectively given by

$$K = \frac{\rho}{2}\left\{ \frac{h^3}{12}\left[\left(\frac{\partial \psi_x}{\partial t}\right)^2 + \left(\frac{\partial \psi_y}{\partial t}\right)^2\right] + h\left(\frac{\partial w}{\partial t}\right)^2 \right\}$$
$$W = \frac{1}{2}\big(m_x \Gamma_x + m_y \Gamma_y + m_{xy}\Gamma_{xy} + q_x \Gamma_{xz} + q_y \Gamma_{yz}\big)$$

where

$$\begin{array}{rcl} \Gamma_x &=& -\dfrac{\partial \psi_x}{\partial x} \\ \Gamma_y &=& -\dfrac{\partial \psi_y}{\partial y} \\ \Gamma_{xy} &=& -\left(\dfrac{\partial \psi_y}{\partial x} + \dfrac{\partial \psi_x}{\partial y}\right) \\ \Gamma_{xz} &=& -\psi_x + \dfrac{\partial w}{\partial x} \\ \Gamma_{yz} &=& -\psi_y + \dfrac{\partial w}{\partial y} \end{array} \tag{11}$$

are the the strain variables, [14].

IV. dpH formulation of the Mindlin plate

In order to properly obtain the port Hamiltonian model of a physical system, it is necessary to determine the right set of energy (state) variables χ and to define the corresponding Stokes–Dirac structure (i.e. the skew-adjoint operator J of Prop. 2.2). As discussed in [10] and from an analysis of the energy function (10), it seems natural to assume as energy variables the (translational and rotational) momenta and the strain terms (11), that is

$$\chi := \left[\rho h v, \Gamma_{xz}, \Gamma_{yz}, \rho\frac{h^3}{12}\omega_x, \rho\frac{h^3}{12}\omega_y, \Gamma_x, \Gamma_y, \Gamma_{xy}\right]^{\mathrm{T}} \tag{12}$$

with $v = \dot{w}$, $\omega_x = \dot{\psi}_x$ and $\omega_y = \dot{\psi}_y$. Consequently, the flows are related to the time derivatives of the energy variables, that is $f = -\dot{\chi}$, as in (6). Moreover, the efforts are the co-energy variables given by

$$e := [v, q_x, q_y, \omega_x, \omega_y, m_x, m_y, m_{xy}]^{\mathrm{T}} \tag{13}$$

It is easy to check that, if the Hamiltonian function is chosen as in (10), then $e = \delta_\chi \mathcal{H}$, as in (6).

The equations of motion (9) can be re-written in terms of the energy and co-energy variables (12) and (13), so that the system (14) can be obtained. This system is the dpH formulation of the Mindlin model of an elastic plate. The operator J introduced in (14) is skew-adjoint. Then, it is possible to define a Stokes–Dirac structure in the form (4). Clearly, it is necessary to determine the operator $B_\mathcal{Z}$, which

$$
\underbrace{\begin{pmatrix} \rho h \dot{v} \\ \dot{\Gamma}_{xz} \\ \dot{\Gamma}_{yz} \\ \rho \frac{h^3}{12} \dot{\omega}_x \\ \rho \frac{h^3}{12} \dot{\omega}_y \\ \dot{\Gamma}_x \\ \dot{\Gamma}_y \\ \dot{\Gamma}_{xy} \end{pmatrix}}_{f=-\dot{\chi}} = \underbrace{\begin{pmatrix} 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial x} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial y} \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 & 0 & 0 \end{pmatrix}}_{-J} \underbrace{\begin{pmatrix} v \\ q_x \\ q_y \\ \omega_x \\ \omega_y \\ m_x \\ m_y \\ m_{xy} \end{pmatrix}}_{e=\delta_\chi \mathcal{H}} \tag{14}
$$

