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Input-output decoupling for m-inputs m-outputs linear mechanical systems through interconnection

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Abstract— **This paper deals with the input-output decoupling problem with asymptotic stability for a class of m-inputs m-outputs linear mechanical systems, through parallel connection with another mechanical system, called the controller. The paper gives a procedure for the design of a controller, which solves the above problem under some mild sufficient conditions, thus extending [1] where only 2-inputs 2-outputs linear mechanical systems were considered.**

I. INTRODUCTION

This paper is concerned with the input-output decoupling problem for linear mechanical systems having m-inputs and m-outputs ($m > 2$) with the approach used in the paper [1] for systems having 2-inputs and 2-outputs. The approach used in these two papers is quite different from the standard one: as a matter of fact we require that the controller be another mechanical system to be physically connected to the given one. Hence, with the approach proposed here, the controlled system is just another mechanical system, having in general more degrees of freedom than the given one, with the desired properties: it is asymptotically stable and it is input-output decoupled. The solution of the 2-inputs and 2-outputs case is a controller that copies some parts of the given system and includes a speed reducer. In general, it is necessary to duplicate only those parts that are essential for decoupling, since copying a larger portion of the system results in a loss of the structural properties of reachability and observability, and, consequently, of the possibility of stabilizing the overall system. The extension from the case of 2-inputs and 2-outputs systems to the generic case of $m > 2$ is not trivial: it turns out that if one tries to replicate the same procedure, in almost all cases there is a loss of reachability and observability. Hence, in this paper we propose a different iterative algorithm for the design of a controller. Based on the algorithm we obtain sufficient conditions for the solvability of the problem with the mentioned approach.

With respect to the standard way of designing a controller, constituted by a generic dynamical system taking as input the available outputs of the system (often the whole state), and giving as output the forces or torques to be applied to the actuated bodies, the approach taken here has many differences, that render it interesting. Some of such differences are actually restrictions, in fact the proposed controller has to be a very special dynamical system, with a strong structure: this limits severely the possible choices for the designer. On the other hand, with this, it is possible to use non-causal controllers, which is quite unusual in control theory.

Many of the concepts used in this paper are inspired by the classical tradition of studying mechanical systems through the analogy with suitable electric circuits (see, e.g., [2]) and by the use of passivity concepts [3].

For readability, the following section reports some preliminaries, already present in [1], concerned mainly with stability and stabilization of mechanical systems; such a problem is widely studied in the control literature, for recent results and for more references see [4], [5] and [6],[7].

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a linear mechanical system constituted by ideal point bodies, linear springs and linear dampers, moving on a line. Let $q_i(t)$ be the position at time $t \in \mathbb{R}$ with respect to an inertial reference frame of the i-th body, $i = 1, 2, ..., n$, where n is the number of the bodies and let $\mathbf{q}(t) := \begin{bmatrix} q_1(t) & q_2(t) & \cdots & q_n(t) \end{bmatrix}^T$; let $M_i \in \mathbb{R}, M_i > 0$, be the mass of the i-th body, $i = 1, 2, ..., n$. When present, let $K_{i,j} \in \mathbb{R}$ ($D_{i,j} \in \mathbb{R}$, $D_{i,j} \geq 0$) be the coefficient of elasticity (the damping factor) of the spring (the damper) possibly connecting the i-th body with the j-th one, $i = 1, 2, ..., n, j = i + 1, ..., n$; when present, let $K_{0,i} \in \mathbb{R}$ ($D_{0,i} \in \mathbb{R}$, $D_{0,i} \geq 0$) be the coefficient of elasticity (the damping coefficient) of the spring (the damper) possibly connecting the i-th body ($i = 1, 2, ..., n$) with the ground, constituted by an infinitely massive body (numbered with the index 0). Without loss of generality, in all the paper the length at rest of all the springs will be considered null.

Notation 1: $\mathbf{A} > 0$ (respectively, $\mathbf{A} \geq 0$) means that matrix $\mathbf{A}$ is real, symmetric and positive definite (respectively, semi-definite).

Let the system be described by the following kinetic and potential energies and by the following dissipation function, respectively: $\mathcal{T} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{B}\, \dot{\mathbf{q}} = \frac{1}{2}\sum_{i=1}^{n} M_i \dot{q}_i^2$, $\mathcal{U} = \frac{1}{2}\mathbf{q}^T \mathbf{H}\, \mathbf{q} = \frac{1}{2}\sum_{i=1}^{n} K_{0,i} q_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} K_{i,j}\left(q_i - q_j\right)^2$, $\mathcal{F} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{D}\, \dot{\mathbf{q}} = \frac{1}{2}\sum_{i=1}^{n} D_{0,i}\dot{q}_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} D_{i,j}\left(\dot{q}_i - \dot{q}_j\right)^2$, where $\mathbf{B}$ is the generalized inertia matrix which is diagonal and positive definite (since all the bodies have non-null mass), $\mathbf{D}$ is symmetric and semi-definite positive, and $\mathbf{H}$ is symmetric

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and semi-definite positive if all the springs have non-negative coefficients of elasticity.

