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Optimality Condition for the Receding Horizon Control of Markov Jump Linear Systems with Non-observed Chain and Linear Feedback Controls

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***Abstract*— We demonstrate here that a necessary condition of optimality studied in a previous paper is in fact a necessary and sufficient condition of optimality for the receding horizon control problem of discrete-time Markov jump linear systems subject to noisy inputs. The performance index is quadratic and the information available to the controller does not involve observations of Markov chain states. Sequences of linear feedback gains that are independent of the Markov state is adopted, in accordance with the information available to the controller. We make use of an equivalent deterministic form of expressing the stochastic problem, and the complete solution given in feedback form, is obtained by dynamic programming arguments and by the benefit of some quadratic convex relations.**

I. INTRODUCTION

In [1], the authors have developed a necessary optimality condition of the receding horizon control problem for Markov Jump Linear Systems (MJLS) with unobservable Markov state chain, subject to exogenous stationary noise. The admissible control sequences should be given in linear feedback form that are independent of the Markov state, an assumption that is well-matched with the fact that the Markov state is unaccessible. In addition, that assumption renders a problem with restrict complexity, which is solvable by recursive methods. In the present paper we return to the subject, showing that those optimality conditions are not only necessary but also sufficient conditions of optimality.

The MJLS class comprises an important class of stochastic linear systems, and considerable interest has been focused on these systems over the last decades. The class is suited to model dynamic systems subject to random phenomena that presents abrupt changes in its structure or parameters, such as component failures or repairs, sudden environment changes, modification of the operating point of a system, etc. Several results can be found nowadays in the current literature concerning applications, stability conditions and optimal control problems. We can cite the monograph [2] and the articles [3], [4], [5], [6] as important contributions on the theoretical development of MJLS.

Receding horizon control problems have been studied in a vast literature of deterministic systems connected with the model predictive control (MPC) technique. Results for linear deterministic systems are amalgamated notably in [7] and [8]. For nonlinear systems one can observe an strong output drive in the present-day research; see the recent surveys [9], [10]. Receding horizon results applied in MJLS are relatively new, see [11], [12], [13], [14].

The aim of the present paper is to complete the solution to the problem studied in [1], by showing that the optimality conditions presented therein are not only necessary but also sufficient. These results involve a dynamic representation for the expectation of the process expressed in a convenient form, which yields an associated calculus to express the cost and dynamics in a deterministic equivalent form. The method in the former paper was variational, whereas here, we adopt standard dynamic programming to prove the sufficiency part of the optimal control characterization. We show that these conditions in fact produce the optimal solution in form of a feedback law defined in an appropriate state space.

In Section II we give some basic definitions and notation, in Section III we present the problem formulation, feedback concepts and provide a representation for the problem in terms of some linear operators that are quadratic with respect to the control variable. In Section IV we show a necessary and sufficient condition of optimality given by dynamic programming technique, which constitutes the complete solution to the studied problem. Finally, in Section V, we present some conclusions.

II. Definitions and Basic Concepts

Let $\mathcal{M}^{r,s}$ ($\mathcal{M}^r$) represent the linear space formed by all $r \times s$ ($r \times r$) real matrices. Let $\mathcal{S}^r$ represent the normed linear subspace of $\mathcal{M}^r$ of symmetric matrices such as $\{U \in \mathcal{M}^r : U = U'\}$, where U' denotes the transpose of U. Consider also $\mathcal{S}^{r0}$ ($\mathcal{S}^{r+}$) its closed (open) convex cone of positive semidefinite (definite) matrices $\{U \in \mathcal{S}^r : U \geq 0\ (> 0)\}$. Let $\mathscr{N} := \{1, \ldots, \eta\}$ be a finite set, and let $\mathbb{S}^r$ denote the linear space of all $\mathscr{N}$-sequences of matrices

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such that $\mathbb{S}^r = \{U = (U_1, \ldots, U_n) : U_i \in \mathcal{S}^r, i \in \mathcal{N}\}$. We also write $\mathbb{S}^{r0}$ ($\mathbb{S}^{r+}$) when $U_i \in \mathcal{S}^{r0}$ ($\in \mathcal{S}^{r+}$) for all $i \in \mathcal{N}$.

