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Robust Stability in a Classical and Modern Context

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***Abstract*— Excluding the open loop gain of a feedback system from a region about the origin of the Nichols chart yields a natural guarantee of robust stability which is commonly used in the analysis of flight control systems.**

This paper shows that a generalised form of M-circle, based upon a contour of constant $|S| + |T|$, is symmetrical on the Nichols chart and yields a stability margin consistent with the gain and phase margin requirements of the USAF.

This new margin, motivated on purely classical grounds, is shown to be equal to the robust stability margin associated with the loopshaping design procedure of McFarlane and Glover.

A corresponding interpretation is obtained for the multivariable case.

Notation: $\mathbb{C}$ denotes the complex field with $\mathbb{C}^+$, $\bar{\mathbb{C}}^+$ the open and closed right half plane respectively. If f is a function of a scalar complex variable then $p_{\bar{\mathbb{C}}^+}(f)$ denotes the number of poles in $\bar{\mathbb{C}}^+$ and $\overline{\text{wno}}(f)$ the winding number evaluated on the imaginary axis with indentations taken into the left half plane around any imaginary axis poles of f.

If P and C are matrix transfer functions, $[P, C]$ denotes the eight closed loop transfer functions which result from the feedback interconnection.

$\mathscr{H}_\infty$ denotes the standard Hardy space with norm $\|F\|_{\mathscr{H}_\infty} := \sup_{s \in \mathbb{C}^+} \bar{\sigma}(F(s))$. A feedback system is termed stable if and only if $[P, C] \in \mathscr{H}_\infty$.

All transfer functions are assumed to be rational.

The abbreviations (n)rcf and (n)lcf are used to stand for (normalised) right and left coprime factorisation.

I. The Classical Perspective

Adopting the positive feedback convention sensitivity and complementary sensitivity are given by

$$S = \frac{1}{1-L}, \qquad T = \frac{L}{1-L},$$

where $L = pc$ is the open loop gain.

Ensuring that both of these transfer functions are small is a trade-off inherent in feedback system design. This is addressed in the classical design technique of loopshaping. Typically c is chosen such that:

1) At low frequency $|L|$ is made large to ensure $|S|$ is small, yielding good disturbance rejection and reference tracking.
2) At high frequency $|L|$ is made small to ensure $|T|$ is small, yielding good noise rejection.

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Inevitably there must be a frequency at which $|L| = 1$. Behaviour in this region is important.

If the loop gain of a stable feedback system is perturbed such that some point $L(j\omega)$ approaches $+1$ on the Nyquist diagram, then both $|S(j\omega)|$ and $|T(j\omega)|$ will increase, becoming unbounded when $L(j\omega) = +1$. This corresponds to the violation of the Nyquist stability criterion, with closed loop poles crossing the stability boundary at $\pm j\omega$.

The loop gain must avoid the neighbourhood of $+1$ on the Nyquist diagram or Nichols chart in order to obtain:

1) Performance, that is suitable bounds on $|S|$ and $|T|$.
2) Robust stability, in the sense that the feedback system is robust with respect to "small" perturbations in the loop gain at any frequency.

A stability margin is an attempt to measure the degree of robust stability.

Gain and phase margin measure the closest approach of the loop gain to the $+1$ point in particular directions and can therefore yield only a necessary condition for robust stability.

To obtain a necessary and sufficient condition it is natural to define a margin in terms of the maximum size of some region about $+1$ from which the loop gain is excluded.

Experience in the design and clearance of aircraft control systems has resulted in the use of symmetrical exclusion regions on the Nichols chart [1], and stability margins based upon the greatest factor by which the exclusion region may be scaled before violation. In [1], [2] the region is defined by a polygon.

M-circles are contours of constant $|S|$ or $|T|$ which, if used to define the exclusion region, combine the complementary objectives of ensuring robust stability and bounding worst case performance [3, Chapter 1]. The notion leads to the following margins:

$$m_S := \begin{cases} \inf_\omega \frac{1}{|S|}, & \text{if } [p, c] \in \mathscr{H}_\infty, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

$$m_T := \begin{cases} \inf_\omega \frac{1}{|T|}, & \text{if } [p, c] \in \mathscr{H}_\infty, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

A small value of m_S corresponds to the loop gain penetrating a high value $|S|$ contour over some frequency range. The sensitivity response will then exhibit a large peak at these frequencies resulting from some lightly damped closed loop poles approaching the stability boundary. Similarly for m_T and $|T|$.