Now, assume that m bodies (without loss of generality, the first m ones) are actuated by external forces $u_i(t), i = 1, ..., m$, and let $\mathbf{u}(t) := \begin{bmatrix} u_1(t) & \cdots & u_m(t) \end{bmatrix}^T$ be the input of the system. The relevant outputs of the system are both the positions $\mathbf{y}_q(t) = \begin{bmatrix} q_1(t) & \cdots & q_m(t) \end{bmatrix}^T$ of the first m bodies and their velocities $\mathbf{y}_v(t) = \begin{bmatrix} \dot{q}_1(t) & \cdots & \dot{q}_m(t) \end{bmatrix}^T$. The considered mechanical system is then described by the following equations:

$$\begin{aligned} \mathbf{B}\ \ddot{\mathbf{q}}(t) + \mathbf{D}\ \dot{\mathbf{q}}(t) + \mathbf{H}\ \mathbf{q}(t) &= \mathbf{E}\ \mathbf{u}(t), && (1)\\ \mathbf{y}_q(t) &= \mathbf{E}^T\ \mathbf{q}(t), && (2)\\ \mathbf{y}_v(t) &= \mathbf{E}^T\ \dot{\mathbf{q}}(t), && (3) \end{aligned}$$

where $\mathbf{E} \in \mathbb{R}^{n \times m}, \mathbf{E} = \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \end{bmatrix}^T$.

Notice that $\det\left(\mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H}\right)$ is not the null function since $\mathbf{B}$ is non-singular. By Laplace transformation, we have:

$$\begin{aligned} \mathbf{y}_q(s) &= \mathbf{E}^T\ (\mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H})^{-1}\mathbf{E}\ \mathbf{u}(s), \\ \mathbf{y}_v(s) &= \mathbf{E}^T\ (\mathbf{B}\ s + \mathbf{D} + \mathbf{H}\ \frac{1}{s})^{-1}\mathbf{E}\ \mathbf{u}(s), \end{aligned}$$

where $\mathbf{y}_q(s) = \mathcal{L}\left\{\mathbf{E}^T\ \mathbf{q}(t)\right\}$, $\mathbf{y}_v(s) = \mathcal{L}\left\{\mathbf{E}^T\ \dot{\mathbf{q}}(t)\right\}$, $\mathbf{u}(s) = \mathcal{L}\left\{\mathbf{u}(t)\right\}$. In the following, the **impedance matrix** $\mathbf{Z}(s) = \mathbf{E}^T\ (\mathbf{B}\ s + \mathbf{D} + \mathbf{H}\ \frac{1}{s})^{-1}\mathbf{E}$ and the **admittance matrix** $\mathbf{Y}(s) = \mathbf{Z}^{-1}(s)$ will be used repeatedly.

It is well known that a square rational matrix function $\mathbf{Z}(s)$ is **positive real** if $\mathrm{Re}(\mathbf{Z}(s))$ is positive semi-definite for all s having $\mathrm{Re}(s) \geq 0$; $\mathbf{Z}(s)$ is **BIBO stable** if each entry $Z_{i,j}(s) = \dfrac{N_{i,j}(s)}{D_{i,j}(s)}$, with $N_{i,j}(s)$ and $D_{i,j}(s)$ being coprime polynomials, is proper and its denominator $D_{i,j}(s)$ has all the roots with negative real part; the system (1), (2), (3) described by the impedance $\mathbf{Z}(s)$ is asymptotically stable if all the roots of $\det(\mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H}) = 0$ have negative real part.

It can be see (see also [1]) that, if $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$ are positive semi-definite, then both the impedance and the admittance of system (1), (2), (3) are positive real.