We employ the ordering $U > V$ ($U \geq V$) for elements of $\mathbb{S}^r$, meaning that $U_i - V_i$ is positive definite (semi-definite) for all $i \in \mathcal{N}$. For $U \in \mathbb{S}^r$ we use the following norm $\|\cdot\|_2$:

$$\|U\|_2^2 = \sum_{i \in \mathcal{N}} \text{ tr } \{U_i' U_i\},$$

where tr $\{\cdot\}$ represents the trace operator. It is known that $\mathbb{S}^r$ equipped with the above norm forms a Hilbert space with an inner product given by

$$\langle U, V \rangle = \sum_{i \in \mathcal{N}} \text{ tr } \{U_i' V_i\}.$$

We consider a controlled Markov jump linear system described as follows. Let $A := \{A_i \in \mathcal{M}^r : i \in \mathcal{N}\}$, $B := \{B_i \in \mathcal{M}^{r,s} : i \in \mathcal{N}\}$, $H := \{H_i \in \mathcal{M}^{r,\ell} : i \in \mathcal{N}\}$, $Q := \{Q_i \in \mathcal{S}^{r0} : i \in \mathcal{N}\}$, $R := \{R_i \in \mathcal{S}^{s+} : i \in \mathcal{N}\}$ and $F := \{F_i \in \mathcal{S}^{r0} : i \in \mathcal{N}\}$ be some associated finite set of matrices.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}, \mathbb{P})$ be the fundamental probability space. Let $\Theta := \{\theta(k); k = 0, 1, \ldots\}$ be the discrete-time homogeneous Markov chain, with $\mathcal{N}$ as state space, having $\mathbb{P} = [p_{ij}], \forall i, j \in \mathcal{N}$ as the transition probability matrix. The state of the Markov chain at a certain future time t, conditioned to the knowledge of the state at time instant k, is determined according to an associated probability distribution $\mu_{t|k}$ on $\mathcal{N}$, namely, $\mu_{t|k}(i) := \Pr(\theta(t) = i \mid \mathcal{F}_k)$. Considering the n-dimensional vector $\mu_{t|k} = [\mu_{t|k}(0), \ldots, \mu_{t|k}(i), \ldots, \mu_{t|k}(\eta)]'$, the state distribution of the chain, $\mu_{t|k}$, is defined as $\mu_{t|k} = (\mathbb{P}')^t \mu_{k|k}$. In the sequel we often set $k = 0$, thus we denote $\mu_{t|0}$ simply by μ_t.

III. Problem Formulation

Consider the process $\mathcal{G}$ in an underlying probability space defined by:

$$\mathcal{G} : \begin{cases} x(k+1) = A_{\theta(k)} x(k) + B_{\theta(k)} u(k) + H_{\theta(k)} w(k) \\ q(k) = x(k)' Q_{\theta(k)} x(k) + u(k)' R_{\theta(k)} u(k) \\ p(k) = x(k)' F_{\theta(k)} x(k) \\ k \geq k_0,\ x(k_0) = x_0,\ \theta(k_0) \sim \mu_0, \end{cases}$$

where x is an r-dimensional state vector and u is an s-dimensional control vector. The Markov chain is indexed by θ and the joint process $\{x, \theta\}$ is a Markovian process. The second and third expressions in $\mathcal{G}$ represent the cost by stage q and the final cost p, respectively. The model $\mathcal{G}$ is a discrete-time stochastic linear system subject to Markov jumps in the parameters, written in state variable form. The stochastic process $\{w(k); k \geq k_0\}$ is a second-order i.i.d. sequence of ℓ-dimensional random vectors with zero mean values and covariance matrix $\Sigma := E[w(k)w(k)'] \in \mathcal{S}^{r0}, \forall k \geq k_0$, where $E[\cdot]$ represents the expected value. We also know that $\{w(k); k \geq k_0\}$ is independent from $\{\theta(k); k \geq k_0\}$. In particular, $x(k)$ and $w(k)$ are independent random vectors. Notice that the matrices $A_{\theta(k)}$, $B_{\theta(k)}$ and $H_{\theta(k)}$ are functions of the process $\Theta = \{\theta(k); k \geq k_0\}$. Thus, whenever $\theta(k) = i$, $i \in \mathcal{N}$, one has that A_i, B_i and H_i.

The performance index associated with $\mathcal{G}$ is a standard quadratic cost functional with a horizon of N stages defined by

$$J^{k,N} := E_{x_k, \mu_{k|k}} \left[\sum_{\ell=0}^{N-1} q(k+\ell) + p(k+N) \right], \tag{1}$$

where $E_{x_k, \mu_{k|k}}[\ \cdot\ ] \equiv E[\ \cdot\ | x(k) = x_k, \theta(k) \sim \mu_{k|k}]$ and $N > 0$. The receding horizon control principle states that the cost functional (1) should be minimized at each time instant $k = k_0, \ldots, k_1$. The current input $u(k)$ is obtained by determining the input sequence $\{\hat{u}(k), \ldots, \hat{u}(k+N-1)\}$ that minimizes $J^{k,N}$, by setting $u(k) = \hat{u}(k)$. The remaining sequence is discarded and this procedure is repeated subsequently at each time instant.

The model is valid solely when the time index k is such that $k_0 \leq k \leq k_1 + N$, where $k_0, \ldots, k_1 + N$ represents the certainty range of $\mathcal{G}$.