$$
\begin{aligned}
-e^{\mathrm{T}} f = {} & v \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right) + q_x \left(\frac{\partial v}{\partial x} - \omega_x \right) + q_y \left(\frac{\partial v}{\partial y} - \omega_y \right) + \omega_x \left(q_x - \frac{\partial m_x}{\partial x} - \frac{\partial m_{xy}}{\partial y} \right) \\
& + \omega_y \left(q_y - \frac{\partial m_{xy}}{\partial x} - \frac{\partial m_y}{\partial y} \right) - m_x \frac{\partial \omega_x}{\partial x} - m_y \frac{\partial \omega_y}{\partial y} - m_{xy} \left(\frac{\partial \omega_x}{\partial y} + \frac{\partial \omega_y}{\partial x} \right) \\
= {} & \frac{\partial}{\partial x} \left(v q_x - \omega_x m_x - \omega_y m_{xy} \right) + \frac{\partial}{\partial y} \left(v q_y - \omega_y m_y - \omega_x m_{xy} \right)
\end{aligned} \tag{15}
$$

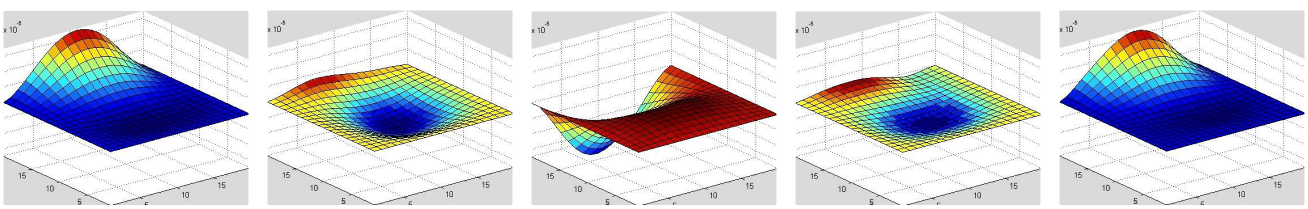

Fig. 1. Simulation of the uncontrolled Mindlin plate. The system is clamped except along the $x = L_x$ side.

provides the set of boundary variables, and the operator B_J, which is necessary in order to compute the boundary energy flow.

From (15) and the Stokes' Theorem, we have that

$$
-\int_{\mathcal{Z}} e^{\mathrm{T}} f = \oint_{\partial \mathcal{Z}} \left(v q_n - \omega_n m_n - \omega_t m_{nt} \right) \tag{16}
$$

where the subscripts *n* and *t* denote the normal and tangential directions to the border respectively. Consequently, the boundary terms or, equivalently, the operator $B_{\mathcal{Z}}$ are given by

$$
w = B_{\mathcal{Z}}(e) = [v, \omega_n, \omega_t, m_{nt}, m_n, q_n]^{\mathrm{T}}
$$

while the power flow through the boundary can be determined by integrating $\frac{1}{2} B_J(w, w) = \frac{1}{2} w^{\mathrm{T}} B_J w$ on $\partial \mathcal{Z}$, with the symmetric matrix B_J given by

$$
B_J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

Note 4.1: By simply analyzing the corresponding Stokes–Dirac structures, i.e. the operator J, it is easy to note the similarities between Mindlin plate and Timoshenko beam. In both cases, the effect of rotatory inertia and the presence of deformation due to shear are revealed by the presence of a power flow that couples the vertical motion with the rotational motion of the cross section.

V. Control by Damping Injection

A. Boundary control

In this section, the control by damping injection [17] of the Mindlin plate is briefly discussed. The energy function (10) assumes its minimum in the *zero* configuration ($\chi = 0$), the undeformed state. As any mechanical system, if a dissipative effect is present or introduced by means of a controller, it is possible to drive the state to the configuration in which the (open loop) energy function assumes a local minimum. In the case of the Mindlin plate, a dissipative controller can make the undeformed state (asymptotically) stable. If the controller can interact with the system through the border, dissipation can be introduced by terminating the boundary ports with a *dissipative element*, i.e. by a generalized *impedance* simulated by the control algorithm.

For simplicity, let us assume that the spatial domain of the flexible plate is rectangular and given by $\mathcal{Z} = [0, L_x] \times [0, L_y]$, with $L_x, L_y > 0$.