Contours of constant m_S and m_T are plotted in Figure 1. Note that neither contour is symmetrical with respect to

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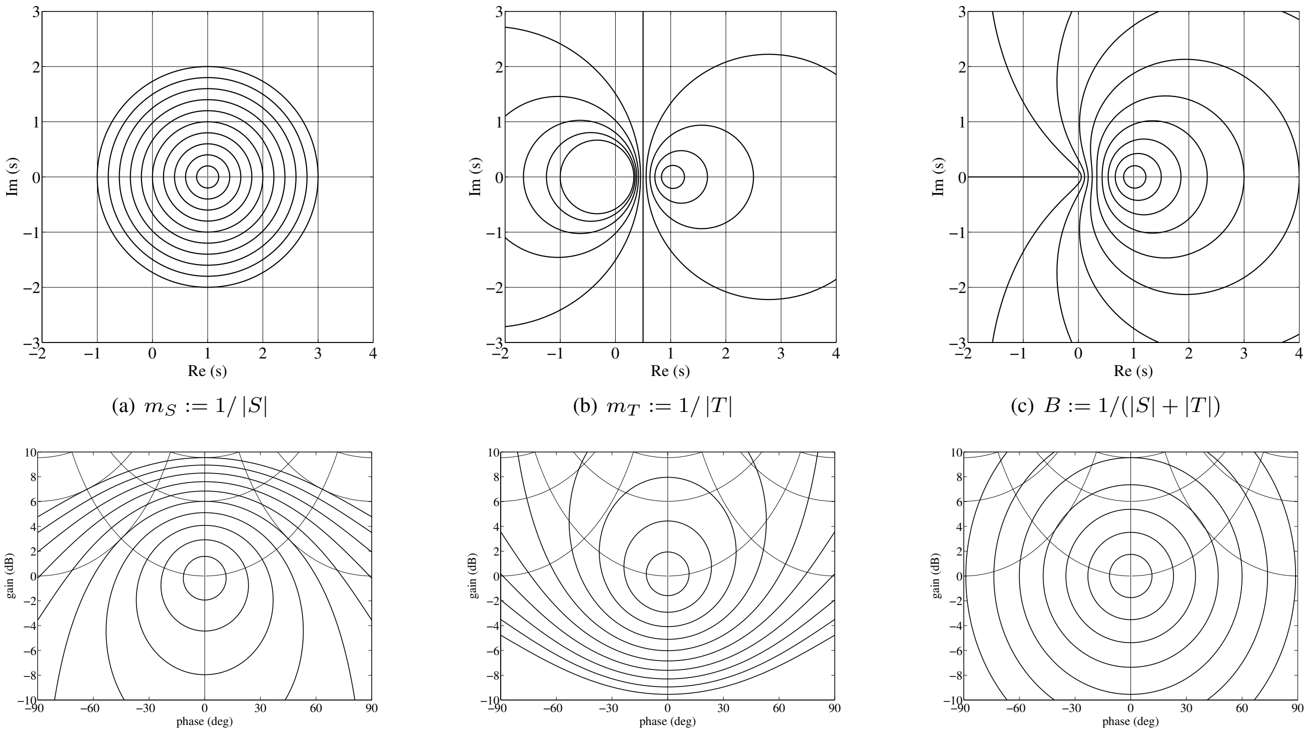

(a) $m_S := 1/|S|$ (b) $m_T := 1/|T|$ (c) $B := 1/(|S|+|T|)$

(d) $m_S := 1/|S|$ (e) $m_T := 1/|T|$ (f) $B := 1/(|S|+|T|)$

Fig. 1. Comparison between stability margin contours on Nyquist diagram and Nichols chart

gain in dB's and hence neither margin is consistent with the classical notion of gain margin.

As feedback system design concerns the trade-off of $|S|$ and $|T|$ a more appropriate margin may be defined in terms of contours of constant $|S|+|T|$ as follows.

$$B := \begin{cases} \inf_\omega \frac{1}{|S|+|T|}, & \text{if } [p,c] \in \mathscr{H}_\infty, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Note that $B \geq \beta$ if and only if L satisfies

$$|1-L| \geq \beta\,(1+|L|)\,,\ \forall\omega, \tag{4}$$

which characterises the exclusion region about $+1$.

With $L = re^{j\theta}$, r and ϕ satisfy

$$\sin^2\frac{\phi}{2} \geq \frac{(1+\beta-r(1-\beta))\,(-1+\beta+r(1+\beta))}{4r}$$

(part 2, Lemma A.1). Hence the exclusion region is symmetrical about $\pm\phi$ and $\pm\log r$ (symmetrical about $r, 1/r$) and therefore represents a symmetrical exclusion region on the Nichols Chart, Figure 1(f). Inspection of this figure also reveals that the contours are approximately evenly spaced.

A lower bound on the gain and phase margin follows from the characterisation in (4). (The result appears in [4, Theorem 2.10].)