A. Feedback Concepts

A restricted information pattern is imposed in the sense that the present state $x(k)$ is available but Θ cannot be precisely known on the interval from k to $k+N$, the distribution $\mu_{k|k}$ being the only information available. In this regard, we consider the regulation problem of receding horizon control with noisy input for $\mathcal{G}$, assuming that admissible controls are in the linear state feedback form as

$$u(k) = K^k x(k), \tag{2}$$

for each $k = k_0, \ldots, k_1$.

Let $\mathbf{K} := \{K^t \in \mathcal{M}^{s,r}, t = 0, \ldots, N-1\}$ be a sequence of feedback gains. Recall the cost functional $J^{k,N}$ in (1). In accordance with (2), we associate the sequence $\mathbf{K}$ with the receding state feedback as

$$\hat{u}(t+k) = K^t x(t+k), \quad t = 0, \ldots, N-1, \tag{3}$$

and denote accordingly, the system $\mathcal{G}_{\mathbf{K}}$ and the cost $J_{\mathbf{K}}^{k,N}$. For notational simplicity we set the initial stage k to coincide with the time origin, whenever possible, and we refer to the cost functional by $J_{\mathbf{K}}^N$, that is

$$J_{\mathbf{K}}^N = \mathrm{E}_{x_0, \mu_0} \left[\sum_{\ell=0}^{N-1} q(\ell) + p(N) \right], \tag{4}$$

with controls in the form $u(t) = K^t x(t)$, $t = 0, \ldots, N-1$. The class of all possible controls is denoted by $\mathcal{K}$.

B. Associated Functionals and Operators

We provide in this section an equivalent deterministic form of expressing the cost $J_{\mathbf{K}}^N$ that is convenient for optimization. As in [1], we define a set of matrices of conditional second moments of the state $X^t = \{X_i^t, i \in \mathcal{N}\}$ as

$$X_i^t = \mathrm{E}_{x_0, \mu_0}[x(t)x(t)' \mathbb{1}_{\{\theta(t)=i\}}],\ \forall i \in \mathcal{N}, \tag{5}$$

$t=0,\ldots,N$, where μ_0 and $x(0)=x_0$ are known vectors and $\mathbb{1}_{\mathscr{C}}$ represents the indicator function of the set $\mathscr{C}$. Recall that μ_t express the distribution of the Markov chain at a certain time t given μ_0. Let $\Psi^t \in \mathbb{S}^{r0}$ be

$$\Psi_i^t := \sum_{j\in\mathscr{N}} p_{ji}\mu_t(j)H_j\Sigma H_j', \ \forall i \in \mathscr{N}.$$

Let us introduce the following operators $\mathscr{E}:\mathbb{S}^{r0}\to\mathbb{S}^{r0}$ and $\mathscr{L},\mathscr{T}:\mathbb{S}^{r0}\times\mathbb{M}^{s,r}\to\mathbb{S}^{r0}$ defined, respectively, as

$$\mathscr{E}_i(\phi) := \sum_{j\in\mathscr{N}} p_{ij}\phi_j, \ \forall i\in\mathscr{N},$$
$$\mathscr{L}_i(\phi,\psi) := (A_i+B_i\psi)'\mathscr{E}_i(\phi)(A_i+B_i\psi), \ \forall i\in\mathscr{N},$$
$$\mathscr{T}_i(\phi,\psi) := \sum_{j\in\mathscr{N}} p_{ji}(A_j+B_j\psi)\phi_j(A_j+B_j\psi)', \ \forall i\in\mathscr{N}.$$

for any $\phi\in\mathbb{S}^{r0}$ and $\psi\in\mathbb{M}^{s,r}$.

The next representation result establishes the dynamics of $X_i^t, 0\le t\le N$ using the operators introduced above. The proof is detailed in [1].

Proposition 3.1: For any $\mathbf{K}\in\mathcal{K}$ and $X_i^0 = \mu_0(i)x(0)x(0)', \forall i\in\mathscr{N}$,

$$X_i^{t+1} = \mathscr{T}_i(X^t,K^t)+\Psi_i^t, \tag{6}$$

$\forall i\in\mathscr{N}$ and $t=0,\ldots,N-1$.

The above representation shows that the second moment dynamics (6) is non-linear (quadratic) in K^t, and given any sequence $\{K^0,\ldots,K^{N-1}\}$ in $\mathcal{K}$, the corresponding trajectory $X^0,\ldots,X^N$ is uniquely determined.

Next, we state the representation of the cost in terms of the trajectories $X^t=(X_1^t,\ldots,X_\eta^t)$.