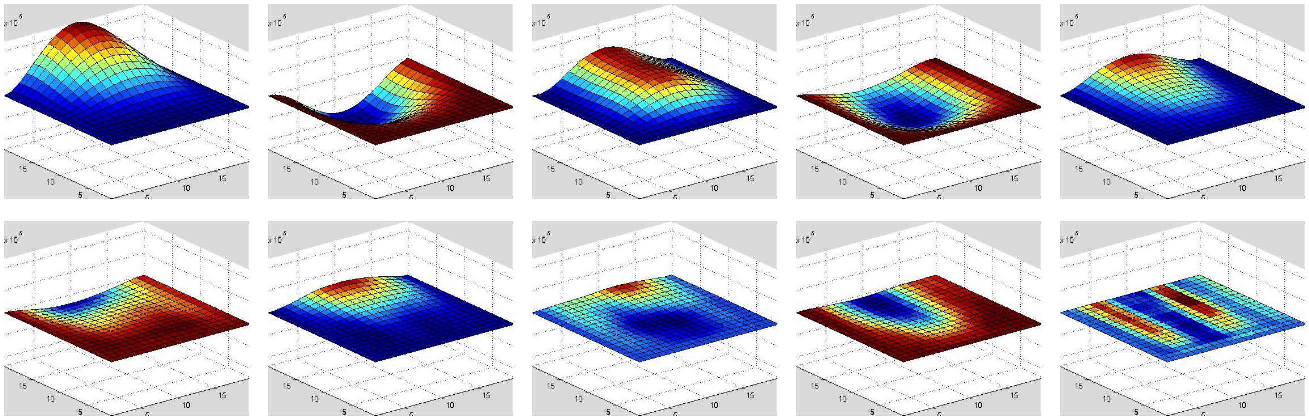

Fig. 2. Simulation of the boundary controller by damping injection. The controller is interconnected to the system along the $x = L_x$ side, while the plate is clamped on the others.

The control law

$$\begin{cases} v(L_x, y, t) = -\beta_t q_x(L_x, y, t) \\ \omega_x(L_x, y, t) = -\beta_{rn} m_x(L_x, y, t) \\ \omega_y(L_x, y, t) = -\beta_{rt} m_{xy}(L_x, y, t) \\ w(0, y, t) = \psi_x(0, y, t) = \psi_y(0, y, t) = 0 \\ w(x, 0, t) = \psi_x(x, 0, t) = \psi_y(x, 0, t) = 0 \\ w(x, L_y, t) = \psi_x(x, L_y, t) = \psi_y(x, L_y, t) = 0 \end{cases} \tag{17}$$

with β_t, β_{rn}, $\beta_{rt} > 0$, is able to (asymptotically) stabilize the plate around the undeformed configuration. More in detail, relation (17) states that the controller is interconnected to the system along the $x = L_x$ side, while the plate is clamped on the others. Furthermore, the controller can apply a torque and a force, thus allowing the regulation of both vertical displacement and rotation of the cross section. If it is assumed that $\beta_t = 0$, then the controller can directly act only on the angular deformation of the plate but still assuring the stability of the scheme. If no control action is applied and the $x = L_x$ side is free to move, i.e. no force/torque action is applied, the system is not asymptotically stable, as shown in the simulation of Fig. 1. Here, the distributed parameter system (14) has been spatially discretized by using the so-called *line method*, [18]. According to this method, the spatial domain is approximated by a grid of $n_x \times n_y$ small rectangular elements, while the partial derivatives with respect to the spatial variables x and y are replaced by a discrete approximation. The result is an high order set of ODEs that can be integrated using standard methods (e.g. in Matlab). The parameters of the plate have been reported in Table I.

Stability of the control scheme (17) can be easily proved by considering (8), (16) and (17). We have that

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = -\int_0^{L_x} \left(\beta_t q_x^2 + \beta_{rn} m_x^2 + \beta_{rt} m_{xy}^2\right) \mathrm{d}y \leq 0$$

relation showing that the energy of the system is not increasing along system trajectories. Moreover, asymptotic stability can be proved by applying a generalization of La Salle Theorem to the distributed parameter case, as discussed in [19] or by reformulating the problem within the framework proposed in [20].