Lemma 1.1 (Guaranteed margins): If $[p,c] \in \mathscr{H}_\infty$ and $|1-L| \geq \beta\,(1+|L|)$ for all ω then

$$|\text{GM}| \geq 20\log_{10}\left(\frac{1+\beta}{1-\beta}\right) \quad \text{and} \quad |\text{PM}| \geq 2\arcsin\beta.$$

Proof: (GM) For $L = k \in \mathbb{R}^+ > 1$

$$|1-k| = k-1 \geq \beta(1+k)$$
$$k \geq \frac{1+\beta}{1-\beta}.$$

(PM) For $L = e^{j\theta}$, $\theta > 0$

$$\left|1-e^{j\theta}\right| \geq \beta\left(1+\left|e^{j\theta}\right|\right)$$

$$\text{LHS} = \left((1-\cos\theta)^2 + \sin^2\theta\right)^{1/2} = 2\sin\tfrac{\theta}{2}$$

$$\sin\frac{\theta}{2} \geq \beta.$$

The results follow from the fact that $\log_{10} x$ and $\arcsin x$ are monotonically increasing and that the exclusion region is symmetrical with respect to logarithmic gain and phase. ■

Comparison with USAF stability margin specifications for manned piloted aircraft Table I, which encapsulate extensive experience in flight control system design, indicates that B-margin is consistent with the two least stringent conditions.

In conclusion B is motivated as a measure of both robust stability and worst case performance on the basis that:

1) It bounds worst case performance in terms of $|S|+|T|$.
2) It guarantees the exclusion of the loop gain from a region about $+1$ which is symmetrical on the Nichols chart and approximately scales in proportion to B.
3) It yields lower bounds on gain and phase margin which are consistent with the two least stringent specifications in [5].

TABLE I

STABILITY MARGINS

(a) B-margin			(b) USAF [5, Table III]	
B	GM (dB) $\geq$	PM (deg) $\geq$	GM (dB) $\geq$	PM (deg) $\geq$
0.174	± 3	± 20	± 3	± 20
0.259	± 4.6	± 30	± 4.5	± 30
0.383	± 7	± 45	± 6	± 45
0.500	± 9.5	± 60	± 8	± 60

II. THE MODERN APPROACH

$\mathscr{H}_\infty$-loopshaping [6] is a multivariable design procedure in which the plant P is initially weighted, to form $P_s = W_2 P W_1$, before a compensator C_∞ is synthesised to maximise the robust stability margin

$$b_{P_s,C_\infty} := \left\| \begin{bmatrix} P_s \\ I \end{bmatrix} (I - P_s C_\infty)^{-1} \begin{bmatrix} -C_\infty & I \end{bmatrix} \right\|_{\mathscr{H}_\infty}^{-1}.$$

Provided that a sufficient value of b_{P_s,C_∞} is achieved ($\gtrapprox 0.2$) then useful guarantees regarding robust stability and worst case performance are recovered. The guidelines for choosing W_1, W_2 are in line with the classical notion of loopshaping.

A. SISO case

Note that b_{p_s,c_∞} only yields a useful measure if the open loop gain of p_s is such that the guarantees recovered from a bound on b_{p_s,c_∞} are consistent with the design objectives. In the analysis of a general feedback system $[p, c]$ the measure $b_{wp,c/w}$ may be used, where w, $1/w \in \mathscr{H}_\infty$ is chosen either to capture the design objectives or to maximise $b_{wp,c/w}$.

The following theorem proves that $\max_{w,1/w \in \mathscr{H}_\infty} b_{wp,c/w}$ is equal to the B-margin introduced in Section I and therefore yields a guarantee that the loop gain satisfies an exclusion region about the $+1$ point.

Theorem 2.1 (B-margin and $b_{p,c}$): For a SISO feedback system $[p, c] \in \mathscr{H}_\infty$ with loop gain $L = pc$

$$\max_{w,1/w \in \mathscr{H}_\infty} b_{wp,c/w} = \inf_\omega \frac{|1-L|}{1+|L|} = B.$$

Proof: As $[p, c] \in \mathscr{H}_\infty$, $b_{p,c} = \inf_\omega \rho(p, c) > 0$ where

$$\rho(p, c) := \frac{|1 - pc|}{\sqrt{1 + |p|^2}\sqrt{1 + |c|^2}}.$$

At a particular frequency ω_0

$$\max_{w,1/w \in \mathbb{C}} \rho(wp, c/w) = \max_{w,1/w \in \mathbb{C}} \frac{|1 - pc|}{\sqrt{1 + |w|^2 |p|^2}\sqrt{1 + \frac{|c|^2}{|w|^2}}}.$$

If $p, 1/p, c, 1/c \in \mathbb{C}$ then noting $\min_{x>0}(1 + xa^2)(1 + b^2/x) = (1 + ab)^2$ gives

$$\max_{w,1/w \in \mathbb{C}} \rho(wp, c/w) = \frac{|1-L|}{1+|L|},$$

achieved by $|w| = \sqrt{\frac{|c|}{|p|}}$.

As $\rho(p, c) > 0$ there is no pole-zero cancellation on the imaginary axis. If p or $c \in \{0 \cup \infty\}$ then $\rho(wp, c/w)$ can be made arbitrarily close to 1, which equals $\frac{|1-L|}{1+|L|}$ in this instance.