Proposition 3.2: The cost $J_{\mathbf{K}}^N$ is identical to

$$J_{\mathbf{K}}^N = \sum_{t=0}^{N-1}\langle Q+(K^t)'RK^t, X^t\rangle + \langle F, X^N\rangle. \tag{7}$$

Proof: Let $Z=\{Z_i\in\mathbb{M}^{r0}, i\in\mathscr{N}\}$ be any set. Then

$$\begin{aligned}\mathrm{E}_{x_0,\mu_0}[x(t)'Z_{\theta(t)}x(t)] &= \sum_{i\in\mathscr{N}}\mathrm{E}_{x_0,\mu_0}[x(t)'Z_{\theta(t)}x(t)\mathbb{1}_{\{\theta(t)=i\}}]\\ &= \sum_{i\in\mathscr{N}}\mathrm{tr}\{Z_iX_i^t\} = \langle Z, X^t\rangle.\end{aligned}$$

Applying the result above in (4) with $u(t)=K^tx(t)$ gives (7). ■

Remark 3.1: Note that the stochastic control problem associated to the receding cost can now be solved at each k by considering the problem:

minimizing $J_{\mathbf{K}}^N$ in (7)
subject to (6), $\mathbf{K}\in\mathcal{K}$ and $X_i^0=\mu_{k|k}(i)x(k)x(k)'\forall i\in\mathscr{N}$.

This is an unconstrained optimal control problem involving a linear dynamics (see operator $\mathscr{T}_i$), for which the dependence on the control sequence $\mathbf{K}$ is quadratic. The cost functional possesses a linear dependence with respect to the trajectories $X^t, 0\le t\le N$ and a quadratic dependence on the control sequence $\mathbf{K}$. One direct approach to the problem is to solve it as static optimization problem. This involves substituting the trajectory X^t by its representation in terms of the past sequence $K^0,\ldots,K^{t-1}$ and the initial condition X^0. Then one could apply hill-climbing techniques to solve it. However this would be an unsatisfactory way of solving it since by throwing away the dynamic structure, the procedure does not lead to any conclusion regarding optimality and the characterization of solutions, as the next simple example shows.

Example 3.1: Consider a SISO Markov jump system

$$x(t+1) = a_{\theta(t)}x(t)+b_{\theta(t)}u(t),$$

with $\mathscr{N}=\{1,2\}$ and parameters $a_1=0.3$, $a_2=0.1$, $b_1=-1$, $b_2=1$. Let $u(t)=g_tx(t)$ be the control input. According to (7) the cost is defined as

$$J^N_{\{g_0,\ldots,g_{N-1}\}} = \sum_{t=0}^{N-1}\Big(\sum_{i\in\mathscr{N}}(q_i+r_ig_t^2)X_i^t + f_iX_i^N\Big),$$

where X_i^t, $i=1,2$ are scalars defined as in (6). The cost parameters are $q_1=q_2=0.4$, $r_1=r_2=1$, $f_1=f_2=0.5$, and we adopt $N=2$, $x(0)=2$, $\mu_0=[0.25 \ \ 0.75]$ and the stochastic transition matrix

$$\mathbb{P} = \begin{bmatrix}0.6 & 0.4\\ 0.2 & 0.8\end{bmatrix}.$$

After some algebraic manipulations, one can obtain the cost explicitly as

$$\begin{aligned}J^2_{\{g_0,g_1\}} &= 1.6+4g_0^2+(0.4+g_1^2)(0.3-g_0)^2\\ &+(1.2+3g_1^2)(0.1+g_0)^2+0.3(0.3-g_1)^2(0.3-g_0)^2\\ &+0.2(0.1+g_1)^2(0.3-g_0)^2+0.1(0.3-g_1)^2(0.1+g_0)^2\\ &+0.4(0.1+g_1)^2(0.1+g_0)^2.\end{aligned} \tag{8}$$

Note that (8) as function of g_0 with g_1 fixed, or vice-versa, is a quadratic and convex function with respect to the remaining variable. However, the multiple variable function (8) is not even convex as can be seen from its contour plot in Fig 1. The figure suggests that the function presents an unique minimum but the solution could be nonunique.

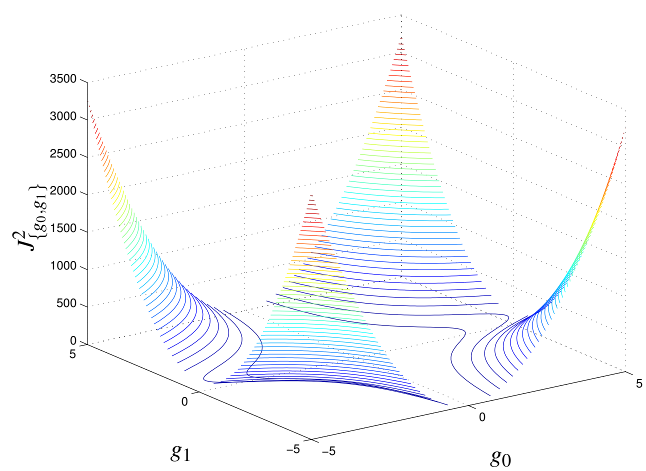

Fig. 1. Three dimensional contours of the cost $J^2_{\{g_0,g_1\}}$.