L_x, L_y	1
G	$3.84 \cdot 10^4$
D	1.14
ν	0.3
h	0.05
ρ	10
k	$\pi^2/12$
n_x, n_y	20
β_t	1
β_{rn}, β_{rt}	5

TABLE I

MINDLIN PLATE AND CONTROLLER PARAMETERS USED IN THE SIMULATIONS.

The behavior of the proposed control technique is presented in Fig. 2 in order to show the validity of the boundary regulator (17) and the asymptotic stability of the closed–loop system. The parameters of the plate and its initial state are the same of the simulation of Fig. 1 while the values of the controller impedances have been reported in Table I.

B. Distributed control

A different way to control the system via damping injection can be to interconnect the regulator along the spatial domain $\mathcal{Z}$. This approach requires the definition of a distributed power port on $\mathcal{Z}$ and, then, the Stokes–Dirac structure (4) has to be modified accordingly. Denote by $\mathcal{F}_d \times \mathcal{E}_d$ the space of power variables associated to the distributed port and by G_d a differential operator in the form (2). Then, it is possible to prove that the following subset

$$\begin{aligned} \mathbb{D} = \Big\{ (f, f_d, e, e_d, w) \in \mathcal{F} \times \mathcal{F}_d \times \mathcal{E} \times \mathcal{E}_d \times \mathcal{W} \ \mid \\ f = -Je - G_d f_d, \ e_d = G_d^* e \\ w = B_{\mathcal{Z}}(e, f_d) \Big\} \end{aligned} \tag{18}$$

is a Stokes–Dirac structure, [5], [6].

In the case of the Mindlin plate, the space of power variables associated with the distributed port is given by $\mathcal{F}_d \times \mathcal{E}_d$, where $\mathcal{F}_d \equiv \mathcal{E}_d = (L_2(\mathcal{Z}))^3$, while the operator G_d is given by

$$G_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Consequently, from (18), given a *distributed* flow $f_d = [f_{d,t}, f_{d,rx}, f_{d,ry}]^{\mathrm{T}}$, the corresponding effort is equal to

$$e_d = \begin{pmatrix} e_{d,t} \\ e_{d,rx} \\ e_{d,ry} \end{pmatrix} = \begin{pmatrix} v \\ \omega_x \\ \omega_y \end{pmatrix}$$

Under the assumption of a zero power flow through the boundary (e.g. the plate is clamped), from the properties of a Dirac structure, the following energy balance relation holds.

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_{\mathcal{Z}} e_d^{\mathrm{T}} f_d \tag{19}$$

Then, a (asymptotically) stabilizing control law based on damping injection is the following:

$$\begin{cases} f_{d,t} = -b_t v \\ f_{d,rx} = -b_{rx}\omega_x \\ f_{d,ry} = -b_{ry}\omega_y \end{cases} \tag{20}$$

with b_t, b_{rx}, $b_{ry} > 0$. The stability of the control law can be easily proved from (19), since

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = -\int_{\mathcal{Z}} \left(b_t v^2 + b_{rx}\omega_x^2 + b_{ry}\omega_y^2\right) \leq 0 \tag{21}$$

while asymptotic stability can be verified by means of classical results on infinite dimensional systems, [21]. Finally, it is interesting to note that the distributed control law (20) can be applied only on a subset $\bar{\mathcal{Z}} \subset \mathcal{Z}$ of the spatial domain if it is supposed that $b_t(z)$, $b_{rx}(z)$ and $b_{ry}(z)$ are greater than 0 for every $z \in \bar{\mathcal{Z}}$ and they are equal to 0 if $z \in \mathcal{Z} \setminus \bar{\mathcal{Z}}$. The power balance relation (21) still holds, but it is sufficient to integrate on $\bar{\mathcal{Z}}$.

VI. Conclusions

In this paper, the Mindlin model of an elastic plate has been described within the framework of distributed port Hamiltonian systems. Once the distributed port Hamiltonian (dpH) model of the plate has been introduced, simple control methodologies based on damping injection have been discussed. An interesting improvement could be the computation of a control actuation distribution (over the free boundary) that minimizes some meaningful cost index (e.g. the norm of certain signals).

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