Take

$$\max_{w,1/w \in \mathscr{H}_\infty} b_{wp,c/w} = \max_{w,1/w \in \mathscr{H}_\infty} \inf_\omega \rho(wp, c/w)$$

and note that $w, 1/w \in \mathscr{H}_\infty$ may be chosen such that ρ is maximised at every frequency to yield

$$\max_{w,1/w \in \mathscr{H}_\infty} b_{wp,c/w} = \inf_\omega \frac{|1-L|}{1+|L|}.$$

∎

$b_{wp,c/w}$ is also a measure of the degree of robust stability to ncf uncertainty in wp. It follows that $\max_{w,1/w \in \mathscr{H}_\infty} b_{wp,c/w} \geq \beta$ is a necessary and sufficient condition for the stability of $[p_\Delta, c]$ for all $p_\Delta \in S_A$ where

$$S_A = \left\{ \frac{(n + \delta_n)}{w(m + \delta_m)} : \left\| \begin{bmatrix} \delta_n \\ \delta_m \end{bmatrix} \right\|_{\mathscr{H}_\infty} < \beta \right\}$$

with (n, m) a ncf of wp.

The following theorem proves that it is also a necessary and sufficient[1] condition for stability with $p_\Delta \in S_B$ where

$$S_B = \left\{ \frac{(1 + \delta)^2}{1 - \beta^2} p : \|\delta\|_{\mathscr{H}_\infty} < \beta \right\} \subset S_A.$$

Firstly recall the Nyquist stability criterion.

Proposition 2.2 (Nyquist Criterion - SISO case): The following are equivalent:

1) $[p, c] \in \mathscr{H}_\infty$.
2) $L \neq +1, \forall\omega$ and $\overline{\mathrm{wno}}(1 - L) + p_{\bar{\mathbb{C}}+}(P) + p_{\bar{\mathbb{C}}+}(C) = 0$.

Theorem 2.3 (Robust Stability): For a SISO feedback system $[p, c]$ with loop gain $L = pc$, the following are equivalent:

1)
$$\max_{w,1/w \in \mathscr{H}_\infty} b_{wp,c/w} \geq \beta.$$

2) $[p, c] \in \mathscr{H}_\infty$ and
$$+1 \notin \left\{ \frac{(1 + \delta)^2}{1 - \beta^2} L : |\delta| < \beta \right\}, \forall\omega.$$

3) $[p_\Delta, c] \in \mathscr{H}_\infty$ for all $p_\Delta \in S_B$ where
$$S_B = \left\{ \frac{(1 + \delta)^2}{1 - \beta^2} p : \|\delta\|_{\mathscr{H}_\infty} < \beta \right\}.$$

Proof: (1⇔2) Follows from Theorem 2.1, Lemma A.1 (part 1 and 3) and the fact that the exclusion region is symmetrical on the Nichols Chart.

(3⇒2) $p \in S_B$ $\left(\delta = -1 + \sqrt{1 - \beta^2}\right) \Rightarrow [p, c] \in \mathscr{H}_\infty$.

[1]Sufficiency is obvious from the fact that S_B is a subset of S_A. This follows by substituting $\delta_\times = \delta_n / n$ and $\delta_\div = \delta_m / m$ into S_A, noting

$$\left\| \begin{bmatrix} \delta_\times & 0 \\ 0 & \delta_\div \end{bmatrix} \begin{bmatrix} n \\ m \end{bmatrix} \right\|_{\mathscr{H}_\infty} \leq \max\left(\|\delta_\times\|_{\mathscr{H}_\infty}, \|\delta_\div\|_{\mathscr{H}_\infty} \right)$$

and applying Fact A.2.

At a frequency ω the loop gain of the perturbed feedback system is given by

$$L_\Delta(j\omega) = \frac{(1+\delta(j\omega))^2}{1-\beta^2} L(j\omega).$$

For $[p_\Delta, c]$ to have a pole at ω requires $L_\Delta(j\omega) = +1$. If this were possible for $|\delta(j\omega)| < \beta$ then a δ could be constructed with $\|\delta\|_{\mathscr{H}_\infty} < \beta$ such that $p_\Delta = \frac{(1+\delta)^2}{1-\beta^2} \in S_B$.

Hence $[p_\Delta, c] \in \mathscr{H}_\infty \Rightarrow L_\Delta(j\omega) \neq +1, \forall\omega$ and $|\delta(j\omega)| < \beta$.