The analysis developed in the sequel is aimed at an explicit feedback control rule for the control problem stated in Remark 3.1. The precise sense is to obtain a set of algebraic function $\varphi^t : \mathbb{S}^{r0} \times \mathcal{M}^{s,r} \to \mathcal{M}^{s,r}$, $t = 0, \ldots, N-1$ such that for any $X \in \mathbb{S}^{r0}$ there exists $K \in \mathcal{M}^{s,r}$ with $\varphi^t(K, X) = 0 \in \mathcal{M}^{s,r}$. Thus, given $X^t = X \in \mathbb{S}^{r0}$, one can get the optimal action K^t from the above relation, and $\varphi^t(\cdot)$ represents the optimal closed loop policy for the optimal control problem stated in Remark 3.1.

IV. Optimality Analysis

We wish to choose a sequence $\mathbf{K} = \{K^0, \ldots, K^{N-1}\} \in \mathscr{K}$ so as to minimize the cost $J^N_{\mathbf{K}}$ in (7). We use dynamic programming technique as follows. Let $V_t(X)$ be the infimum of the functional cost for the intermediate problem starting at $X^t = X \in \mathbb{S}^{r0}$ for some $t \in \{0, \ldots, N\}$. This is known as the value function at time t. This definition indicates that V_t ought to satisfy the optimality principle [15]:

$$V_t(X) = \inf_G \left[\langle Q + G'RG, X \rangle + V_{t+1}(\mathscr{T}(X,G) + \Psi^t) \right] \quad (9)$$

where the infimum is over all admissible $G \in \mathscr{M}^{s,r}$. To check this, set $X^t = X$ and feedback gain $K^t = G$. Then:
a) The cost paid at time t is $\langle Q + G'RG, X \rangle$.
b) The next state is $X_i^{t+1} = \mathscr{T}_i(X,G) + \Psi_i^t, \forall i \in \mathscr{N}$.
Thus $V_{t+1}(\mathscr{T}(X,G) + \Psi^t)$ is the minimal cost for the rest of the problem if gain value G is applied at stage t, and obviously,

$$V_t(X) \le \langle Q + G'RG, X \rangle + V_{t+1}(\mathscr{T}(X,G) + \Psi^t) \quad (10)$$

and this holds for any value of G.

On the other hand, suppose that $\{\bar{X}^k, \bar{K}^k\}$ is optimal over the stages $t \le k \le N$ starting at $\bar{X}^t = X$. The principle of optimality indicates that

$$V_\ell(\bar{X}^\ell) = \sum_{k=\ell}^{N-1} \langle Q + (\bar{K}^k)' R \bar{K}^k, \bar{X}^k \rangle + \langle F, \bar{X}^N \rangle.$$

where ℓ is either t or $t+1$. Hence,

$$V_t(X) = \langle Q + (\bar{K}^t)' R \bar{K}^t, X \rangle + V_{t+1}(\mathscr{T}(X, \bar{K}^t) + \Psi^t) \quad (11)$$

since $\bar{X}^t = X$. Now, (10) and (11) together imply that (9) holds. Note that at the terminal time N the value function is

$$V_N(X) = \langle F, X \rangle. \quad (12)$$

The proof of the next result is addressed in the appendix.

Theorem 4.1: (Verification theorem) Suppose $V_N, V_{N-1}, \ldots, V_0$ satisfy the Bellman equation (9) with terminal condition (12). Suppose that for each $0 \le t \le N-1$, the infimum in (9) is achieved by $G = \bar{K}^t$ that satisfies $\varphi^t(G, \bar{X}^t) = 0$, where $\varphi^t(\cdot)$ is some specified function. Now define $(\bar{X}^t, \bar{K}^t)$ recursively as follows:

$$\bar{X}^0 = X^0, \quad \varphi^t(\bar{K}^t, \bar{X}^t) = 0,$$
$$\bar{X}^{t+1} = \mathscr{T}(\bar{X}^t, \bar{K}^t) + \Psi^t, \; t = 0, \ldots, N-1.$$

Then $\bar{\mathbf{K}} = \{\bar{K}^0, \ldots, \bar{K}^{N-1}\}$ is an optimal feedback gain sequence and the minimum cost is $V_0(X^0)$.

The next result establishes the optimal gain sequence for the overall control problem.