In this paper, the controller will not be a generic dynamical system taking as input $\mathbf{y}_q(t)$ and/or $\mathbf{y}_v(t)$ and giving as output $\mathbf{u}(t)$, but, rather, the controller will be another mechanical system having m terminal points to be physically connected to the first m bodies of the system. The connection can be either a direct one (i.e., the terminal point is glued to the mass of the body) or through an (ideal) speed reducer (e.g., an ideal gear reduction unit). The speed reducer, represented schematically in Figure 1, is a two terminal points object, without mass, friction and elasticity, characterized by the transmission ratio r. Denoting by v_i and u_i, $i \in \{a, b\}$, respectively, the velocity and the force applied to the i-th terminal point of the speed reducer, the equations describing its behaviour are

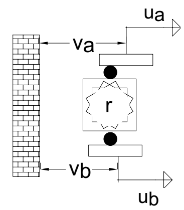

Fig. 1. The pictorial representation of the speed reducer.

$$\begin{aligned} v_b &= r\ v_a, && (4)\\ u_b &= \frac{1}{r}\ u_a. && (5) \end{aligned}$$

By integrating equation (4), if q_i, $i \in \{a, b\}$, denotes the position of the i-th terminal point of the speed reducer, we have $q_b = r\ q_a + c$, with c being an arbitrary constant that in this paper is taken equal to 0, without loss of generality. In the special case where $r = 1$, the speed reducer is equivalent to the direct connection, whereas when $r = -1$, it corresponds to inverting the velocity. In particular, if $r_1, ..., r_m$ are the transmission ratios of the reducers used for the connection (possibly, equal to 1), the controller is described by:

$$\mathbf{B}_c\ddot{\mathbf{q}}_c(t) + \mathbf{D}_c\dot{\mathbf{q}}_c(t) + \mathbf{H}_c\mathbf{q}_c(t) = \mathbf{0}, \quad (6)$$

$$\mathbf{y}_{c,q}(t) = \mathbf{E}_c^T\mathbf{q}_c(t), \quad (7)$$

with $\mathbf{q}_c(t) \in \mathbb{R}^{n_c}, \mathbf{E}_c \in \mathbb{R}^{n_c \times m}, \mathbf{E}_c = \begin{bmatrix} \mathbf{R} & \mathbf{0} \end{bmatrix}^T$, $\mathbf{R} = \mathrm{diag}(r_1, ..., r_m)$, $\mathbf{B}_c$ diagonal and positive semi-definite, $\mathbf{D}_c$ symmetric and positive semi-definite, $\mathbf{H}_c$ symmetric and $\det(\mathbf{B}_c\ s^2 + \mathbf{D}_c\ s + \mathbf{H}_c)$ being not the null function. The overall system is then described by the following equations:

$$\mathbf{B}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{H}\mathbf{q}(t) = \mathbf{E}\mathbf{u}(t) + \mathbf{E}\lambda(t), \quad (8)$$

$$\mathbf{B}_c\ddot{\mathbf{q}}_c(t) + \mathbf{D}_c\dot{\mathbf{q}}_c(t) + \mathbf{H}_c\mathbf{q}_c(t) = -\mathbf{E}_c\lambda(t), \quad (9)$$

$$\mathbf{y}_q(t) = \mathbf{y}_{c,q}(t), \quad (10)$$

where $\lambda(t)$ is the vector of the Lagrange multipliers used in order to take into account the equality constraint (10), which represents the forces exchanged between the system and the controller. Notice that, by eliminating the Lagrange multipliers and using the equality constraint (10), the overall system can be rewritten in the form (1), i.e., as an unconstrained mechanical system having $n + n_c - m$ degrees of freedom. The input of the overall system (8), (9), (10) is still $\mathbf{u}(t)$ and the relevant outputs are still $\mathbf{y}_q(t)$ and $\mathbf{y}_v(t)$. The control problem studied in this paper is stated formally as follows.

Problem 1: Find, if any, a controller of the form (6), (7) such that the overall system (8), (9), (10) is asymptotically stable and input-output decoupled (the latter being equivalent to be have a non-singular and diagonal impedance/admittance matrix).

The overall system (8), (9), (10) will be called the (**mechanical**) **parallel connection** of the system and the controller, because if $\mathbf{Y}(s)$ and $\mathbf{Y}_c(s)$ are the admittances of the mechanical system and of the controller, respectively, then the admittance of the parallel connection is $\mathbf{Y}_p(s) = \mathbf{Y}(s) + \mathbf{Y}_c(s)$. In addition, as for the impedance $\mathbf{Z}_p(s)$ of the parallel connection, it can be easily seen that, if $\mathbf{Z}_c(s)$ denotes the impedance of the controller,

$$\mathbf{Z}_p(s) = \mathbf{Z}(s)\left(\mathbf{I} + \mathbf{Z}_c^{-1}(s)\mathbf{Z}(s)\right)^{-1},$$

i.e. the parallel connection can be seen as a feedback system from the output $\mathbf{y}_v(t)$, in which the transfer matrix of the controller is $\mathbf{Z}_c^{-1}(s)$. Notice that $\mathbf{Z}_c^{-1}(s)$ is not necessarily proper and, moreover, that we are interested in a controller whose inverse be the impedance of a mechanical system, whence the classical tools for designing a controller that guarantees input-output decoupling with stability cannot be used.