(2⇒3) By Theorem 2.2

$$[p,c] \in \mathscr{H}_\infty \Leftrightarrow \overline{\text{wno}}(1-L) + p_{\bar{\mathbb{C}}^+}(P) + p_{\bar{\mathbb{C}}^+}(C) = 0, \quad (5)$$

$$[p_\Delta,c] \in \mathscr{H}_\infty \Leftrightarrow \overline{\text{wno}}(1-L_\Delta) + p_{\bar{\mathbb{C}}^+}(P_\Delta) + p_{\bar{\mathbb{C}}^+}(C) = 0. \quad (6)$$

Noting that $p_\Delta \in S_B$ satisfies

$$p_\Delta = \frac{(1+\delta)^2}{1-\beta^2} p, \|\delta\|_{\mathscr{H}_\infty} < \beta \leq 1$$

implies $p_{\bar{\mathbb{C}}^+}(p_\Delta) = p_{\bar{\mathbb{C}}^+}(p)$, as $\frac{(1+\delta)^2}{1-\beta^2}$ has no poles or zeros in $\bar{\mathbb{C}}^+$. Hence

$$\overline{\text{wno}}(1-L_\Delta) = \overline{\text{wno}}(1-L)$$

is a condition for the equivalence of (5) and (6).

Noting that $L_\Delta = L$ for $\delta = -1 + \sqrt{1-\beta^2}$ and that by part 2, $1 - L_\Delta \neq 0$ for any ω and $|\delta| < \beta$, it follows from a homotopy argument that

$$\overline{\text{wno}}(1-L_\Delta) = \overline{\text{wno}}(1-L), \ \forall p_\Delta \in S_B.$$

Thus $[p_\Delta, c] \in \mathscr{H}_\infty, \forall p_\Delta \in S_B$. ∎

Therefore $\max_{w,1/w \in \mathscr{H}_\infty} b_{wp,c/w} \geq \beta$ is equivalent to robust stability with respect to multiplicative loop gain perturbations of the form

$$\left\{ \frac{(1+\delta)^2}{1-\beta^2} : \|\delta\|_{\mathscr{H}_\infty} < \beta \right\}.$$

On the occasion that c is synthesised to uniformly optimise $\rho(p,c)$ over all stabilising compensators, the following theorem asserts that, provided there is sufficient variation in the magnitude of the specified loopshape with respect to 1, then $b_{p,c} = \max_{w,1/w \in \mathscr{H}_\infty} b_{wp,c/w}$ and is therefore equal to the B-margin.

Theorem 2.4 (B-margin and $b_{p,c}$ - special case): If $[p,c] \in \mathscr{H}_\infty$ and $\rho(p,c) = b_{p,c}, \forall\omega$ and $|p(\omega_1)| \geq \sqrt{\frac{1+b_{p,c}}{1-b_{p,c}}}$, $|p(\omega_2)| \leq \sqrt{\frac{1-b_{p,c}}{1+b_{p,c}}}$ for any ω_1, ω_2 then

$$b_{p,c} = \max_{w,1/w \in \mathscr{H}_\infty} b_{wp,c/w} = B.$$

Proof: $\rho(p,c)$ is the chordal distance between the projections of the points p and $1/c$ onto the Riemann Sphere. Noting that this is equal to $b_{p,c}$ at all frequencies and that the chordal distance between $\sqrt{\frac{1+b_{p,c}}{1-b_{p,c}}}$ and $\sqrt{\frac{1-b_{p,c}}{1+b_{p,c}}}$ is also equal to $b_{p,c}$ it follows that

$$|p(\omega_1)| \geq \sqrt{\frac{1+b_{p,c}}{1-b_{p,c}}} \geq |c(\omega_1)|,$$

$$|p(\omega_2)| \leq \sqrt{\frac{1-b_{p,c}}{1+b_{p,c}}} \leq |c(\omega_2)|.$$

As p, c are assumed to be rational then by continuity there exists an ω_0 such that $|p(\omega_0)| = |c(\omega_0)|$.

Since

$$\arg \max_{w,1/w \in \mathbb{C}} \rho\left(wp(\omega_0), \frac{c(\omega_0)}{w}\right) = \sqrt{\frac{|c(\omega_0)|}{|p(\omega_0)|}} = 1$$

the pointwise margin is optimal with respect to the weighting at ω_0 and cannot therefore be improved. Thus

$$\begin{aligned} \max_{w,1/w \in \mathscr{H}_\infty} b_{wp,c/w} &= \max_{w,1/w \in \mathscr{H}_\infty} \inf_\omega \rho(wp, c/w) \\ &= \inf_\omega \max_{w,1/w \in \mathscr{H}_\infty} \rho(wp, c/w) = b_{p,c}. \end{aligned}$$

∎

B. MIMO case

Corresponding results are obtained in the multivariable setting by restricting the weighting functions to a diagonal structure.

Lemma 2.5 (Subset of ncf uncertainty): For

$$\begin{aligned} S_A = \Bigg\{ & W_2^{-1}(N+\Delta_N)(M+\Delta_M)^{-1}W_1^{-1} : \\ & \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_{\mathscr{H}_\infty} < \beta, (M+\Delta_M) \text{ invertible} \Bigg\}, \end{aligned}$$

$$\begin{aligned} S_B = \{ & (I+\Delta_\times)P(I+\Delta_\div)^{-1} : \Delta_\times, \Delta_\div \text{ diagonal}, \\ & \|\Delta_\times\|_{\mathscr{H}_\infty}, \|\Delta_\div\|_{\mathscr{H}_\infty} < \beta, (I+\Delta_\div) \text{ invertible} \}, \end{aligned}$$

where (N, M) is a nrcf of W_2PW_1 and W_1, W_1^{-1}, W_2, $W_2^{-1} \in \mathscr{H}_\infty$ diagonal, then $S_A \supset S_B$.