Theorem 4.2: The solution of the Bellman equation (9) for the receding horizon control problem is given by

$$V_t(X) = \inf_{K^t} \langle L^t, X \rangle + \mu_t' \omega^t, \quad t = 0, \ldots, N-1, \quad (13)$$

with terminal value $V_N(X) = \langle L^N, X \rangle$, where L^t and w^t satisfies the following recurrences

$$L_i^t = Q_i + (K^t)' R_i K^t + \mathscr{L}_i(L^{t+1}, K^t), \; \forall i \in \mathscr{N}, \quad (14)$$
$$\omega_i^t = \mathscr{E}_i(\omega^{t+1}) + \mathrm{tr}\{\mathscr{E}_i(L^{t+1}) H_i \Sigma H_i'\}, \; \forall i \in \mathscr{N}, \quad (15)$$

with terminal condition $L_i^N = F_i$, $\omega_i^N = 0$, $\forall i \in \mathscr{N}$. The optimal feedback gain sequence $\mathbf{K} = \{K^0, \ldots, K^{N-1}\}$ satisfies $\varphi^t(K^t, X^t) = 0$, $t = 0, \ldots, N-1$, with

$$\varphi^t(K, X) = \sum_{i \in \mathscr{N}} [(R_i + B_i' \mathscr{E}_i(L^{t+1}) B_i) K + B_i' \mathscr{E}_i(L^{t+1}) A_i] X_i \quad (16)$$

Moreover, the minimal cost is given by

$$V_0(X^0) = \langle L^0, X^0 \rangle + \mu_0' \omega^0. \quad (17)$$

Remark 4.1: Note that the feedback control given by the law $\varphi^t(K, X) = 0$ in (16) may not be unique; however it is always possible to provide an optimal solution. To see this we write $\varphi^t(K, X) = 0$ equivalently as

$$\left\{ \sum_{i \in \mathscr{N}} X_i \otimes (R_i + B_i' \mathscr{E}_i(L^{t+1}) B_i) \right\} \mathrm{vec}(K) = - \mathrm{vec} \left\{ \sum_{i \in \mathscr{N}} B_i' \mathscr{E}_i(L^{t+1}) A_i X_i \right\} \quad (18)$$

which is an square set of linear equations. Here we denote by $M \otimes N = [m_{ij} N] \in \mathcal{M}^{qs,rt}$ the Kronecker product of matrices $M = [m_{ij}] \in \mathcal{M}^{q,r}$ and $N \in \mathcal{M}^{s,t}$, and we set $\mathrm{vec}(M) = [m_{11} \; m_{21} \; \ldots \; m_{q1} \; m_{12} \; \ldots \; m_{q2} \; m_{13} \; \ldots \; m_{qr}]' \in \mathcal{M}^{qr,1}$. We get (18) by using the identity $\mathrm{vec}(MZN) = N' \otimes M \cdot \mathrm{vec}(Z)$.

A. Proof of Theorem 4.2

We present some preliminaries to the proof of Theorem 4.2. Let $f(X)$ be a differentiable real valued function of the matrix $X = [x_{ij}] \in \mathcal{M}^{m,n}$. The matrix of first order partial derivatives of $f(X)$ is defined by $\partial f(X) / \partial X = [\partial f(X) / \partial x_{ij}] \in \mathcal{M}^{m,n}$, and the Hessian matrix of second order partial derivatives is defined by $\partial^2 f(X) / \partial \mathrm{vec}(X) \partial \mathrm{vec}(X)' \in \mathcal{M}^{mn,mn}$. Now we can present the following result.

Lemma 4.1: (*i*) [16], [17]. Let $N \in \mathcal{M}^{n,n}, M \in \mathcal{M}^{m,m}$ and $Z \in \mathcal{M}^{m,n}$ be any matrices. Then

$$\begin{aligned} \mathrm{tr}\{ZNZ'M\} &= \mathrm{vec}(Z)'(N' \otimes M)\mathrm{vec}(Z) \\ &= \mathrm{vec}(Z)'(N \otimes M')\mathrm{vec}(Z) \end{aligned} \quad (19)$$

$$\frac{\partial \mathrm{tr}\{ZNZ'M\}}{\partial Z} = M'ZN' + MZN \quad (20)$$

$$\frac{\partial^2 \mathrm{tr}\{ZNZ'M\}}{\partial \mathrm{vec}(Z) \partial \mathrm{vec}(Z)'} = N \otimes M' + N' \otimes M. \quad (21)$$

(ii) [18]. Let $N \in \mathcal{M}^{n,n}$ and $M \in \mathcal{M}^{m,m}$ be any symmetric positive semi-definite matrices. Then $N \otimes M \geq 0$.

The next result asserts the convexity of $f(Z) = \mathrm{tr}\{ZNZ'M\}$.

Lemma 4.2: Let $N \in \mathcal{M}^{n,n}$ and $M \in \mathcal{M}^{m,m}$ be symmetric positive semi-definite matrices and $Z \in \mathcal{M}^{m,n}$ be any matrix. If $f(Z) = \mathrm{tr}\{ZNZ'M\}$, then $f(Z)$ is a convex function.