A crucial property of the parallel connection of two mechanical systems is that if two systems having positive real impedance matrices $\mathbf{Z}_1(s)$ and $\mathbf{Z}_2(s)$ are connected in parallel, the impedance matrix of the parallel connection is still positive real. However, special care is to be used when the property of interest is the asymptotic stability of the system, which is stronger than the real positivity. It is stressed that the parallel connection of two asymptotically stable mechanical systems needs not be asymptotically stable.

We recall the well known fact (see [8]) that, for mechanical systems of the form (1), (2), (3), the stabilizability from the input $\mathbf{u}(t)$ and the detectability from the output $\mathbf{y}_v(t)$ can be tested, respectively, by means of the following two necessary and sufficient conditions:

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{B}\,s^2 + \mathbf{D}\,s + \mathbf{H} & \mathbf{E} \end{bmatrix}\right) = n, \quad \forall s \in \mathbb{C}, \operatorname{Re}(s) \geq 0, \tag{11}$$

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{B}\,s^2 + \mathbf{D}\,s + \mathbf{H} \\ s\,\mathbf{E}^T \end{bmatrix}\right) = n, \quad \forall s \in \mathbb{C}, \operatorname{Re}(s) \geq 0. \tag{12}$$

Remark 1: If $\det(\mathbf{H}) \neq 0$, then the stabilizability and detectability conditions (11), (12) are equivalent.

Remark 2: The structural properties of stabilizability and detectability can be lost by the mechanical parallel connection even if the original mechanical system and the controller are stabilizable and detectable.

The goal of this paper is to find a controller having admittance matrix $\mathbf{Y}_c(s)$ such that the overall system is input-output decoupled and asymptotically stable. The next three lemmas recall important facts that will be useful in the proof of the main result. The first one is concerned with the possibility of stabilizing a mechanical system — having positive real impedance $\mathbf{Z}(s)$ — by connecting the m actuated bodies with the ground by means of m identical dampers having damping coefficient equal to $D > 0$. Such a connection can be seen as the parallel connection of the given mechanical system and of the controller with singular $\mathbf{B}_c$ constituted by just the m dampers, having admittance matrix $\mathbf{Y}_c(s) = [\operatorname{diag}\{D, ..., D\}] = \mathbf{Z}_c^{-1}(s)$.

Lemma 1: If $D > 0$ and $\mathbf{Z}(s)$ is positive real, then $\mathbf{Z}_p(s) = \mathbf{Z}(s)(\mathbf{I}+D\,\mathbf{Z}(s))^{-1}$ is BIBO stable. If the stabilizability and detectability conditions (11) and (12) hold, then the parallel connection having $\mathbf{Z}_p(s)$ as impedance matrix is asymptotically stable.

The second intermediate result (Lemmas 2 and 3) is concerned with the possibility of rendering positive real the impedance matrix of a mechanical system by connecting the m actuated bodies with the ground by means of m identical springs having a positive and sufficiently high coefficient of elasticity K. Such a connection can be seen as the parallel connection of the given mechanical system and of the controller (with a singular $\mathbf{B}_c$) constituted by just the m springs, having admittance matrix $\mathbf{Y}_c(s) = \frac{K}{s}\mathbf{I} = \mathbf{Z}_c^{-1}(s)$. Moreover, the description of the parallel connection in the form (1), (2), (3) has the same $\mathbf{B}$ and $\mathbf{D}$ matrices of the given mechanical system, whereas for its matrix $\mathbf{H}_p$ we have:

$$\mathbf{H}_p = \mathbf{H} + \operatorname{diag}\left(\underbrace{K, ..., K}_{m \text{ times}}, \underbrace{0, ..., 0}_{n-m \text{ times}}\right). \tag{13}$$

The following is a necessary and sufficient condition for the impedance of the mechanical system to be positive real.

Lemma 2: If $\mathbf{B} > 0$, $\mathbf{D} \geq 0$ and the stabilizability and detectability conditions (11) and (12) hold, then $\mathbf{E}^T(\mathbf{B}\,s + \mathbf{D} + \mathbf{H}\,\frac{1}{s})^{-1}\mathbf{E}$ is positive real if and only if $\mathbf{H} \geq 0$.

As for the possibility of rendering matrix $\mathbf{H}_p$ positive definite through an appropriate choice of K, a necessary and sufficient condition is given by the following lemma (whose proof can be found in [9]).