Proof: Take any

$$P_B = (I+\Delta_\times) P (I+\Delta_\div)^{-1} \in S_B.$$

W_1 commutes with $(I+\Delta_\div)^{-1}$ and W_2 commutes with $(I+\Delta_\times)^{-1}$ by virtue of the fact that they are diagonal to yield

$$\begin{aligned} P_B &= W_2^{-1}(I+\Delta_\times) W_2PW_1 (I+\Delta_\div)^{-1} W_1^{-1} \\ &= W_2^{-1}(N+\Delta_\times N)(M+\Delta_\div M)^{-1} W_1^{-1}, \end{aligned}$$

defining $\Delta_N := \Delta_\times N$ and $\Delta_M := \Delta_\div M$

$$= W_2^{-1}(N+\Delta_N)(M+\Delta_M)^{-1} W_1^{-1}.$$

Now (Δ_N, Δ_M) satisfies

$$\begin{aligned} \left\| \left[\begin{smallmatrix} \Delta_N \\ \Delta_M \end{smallmatrix}\right] \right\|_{\mathscr{H}_\infty} &= \left\| \left[\begin{smallmatrix} \Delta_\times N \\ \Delta_\div M \end{smallmatrix}\right] \right\|_{\mathscr{H}_\infty} \\ &\leq \max\left\{ \|\Delta_\times\|_{\mathscr{H}_\infty}, \|\Delta_\div\|_{\mathscr{H}_\infty} \right\} < \beta \end{aligned}$$

and $M, (I+\Delta_\div)$ invertible $\Rightarrow (M+\Delta_M)$ invertible, so $P_B \in S_A$. ∎

A guarantee of stability is obtained with respect to simultaneous input and output channel perturbations which are the square root of that for which a stability guarantee was obtained in the single loop case.

Theorem 2.6 (Stability under simultaneous perturbations): If $1 > b_{W_2PW_1,W_1^{-1}CW_2^{-1}} \geq \beta$, with W_1, W_1^{-1}, W_2, $W_2^{-1} \in \mathscr{H}_\infty$ diagonal then $[P,C]$ is stable with respect to *simultaneous* perturbations of the form

$$\left\{\frac{1+\delta}{\sqrt{1-\beta^2}} : \|\delta\|_{\mathscr{H}_\infty} < \beta\right\},$$

at each input and output channel of P.

Proof: $b_{W_2PW_1,W_1^{-1}CW_2^{-1}} \geq \beta \Rightarrow [P_\Delta, C]$ is stable for all $P_\Delta \in S_A$ where

$$S_A = \left\{W_2^{-1}(N+\Delta_N)(M+\Delta_M)^{-1}W_1^{-1} : \left\|\begin{bmatrix}\Delta_N \\ \Delta_M\end{bmatrix}\right\|_{\mathscr{H}_\infty} < \beta, (M+\Delta_M)\text{invertible}\right\}.$$

As W_1 and W_2 are diagonal, then by Lemma 2.5 $[P_\Delta, C]$ is stable for all $P_\Delta \in S_B$ where

$$S_B = \{(I+\Delta_\times)P(I+\Delta_\div)^{-1} : \Delta_\times, \Delta_\div \text{ diagonal}, \|\Delta_\times\|_{\mathscr{H}_\infty}, \|\Delta_\div\|_{\mathscr{H}_\infty} < \beta, (I+\Delta_\div) \text{ invertible}\}. \quad (7)$$

Note that

$$S_B = \left\{\frac{(I+\Delta_\times)}{\sqrt{1-\beta^2}}P(I+\Delta_\div)^{-1}\sqrt{1-\beta^2} : \Delta_\times, \Delta_\div \text{ diagonal}, \|\Delta_\times\|_{\mathscr{H}_\infty}, \|\Delta_\div\|_{\mathscr{H}_\infty} < \beta, (I+\Delta_\div) \text{ invertible}\right\},$$

whereupon applying Fact A.2 yields

$$S_B = \left\{\frac{(I+\Delta_{\times_1})}{\sqrt{1-\beta^2}}P\frac{(I+\Delta_{\times_2})}{\sqrt{1-\beta^2}} : \Delta_{\times_1}, \Delta_{\times_2} \text{ diagonal}, \|\Delta_{\times_1}\|_{\mathscr{H}_\infty}, \|\Delta_{\times_2}\|_{\mathscr{H}_\infty} < \beta\right\},$$

which is symmetrical with respect to input and output uncertainty and gives the result. ∎

Note that in contrast to the SISO case $b_{W_2PW_1,W_1^{-1}CW_2^{-1}} \geq \beta$ is only a sufficient condition for satisfying the exclusion region.