Proof: We can see from (19) that $f(Z) = \mathrm{tr}\{ZNZ'M\}$ is a quadratic function with respect to Z. Moreover, (21) yields that the Hessian matrix of $f(Z)$ is $N \otimes M' + N' \otimes M$; from Lemma 4.1($ii$) we conclude that it is a positive semi-definite matrix. The result follows from the well-known fact that any quadratic function with positive semi-definite Hessian matrix is a convex function. ■

Next we develop some useful equivalences considering an arbitrary $X^t = X \in \mathbb{S}^{r0}$.

Lemma 4.3:

$$(i)\ \langle L^{t+1}, \mathscr{T}(X,G) + \Psi^t \rangle = \sum_{i \in \mathscr{N}} \Big[\mathrm{tr}\{\mathscr{L}_i(L^{t+1},G)X_i\} + \mathrm{tr}\{\mathscr{E}_i(L^{t+1})H_i\Sigma H_i'\}\mu_t(i) \Big] \quad (22)$$

$$(ii)\ \mu_{t+1}'\omega^{t+1} = \sum_{i \in \mathscr{N}} \mathscr{E}_i(\omega^{t+1})\mu_t(i) \quad (23)$$

Proof: (i) Set $\hat{A}_j := A_j + B_j G, \forall j \in \mathscr{N}$. From the definition of Ψ_i^t and operators $\mathscr{T}_i(\cdot)$, get from basic trace properties that

$$\begin{aligned}
&\langle L^{t+1}, \mathscr{T}(X,G) + \Psi^t \rangle \\
&= \sum_{i \in \mathscr{N}} \mathrm{tr}\{\ L_j^{t+1} \sum_{j \in \mathscr{N}} p_{ji}[\hat{A}_j X_j \hat{A}_j' + \mu_t(j) H_j \Sigma H_j']\ \} \\
&= \sum_{j \in \mathscr{N}} \mathrm{tr}\{\ \sum_{i \in \mathscr{N}} p_{ij} L_j^{t+1} [\hat{A}_i X_i \hat{A}_i' + \mu_t(i) H_i \Sigma H_i']\ \} \\
&= \sum_{i \in \mathscr{N}} \Big[\mathrm{tr}\{\ \hat{A}_i' \sum_{j \in \mathscr{N}} p_{ij} L_j^{t+1} \hat{A}_i X_i\ \} \\
&\qquad + \mathrm{tr}\{ \sum_{j \in \mathscr{N}} p_{ij} L_j^{t+1} \mu_t(i) H_i \Sigma H_i' \} \Big]
\end{aligned}$$

and using operators $\mathscr{E}_i(\cdot)$ and $\mathscr{L}_i(\cdot)$ above we get (22).

$$\begin{aligned}
(ii) \mu_{t+1}'\omega^{t+1} &= \sum_{j \in \mathscr{N}} \mu_{t+1}(j)\omega_j^{t+1} = \sum_{i \in \mathscr{N}} \sum_{j \in \mathscr{N}} \mu_t(i) p_{ij} \omega_j^{t+1} \\
&= \sum_{i \in \mathscr{N}} \mu_t(i) \sum_{j \in \mathscr{N}} p_{ij} \omega_j^{t+1} = \sum_{i \in \mathscr{N}} \mathscr{E}_i(\omega^{t+1}) \mu_t(i).
\end{aligned}$$

■

Now we are prepared to develop the proof of Theorem 4.2.

Proof: The proof is constructed by backwards induction on t and the dynamic programming technique. Note from (12) that $V_N(X) = \langle F, X \rangle$, and the result is certainly true at $t = N$; hence $L^N = F, \omega^N = 0$. Now consider $t = N-1$. For a given $X^{N-1} = X$, the Bellman equation (9), becomes

$$V_{N-1}(X) = \inf_G \big[\langle Q + G'RG, X \rangle + \langle L^N, \mathscr{T}(X,G) + \Psi^{N-1} \rangle \big]. \quad (24)$$

Applying (22) in (24) we get that

$$\begin{aligned}
V_{N-1}(X) &= \inf_G \sum_{i \in \mathscr{N}} \Big[\mathrm{tr}\{\ [Q_i + G'R_iG + \mathscr{L}_i(L^N, G)]X_i \} \\
&\qquad + \mathrm{tr}\{\mathscr{E}_i(L^N) H_i \Sigma H_i'\} \mu_{N-1}(i) \Big] \\
&= \inf_G \langle L^{N-1}, X \rangle + \mu_{N-1}' \omega^{N-1} \quad (25)
\end{aligned}$$