Lemma 3: The matrix $\mathbf{H}_p$ in (13) can be rendered positive definite with a suitable choice of K if and only if the matrix $\mathbf{H}_{mm} \in \mathbb{R}^{(n-m)\times(n-m)}$ obtained by removing the first m rows and columns of $\mathbf{H}$ is positive definite. Moreover, if $\mathbf{H}_{mm} > 0$, then there exists $\bar{K} \geq 0$ such that $\mathbf{H}_p > 0$ for all $K > \bar{K}$.

The concept of a block decoupled admittance matrix is used in the description of the main result.

Definition 1: An $m \times m$ admittance matrix is said to be (p, q)-block decoupled if it is of the form $\operatorname{blockdiag}(\mathbf{Y}_1, \mathbf{Y}_2)$, where $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are, respectively, $p \times p$ and $q \times q$ matrices and $p + q = m$.

III. MAIN RESULT

Now, in order to design a controller solving Problem 1, the (p, q)-block decoupling problem is dealt with first. Consider the pictorial representation of the given mechanical system as a non-directed graph having $n + 1$ vertices, one for each body-mass and one for the ground, and one edge for each spring and damper. The following assumption can be made without loss of generality.

Assumption 1: Assume that the graph associated with the given mechanical system is connected.

Denote by $\overline{M}_1$ the system constituted by the first p actuated bodies and by the springs and dampers connecting such bodies with each other, and by $\overline{M}_2$ the system constituted by the latter $q = m - p$ actuated bodies and by the springs and dampers connecting such bodies with each other.

Denote by $\mathcal{X}_1$ the set of the vertices (masses) that are connected by a path of the graph with the system $\overline{M}_1$ (that is, with a body in $\overline{M}_1$), after removing the vertices corresponding to the bodies in $\overline{M}_2$ and the ground, and all the edges connecting such vertices. Symmetrically, define $\mathcal{X}_2$ by removing $\overline{M}_1$, the ground and the relevant edges. Let $S_{12} = \mathcal{X}_1 \cap \mathcal{X}_2$, $S_1 = \mathcal{X}_1 \setminus \{\mathcal{X}_1 \cap S_{12}\}$ and $S_2 = \mathcal{X}_2 \setminus \{\mathcal{X}_2 \cap S_{12}\}$. Let n_1, n_2 and n_3 be the cardinalities of S_1, S_2 and S_{12}, respectively.

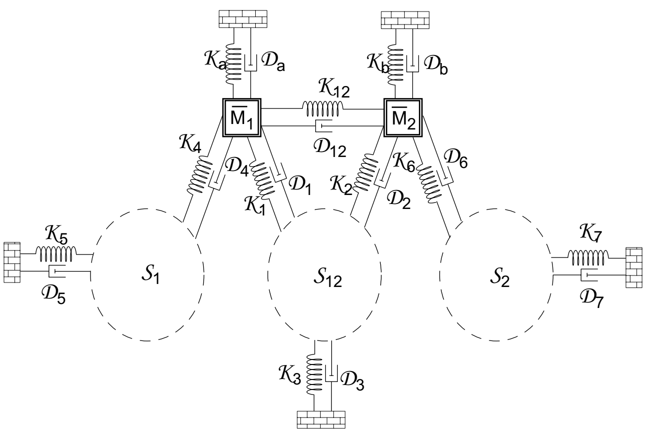

Fig. 2. Decomposition of the given system. For space reasons springs and dampers are depicted in different directions, but the reader should imagine all the motions as horizontal.

In this way, the n degrees of freedom of the given system can be partitioned into 5 sets, represented pictorially in Figure 2, with $n = n_1+n_2+n_3+m$. In Figure 2, the spring labeled by $\mathcal{K}_1$ represents a set of springs with possibly different coefficients of elasticity, each one connecting a different mass of the set S_{12} with $\overline{M}_1$ ($\mathcal{K}_1$ can be understood as the vector of such coefficients of elasticity); the same happens for the springs labeled by $\mathcal{K}_2, ..., \mathcal{K}_7, \mathcal{K}_a, \mathcal{K}_b$ and the dampers labeled by $\mathcal{D}_1, ..., \mathcal{D}_7, \mathcal{D}_a, \mathcal{D}_b$. Furthermore, not all such springs and dampers need to be actually present, since the case when a spring is missing can be considered by letting its coefficient of elasticity be equal to zero, and similarly for the dampers. However, in order to be consistent with the definition of S_1, S_2 and S_{12}, for each $i \in \{1, 2, 4, 6\}$ either $\mathcal{D}_i$ or $\mathcal{K}_i \neq 0$.