A single channel result is recovered by selecting a particular pair of diagonal perturbations in (7). The single channel perturbation with respect to which stability is guaranteed is identical to that obtained in the single loop case.

Corollary 2.7 (Stability under a single perturbation): If $1 > b_{W_2PW_1,W_1^{-1}CW_2^{-1}} \geq \beta$, with W_1, W_1^{-1}, W_2, $W_2^{-1} \in \mathscr{H}_\infty$ diagonal then $[P,C]$ is stable with respect to a *single* perturbation of the form

$$\left\{\frac{(1+\delta)^2}{1-\beta^2} : \|\delta\|_{\mathscr{H}_\infty} < \beta\right\},$$

at any input or output channel of P.

Proof: Choosing

$$I+\Delta_\times = \text{diag}\,(1+\delta_1, 1+\delta_2, \ldots, 1+\delta_2)\,,$$
$$I+\Delta_\div = (1+\delta_2)\,I,$$

in (7) and noting that $(I+\Delta_\div)^{-1}$ commutes with P from input to output implies that $[P_\Delta, C]$ is stable for all $P_\Delta \in S_C$ where

$$S_C = \left\{\text{diag}\left(\frac{1+\delta_1}{1+\delta_2}, 1, \ldots, 1\right)P : \|\delta_1\|_{\mathscr{H}_\infty}, \|\delta_2\|_{\mathscr{H}_\infty} < \beta\right\}.$$

The output channel result then follows from Fact A.2. The input channel result follows in a similar fashion. ∎

III. Conclusion

This paper expands upon the results first published in [7] and [8].

1) A classical robust stability margin (B-margin) is introduced and motivated as a measure of robust stability and worst case performance.
2) It is proven that $B = \max_{w,1/w\in\mathscr{H}_\infty} b_{wp,c/w}$.
3) When c_∞ is optimal with respect to b_{p_s,c_∞}, as in the loopshaping design procedure, then (under some mild assumptions) b_{p_s,c_∞} is equal to the B-margin.

The following robust stability results are presented:

1) In the SISO case $\max_{w,1/w\in\mathscr{H}_\infty} b_{wp,c/w} \geq \beta \Leftrightarrow [p,c]$ is stable with respect to perturbations of the form $\left\{\frac{(1+\delta)^2}{1-\beta^2} : \|\delta\|_{\mathscr{H}_\infty} < \beta\right\}$.
2) In the MIMO case $b_{W_2PW_1,W_1^{-1}CW_2^{-1}} \geq \beta \Rightarrow [P,C]$ is stable with respect to:
   a) Simultaneous perturbations of the form $\left\{\frac{(1+\delta)}{\sqrt{1-\beta^2}} : \|\delta\|_{\mathscr{H}_\infty} < \beta\right\}$ at each input and output of P.
   b) A single perturbation of the form $\left\{\frac{(1+\delta)^2}{1-\beta^2} : \|\delta\|_{\mathscr{H}_\infty} < \beta\right\}$ at any input or output of P.

The above guarantees are in terms of small multiplicative perturbations in the spirit of classical measures of robust stability.

Appendix

The following lemma gives equivalent characterisations of the exclusion region in terms of both a relation between gain and phase and as a set described by a complex perturbation.

Lemma A.1 (Exclusion region): For $L \in \mathbb{C}$, the following are equivalent:

1) L satisfies $|1-L| \geq \beta\,(1+|L|)$.
2) $L = re^{j\phi}$, where r, ϕ satisfy

$$\sin^2\frac{\phi}{2} \geq \frac{(1+\beta-r(1-\beta))\,(-1+\beta+r(1+\beta))}{4r}.$$

3)

$$L \notin \left\{ \frac{(1+\delta)^2}{1-\beta^2} : |\delta| \le \beta \right\}.$$

Proof: (1⇔2) Take

$$|1-L| \ge \beta(1+|L|)$$

and substitute $L = re^{j\phi}$ to give

$$\begin{aligned} \left|1-re^{j\phi}\right|^2 &\ge \beta^2(1+r)^2 \\ (1-r\cos\phi)^2 + r^2\sin^2\phi &\ge \beta^2(1+r)^2 \\ 1-2r\cos\phi + r^2\cos^2\phi + r^2\sin^2\phi &\ge \beta^2(1+r)^2 \\ 2r\underbrace{(1-\cos\phi)}_{2\sin^2\frac{\phi}{2}} + 1 - 2r + r^2 &\ge \beta^2(1+r)^2. \end{aligned}$$