which is an expression identical to (13) for $t = N-1$ and L^{N-1} and ω^{N-1} as in (14) and (15), respectively. Now we evaluate the optimal G that satisfy (25). First note that $\mathrm{tr}\{\mathscr{E}_i(L^N)H_i\Sigma H_i'\}$ does not depend on G. Applying direct differentiation to (25) with respect to G (recall property (20)) we get (16) and $G = K^{N-1}$ sets this derivative to zero, providing in principle a local minimum. On the other hand, (25) is a quadratic function of G as in (19) and by recognizing in Lemma 4.2 $M = R_i + B_i'\mathcal{E}_i(L^N)B_i \geq 0$, $N = X_i \geq 0$ and $Z = G$, we conclude that the expression inside "inf" on the right-hand side of (25) is a quadratic and convex function with Hessian matrix given by $\sum_{i \in \mathscr{N}} 2(R_i + B_i'\mathcal{E}_i(L^N)B_i) \otimes X_i \geq 0$. Thus, $G = K^{N-1}$ that satisfies $\varphi^t(K^{N-1}, X) = 0$ in (16) is a global minimum of (25), which amounts to a necessary and sufficient optimality condition at the stage $t = N-1$ via (16). To show that the result holds for $t < N-1$ we apply induction. Suppose it holds for $t = k+1$. Taking

$$V_{k+1}(X) = \langle L^{k+1}, X \rangle + \mu_{k+1}' \omega^{k+1}$$

with L^{k+1}, ω^{k+1} given, the Bellman equation (9) for any $X^k = X \in \mathbb{S}^{r0}$ becomes

$$V_k(X) = \inf_G \big[\langle Q + G'RG, X \rangle + \langle L^{k+1}, \mathscr{T}(X,G) + \Psi^k \rangle \rangle + \mu_{k+1}' \omega^{k+1} \big]. \quad (26)$$

Using Lemma 4.3 in (26) we get that

$$\begin{aligned}
V_k(X) &= \inf_G \sum_{i \in \mathscr{N}} \Big[\mathrm{tr}\{\ [Q_i + G'R_iG + \mathscr{L}_i(L^{k+1}, G)]X_i\ \} \\
&\qquad + [\mathrm{tr}\{\mathscr{E}_i(L^{k+1}) H_i \Sigma H_i'\} + \mathscr{E}_i(\omega^{k+1})] \mu_k(i) \Big] \\
&= \inf_G \langle L^k, X \rangle + \mu_k' \omega^k \quad (27)
\end{aligned}$$

which is an expression identical to (13) with L^k and ω^k as in (14) and (15), respectively. Each argument developed to show that (25) possesses a global minimum applies equally to (27), implying the fact that $G = K^k$ for which $\varphi^k(K^k, X)$ in (16) is set to zero is an optimal solution at time instant k. These arguments show that the result holds for $t = k$ and the result follows. Finally, (17) holds for the optimal sequence $\mathbf{K}$ according to Theorem 4.1. ■

V. Conclusions

In this paper we characterize the solutions to the receding horizon control problem of MJLS with noisy inputs. In the interest of applications, we assume that the state of the underlying Markov chain is not available to the controller. The controller developed here minimizes the expected cost functional associated to a fixed number of stages, within the admissible class of controls, which comprises affine feedback solutions that are independent of the chain state. The result relies on some evolution operators related to the expected value of second-order moments of the trajectory and cost matrices. The original stochastic control problem is coined in terms of an optimal control problem of deterministic nature, possessing linear dynamics and costs, which is quadratic in the control variable. The dynamic programming framework is employed which provides the optimal solution in feedback form, a result that also benefits from the quadratic convex structure of the problem.

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Appendix – Proof of Theorem 4.1

The Verification Theorem is expressed here in the paper notation.

Proof: Let $\mathbf{K} = \{K^0, \ldots, K^{N-1}\}$ be any control and $X^0, \ldots, X^N$ the corresponding trajectory for some given $X^0 = X$. Then, from (9) we have that

$$V_t(X^t) \leq \langle Q + (K^t)' R K^t, X^t \rangle + V_{t+1}(X^{t+1}). \qquad (28)$$

Hence

$$\begin{aligned} V_N(X^N) - V_0(X^0) &= \sum_{t=0}^{N-1} \big(V_{t+1}(X^{t+1}) - V_t(X^t) \big) \\ &\geq - \sum_{t=0} \langle Q + (K^t)' R K^t, X^t \rangle. \end{aligned} \qquad (29)$$

Since $V_N(X^N) = \langle F, X^N \rangle$ this shows that

$$V_0(X^0) \leq J^N_{\mathbf{K}}. \qquad (30)$$

On the other hand, when $X^t = \bar{X}^t$ and $K^t = \bar{K}^t$ equality in (28) holds by definition, and hence if $K^t = \bar{K}^t, t = 0, \ldots, N-1$, (29) holds with strict equality. Thus,

$$V_0(X^0) = J^N_{\bar{\mathbf{K}}}. \qquad (31)$$

Now (30) and (31) provide that $\bar{\mathbf{K}}$ is optimal and that the minimal cost is $V_0(X^0)$. ∎