The controller proposed to solve the (p, q)-block decoupling problem is a n_c-degrees of freedom mechanical system, with $n_c = n_3+m$, constituted by a copy of the masses in $\overline{M}_1$ and $\overline{M}_2$, whose coordinates will be denoted by $\mathbf{q}_{c,1}$ and $\mathbf{q}_{c,2}$, respectively, and all the masses contained in the set S_{12}, with (i) a copy of all the springs and dampers that in the given system connect such masses with each other and with the ground, (ii) m additional dampers having damping coefficient $D > 0$ connecting the bodies with coordinates $\mathbf{q}_{c,1}$ and $\mathbf{q}_{c,2}$ with the ground and (iii) m additional springs with sufficiently high coefficient of elasticity K connecting the same m bodies with the ground. Such a coefficient of elasticity is to be chosen (as it will be clear in the proof) to guarantee the asymptotic stability of the overall system. Moreover, q speed reducers characterized by $r = -1$ are to be used to connect the bodies having coordinates $\mathbf{q}_{c,2}$ with the corresponding bodies in $\overline{M}_2$, whereas the bodies having coordinates $\mathbf{q}_{c,1}$ are to be glued with the corresponding ones in $\overline{M}_1$. In this way, the matrix $\mathbf{R}$ used in the description of the controller is $\mathbf{R} = \mathrm{blockdiag}(\mathbf{I}_p, -\mathbf{I}_q)$. In order to prove the effectiveness of the proposed controller, let Σ_1 denote the $p \times p$ MIMO mechanical system obtained from the given one by fixing to the ground the masses in $\overline{M}_2$, and removing the masses in S_2 and all the springs and dampers directly connected with the removed masses so to obtain a system with n_1+n_3+p degrees of freedom, whose inputs and outputs are denoted by $\mathbf{u}_1$ and $\mathbf{y}_1$, respectively. Symmetrically, define Σ_2 (a $q \times q$ MIMO system) by fixing the masses in $\overline{M}_1$ and removing all the masses in S_1, with the relevant springs and dampers, so to obtain a system with $n_2 + n_3 + q$ degrees of freedom, whose inputs and outputs are denoted by $\mathbf{u}_2$ and $\mathbf{y}_2$, respectively.

The proof of the following theorem is omitted for brevity.

Theorem 1: Under Assumption 1, if (i) the matrix $\mathbf{H}_{mm}$ defined as in Lemma 3 is positive definite, (ii) Σ_1 and Σ_2 are reachable, then there exists $\bar{K} \geq 0$ such that for each $K > \bar{K}$ the mechanical parallel connection of the given system with the proposed controller is asymptotically stable and (p, q)-block decoupled.

In order to obtain a full input-output decoupled system, the described block decoupling operation has to be iterated on the remaining admittance submatrices. The latter theorem gives only sufficient conditions to perform the first block decoupling operation and gives no information about what could happen if such an operation is iterated on the remaining submatrices. Therefore, in the following, it will be stated that, if, at each block decoupling step, the obtained system satisfies some sufficient conditions, then it will be possible to iterate the process in order to obtain a full input-output decoupled system.

Remark 3: After an admittance matrix has been made (p, q)-block decoupled, the $p \times p$ admittance submatrix on the main diagonal can be seen as the admittance matrix of the system in which the last q inputs have been zeroed, (that is, the last q actuated bodies have been rigidly fixed to the ground) and the masses connected by a path only to the last q fixed bodies have been removed. Obviously, a symmetrical result holds for the $q \times q$ admittance submatrix.

In the following, an algorithm will be presented which, under some conditions, can be used to design a controller that gives an overall parallel connection which input-output decoupled and asymptotically stable. After, its effectiveness will be shown.

Algorithm 1: 1) Let $i = 1$ and $\mathfrak{I}_i = \{1, \ldots, m\}$. Let Sys_i be the given system. Let $m_i = m$.

2) If $i = m$ then STOP (the system is input-output block decoupled). Otherwise, choose an actuated body M_{j_i}, where $j_i \in \mathfrak{I}_i$.
3) Let $\overline{M}_2$ be M_{j_i}. Let $\overline{M}_1$ be constituted by the remaining actuated bodies of Sys_i. In this way, $p = 1$ and $q = m_i - i$.
4) If the system Sys_i satisfies the hypotheses of Theorem 1 (with m_i replacing m), then perform the $(1, m_i - i)$-block decoupling operation on the system Sys_i and go to step 6). Otherwise go to step 5).
5) Choose another actuated body for M_{j_i}, with $j_i \in \mathfrak{I}_i$, and go to step 3). If there are no more different bodies that can be chosen, then FAIL.
6) Consider the system obtained from the parallel connection of the system Sys_i and the controller designed at step 4) by fixing to the ground the chosen M_{j_i} and removing all the masses connected by a path only to M_{j_i} (and the relevant springs and dampers). Let Sys_{i+1} be such system and $\mathfrak{I}_{i+1} = \mathfrak{I}_i \setminus \{j_i\}$. Let $i \leftarrow i + 1$ and $m_i \leftarrow m_i - 1$. Go to step 2.