A relation between the gain and phase of L is thereby obtained

$$\begin{aligned} \sin^2\frac{\phi}{2} &\ge \frac{\beta^2(1+r)^2-(1-r)^2}{4r} \\ \sin^2\frac{\phi}{2} &\ge \frac{(\beta(1+r)+(1-r))(\beta(1+r)-(1-r))}{4r} \\ \sin^2\frac{\phi}{2} &\ge \frac{(1+\beta-r(1-\beta))(-1+\beta+r(1+\beta))}{4r}. \end{aligned}$$

(2⇔3) If $s = re^{j\varphi}$ is an element of the boundary of $\left\{\frac{(1+\delta)^2}{1-\beta^2} : |\delta| < \beta\right\}$ then

$$\begin{aligned} re^{j\varphi} &= \frac{(1+\beta e^{j\theta})^2}{1-\beta^2} \\ e^{j\varphi/2} &= \frac{1+\beta e^{j\theta}}{\sqrt{r(1-\beta^2)}} \\ \cos\frac{\varphi}{2} + j\sin\frac{\varphi}{2} &= \frac{1+\beta(\cos\theta + j\sin\theta)}{\sqrt{r(1-\beta^2)}}. \end{aligned}$$

Equating real and imaginary parts gives

$$\cos^2\frac{\varphi}{2} = \frac{(1+\beta\cos\theta)^2}{r(1-\beta^2)}, \quad (8)$$

$$\sin^2\frac{\varphi}{2} = \frac{\beta^2\sin^2\theta}{r(1-\beta^2)}. \quad (9)$$

Taking the sum of (8) and (9)

$$\begin{aligned} \cos^2\frac{\varphi}{2} + \sin^2\frac{\varphi}{2} &= \frac{(1+\beta\cos\theta)^2 + \beta^2\sin^2\theta}{r(1-\beta^2)} = 1 \\ 1 + 2\beta\cos\theta + \beta^2 &= r(1-\beta^2) \\ \cos\theta &= \frac{r(1-\beta^2)-(1+\beta^2)}{2\beta}. \quad (10) \end{aligned}$$

An expression for $\sin^2\theta$ is obtained from (10)

$$\begin{aligned} \sin^2\theta = 1-\cos^2\theta &= \frac{4\beta^2 - (r(1-\beta^2)-(1+\beta^2))^2}{4\beta^2} \\ &= -(-2r(1-\beta^2)(1+\beta^2) \\ &\quad + r^2(1-\beta^2)^2 + \underbrace{(1+\beta^2)^2 - 4\beta^2}_{(1-\beta^2)^2})/4\beta^2 \\ &= (1-\beta^2)\left(2r(1+\beta^2) - (r^2+1)(1-\beta^2)\right)/4\beta^2 \\ &= (1-\beta^2)\left(\beta^2(1+2r+r^2) - (1-2r+r^2)\right)/4\beta^2 \\ &= (1-\beta^2)\left(\beta^2(1+r)^2 - (1-r)^2\right)/4\beta^2, \end{aligned}$$

which is substituted into (9) to give

$$\begin{aligned} \sin^2\frac{\varphi}{2} &= \frac{\beta^2(1+r)^2-(1-r)^2}{4r} \\ &= \frac{(1+\beta-r(1-\beta))(-1+\beta+r(1+\beta))}{4r}. \end{aligned}$$

The result follows by noting that this is the boundary of the set in part 2. ■

Fact A.2 (Equivalent perturbations): For $\beta < 1$ the following sets are equivalent:

$$S_A = \left\{ \frac{1+\delta_A}{\sqrt{1-\beta^2}} : \|\delta_A\|_{\mathscr{H}_\infty} < \beta \right\}$$

$$S_B = \left\{ \frac{\sqrt{1-\beta^2}}{1+\delta_B} : \|\delta_B\|_{\mathscr{H}_\infty} < \beta \right\}.$$

Proof: $p_A \in S_A$ if and only if

$$p_A = \frac{1}{\sqrt{1-\beta^2}} + \frac{\beta}{\sqrt{1-\beta^2}}(\delta_A/\beta), \ \|\delta_A\|_{\mathscr{H}_\infty} < \beta.$$

$p_B \in S_B$ if and only if

$$p_B = \frac{1}{\sqrt{1-\beta^2}} + \frac{\sqrt{1-\beta^2}}{1+\delta_B} - \frac{1}{\sqrt{1-\beta^2}}, \ \|\delta_B\|_{\mathscr{H}_\infty} < \beta.$$

Equivalently

$$p_B = \frac{1}{\sqrt{1-\beta^2}} + \frac{\beta}{\sqrt{1-\beta^2}}\left(-\frac{\beta + (\delta_B/\beta)}{1+(\delta_B/\beta)}\right).$$

Since transformations of the form $\frac{b+z}{1+b^* z}$ map the unit disc to the unit disc there exists a δ such that

$$p_B = \frac{1}{\sqrt{1-\beta^2}} + \frac{\beta}{\sqrt{1-\beta^2}}(\delta/\beta), \ \|\delta\|_{\mathscr{H}_\infty} < \beta,$$

and so $p_B \in S_A$. ■

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