The strategy at the basis of the proposed procedure consists of isolating, one by one, all the actuated bodies, and trying to find a way that makes possible to apply Theorem 1 at each step, thus obtaining, as the overall controller, the mechanical parallel connection of $m - 1$ subcontrollers, designed as described in Theorem 1. The following proposition shows the effectiveness of the proposed algorithm.

Theorem 2: If Algorithm 1 terminates with STOP, the resulting overall system is input-output decoupled and asymptotically stable.

Proof: The proposition will be proved by induction on the number of already decoupled input-output pairs.

Basis. To show: After choosing M_1, if the given system satisfies the conditions of Theorem 1, the parallel connection of the given system and the first designed controller is $(1, m - 1)$-block decoupled and asymptotically stable.

The proof of the Basis clause is a trivial application of Theorem 1.

Step. Assume: after having performed $i < m - 1$ block decoupling operations through the parallel connection of i subcontrollers, the obtained system is asymptotically stable and is represented by the admittance matrix

$$\overline{\mathbf{Y}}_p^i(s) = \begin{bmatrix} Y_{p,1}(s) & \cdots & 0 & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & Y_{p,i}(s) & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \overline{\mathbf{Y}}_{p,i+1}^i(s) \end{bmatrix},$$

where $\overline{\mathbf{Y}}_{p,i+1}^i(s)$ is a $(m - i) \times (m - i)$ matrix (the superscript i denotes the step number in the decoupling process). To show: if it is possible, as described in Algorithm 1, to choose a new actuated body in a way that Theorem 1 can be applied, then the parallel connection of the system described in the Assume clause and the $(i+1)$-th designed subcontroller is asymptotically stable and is represented by the admittance matrix

$$\overline{\mathbf{Y}}_p^{i+1}(s) = \begin{bmatrix} Y_{p,1}(s) & \cdots & 0 & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & Y_{p,i+1}(s) & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \overline{\mathbf{Y}}_{p,i+2}^{i+1}(s) \end{bmatrix},$$

where $\overline{\mathbf{Y}}_{p,i+2}^{i+1}(s)$ is a $(m - i - 1) \times (m - i - 1)$ matrix.

Consider the system obtained by fixing to the ground the i actuated bodies corresponding to the already decoupled input-output pairs and removing all the masses connected by a path only to such bodies, with the relevant springs and dampers. Such a system is represented by the admittance matrix $\overline{\mathbf{Y}}_{p,i+1}^i(s)$. Since it is possible to choose an actuated body among the remaining $m-i$ in a way that it is possible to perform the block decoupling operation, the parallel connection of this system and the $(i+1)$-th controller will be asymptotically stable and its $(m-i)\times(m-i)$ admittance matrix will be

$$\overline{\mathbf{Y}}_{p,i+1}^{i+1}(s) = \begin{bmatrix} Y_{p,i+1}(s) & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{Y}}_{p,i+2}^{i+1}(s) \end{bmatrix}.$$

Therefore, the overall system will be represented by the following admittance matrix (where the dependence on s is omitted):

$$\overline{\mathbf{Y}}_p^{i+1} = \begin{bmatrix} Y_{p,1} & \cdots & 0 & 0 & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & Y_{p,i} & 0 & \vdots \\ 0 & \cdots & 0 & Y_{p,i+1} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \overline{\mathbf{Y}}_{p,i+2}^{i+1} \end{bmatrix}.$$

The asymptotic stability of such system follows easily and it is also clear that the latter admittance matrix is non-singular, thus completing the proof of the correctness of the proposed algorithm. ∎

In the following example, the proposed algorithm will be illustrated for a non-trivial mechanical system, with four inputs and four outputs.

Example 1: Consider the mechanical system depicted in Figure 3, where $n = 6$, $M_i > 0$, $D_{36} > 0$, and $K_{i,j} > 0$. At step 1, j_1 is chosen to be 1. Therefore, the relevant sets for the design of the first controller are $S_{12} = \{\}$, $S_1 = \{M_5\}$, $S_2 = \{M_6\}$. The first controller is characterized by $n_{c,1} = 4$ and by the values K_1 and D_1 for the spring and damper. At step 2, $j_2 = 2$, and the sets for the design of the second controller are $S_{12} = \{\}$, $S_1 = \{\}$, $S_2 = \{M_6\}$.The second controller is characterized by $n_{c,2} = 3$ and by K_2 and D_2. At step 3, $j_3 = 3$, and the sets are $S_{12} = \{M_6\}$, $S_1 = \{\}$, $S_2 = \{\}$. The third controller has $n_{c,3} = 3$ and K_3 and D_3 as spring and damper. The overall system, i.e., the mechanical parallel connection of the given system and the three designed controllers is depicted in Figure 4. In the figure, the little boxes filled with oblique segments represent the glue that joins two masses together, whereas the small square boxes without text inside represent speed reducers with $r = -1$.

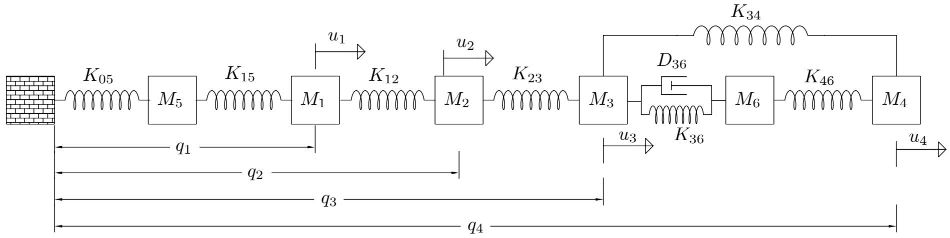

Fig. 3. The mechanical system considered in Example 1.

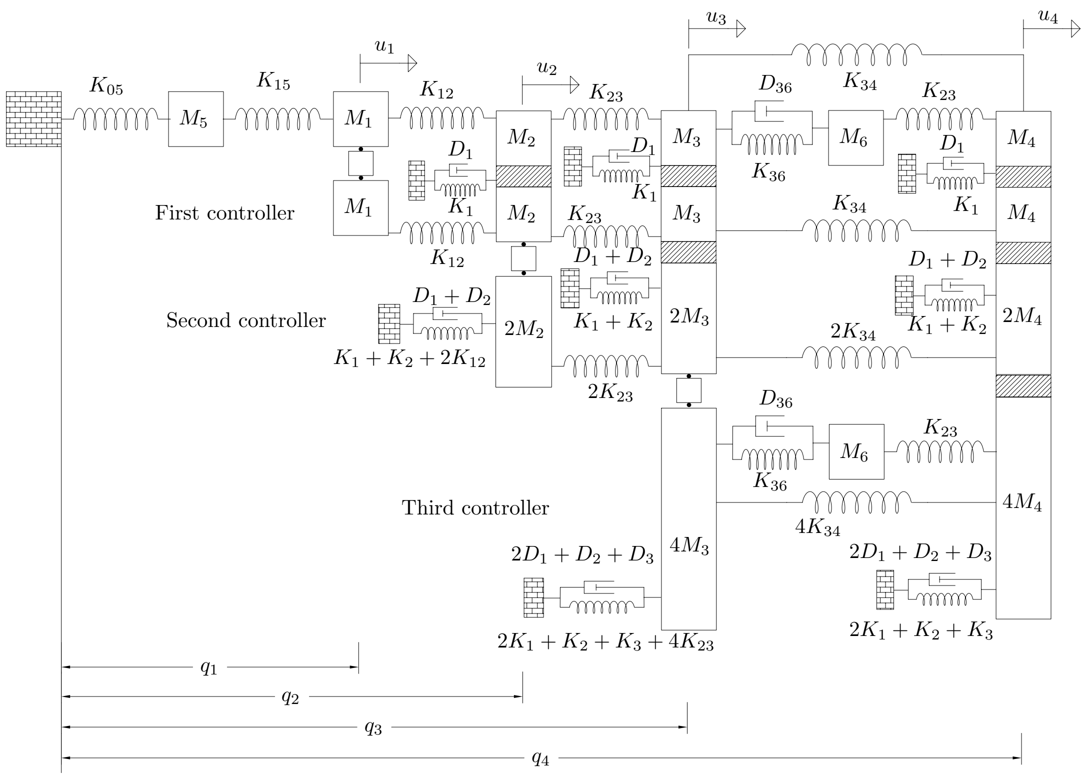

Fig. 4. The overall mechanical system considered in Example 1.

IV. Conclusions

In this paper the problem of input-output decoupling has been dealt with for linear mechanical systems under the requirement that the controller is another mechanical system to be physically connected to the given one. The problem has been solved for m-input m-output systems, under some weak conditions on the structural properties of the system. Further work will be devoted to the case of m-inputs and m-outputs, and to nonlinear mechanical systems.

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