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Integrator Backstepping using Barrier Functions for Systems with Multiple State Constraints

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***Abstract*— The problem of stabilization for a class of feedback linearizable systems with multiple state constraints is addressed. The design procedure is constructive, and yields a continuous final control law which guarantees that all specified states remain within certain bounds for all time. The achieved bounds on the states are independent of the initial conditions. The procedure entails shaping the control Lyapunov function, and propagating hard-bounds imposed on the pertinent stabilising functions and associated error signals through the steps of the backstepping control design framework.**

I. INTRODUCTION

The problem of saturation nonlinearities is by far the most common challenge faced by control engineers as all real systems possess at least one form of saturation nonlinearities [1], [2]. Control problems for constrained linear systems have been extensively studied in the literature due to the hitherto successful use of linear approximations to represent a restricted range of operating conditions of otherwise nonlinear processes. Key approaches include override control [3], set invariance and admissible set control [4], [5], the reference governor approach [6], and Linear Model Predictive Control [7]. Many of those approaches are numerical in nature and/or rely heavily on computationally intensive algorithms to solve the control problems. It is only recently that insights into structural properties of stabilizable constrained linear systems, as well as control design methodologies for such systems, were provided in [8], [9].

All real systems are, however, inherently nonlinear. Factors such as higher product quality specifications, increasing productivity demands, tighter environmental regulations, and demanding economical considerations all require systems to operate over a wider range of operating conditions and often near the boundary of the admissible region. Under these conditions, linear models are no longer sufficient to describe the process dynamics adequately and nonlinear models must be used. Various techniques have been developed to solve the constrained control problems for nonlinear systems, namely artificial potential field [10], invariance control [11], nonlinear reference governor [12], [13], and Nonlinear Model Predictive Control [14]. Works which fall under the 'constructive nonlinear control' framework have focussed on the problem of actuator saturations and led to the modern techniques of forwarding [15], [16].

To tackle output or state saturation, however, it appears more effective to employ the backstepping methodology capable of supplying the high input gain margins, which are required to impose the saturation constraints. Backstepping is a powerful tool for the synthesis of robust and adaptive nonlinear controllers for various important classes of systems with parametric or dynamic nonlinearities and uncertainties [17]–[19]. Freeman and Praly in [20], and later extended to a more general class of nonlinear systems by Mazenc and Iggidr [21], addressed the problem of bounded controls and control rates. To the best of the authors' knowledge, there exists no result on the problem of multiple state constraints in the backstepping paradigm.

In this paper, we present a modified backstepping control design procedure to stabilize a class of feedback linearizable systems with multiple state constraints. The paper is motivated by the consideration of physical motion systems where the non-linear model of the system is only valid for a restricted band of velocities. For example, consider the 4^{th}-order longitudinal dynamics of a conventionally-configured aircraft. The dynamic model comprises altitude, vertical velocity, pitch attitude, and pitch rate [22]. In this case the vertical velocity, or climb rate, is proportional to the angle of attack of the aircraft. This internal state of the system model must be bounded below the stall angle of the aircraft to avoid catastrophic failure of the closed-loop system. For commercial jet aircraft, the "passenger comfort" factor imposes magnitude constraints on both the aircraft's pitch attitude and pitch rate during manoeuvres which can be well below actuator saturation limits. The main features of our design consist of shaping the control Lyapunov function to bound and suppress the propagation of the errors at each stage of the backstepping procedure and the introduction of a barrier-function like term employed to impose hard bound on the associated error signals. The present paper extends an earlier paper [23] that considers imposing a hard-bound on the velocity state to imposing multiple bounds on the states of feedback linearizable systems.

The paper is organized as follows. Section 2 states the problem definition and the control design procedure along

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with the main results. A discussion of the achieved state bounds and how to tune the design constants is given Section 3. Simulation results for a simple 4^{th}-order integrator cascade are presented in Section 4, whilst concluding remarks and possible future work are contained in section 5.

II. Backstepping with Bounded States

A. Notations

The notation $f(x) = \mathbf{O}(g(x))$, read 'f(x) is big O of g(x)', means $|f(x)| \leq \kappa |g(x)|$, for some constant κ.

B. Definitions

A system

$$\dot{x} = f(x,t), \qquad f(x_0,t) = 0, \qquad x_0 \in \mathbb{R}^n \tag{1}$$

with equilibrium point x_0 is termed *domain globally asymptotically stable* (DGAS) to x_0 with domain U if:

1) There exists a set $U \subseteq \mathbb{R}^n$ that is forward invariant under the dynamics of (1) and $x_0 \in U$.
2) The equilibrium point x_0 is Lyapunov stable under the dynamics of (1) restricted to U.
3) For any initial condition $x_0 \in U$, then the solution $x(t)$ of (1) satisfies:

$$\lim_{t\to\infty} x(t) = 0 \tag{2}$$

C. Problem Statement

Consider a feedback linearizable system transformable into the Brunovsky normal form:

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1}, \quad i = 1, ..., n-1 \\ \dot{\xi}_n &= b(\xi) + a(\xi)u, \end{aligned} \tag{3}$$

where $\xi \in \mathbb{R}^n$, $u \in \mathbb{R}$ are the state vector and the control input, respectively. There exist magnitude constraints on the system states due to physical/performance limits as follows:

$$|\xi_{i+1}(t)| \leq B_i, \quad \forall t \geq 0 \tag{4}$$

Our objective is to develop a procedure to design asymptotically stable controllers for system (1) with constraints as defined by (4). No bound is applied to the output state ξ_1. This choice is driven by the motivating example of the longitudinal dynamics of an aircraft. However, it is typical of problems with state rather than output constraints. Problems with unbounded output require a saturated reference trajectory design that is unnecessary for bounded output cases. The case of bounded output and multiple state constraints may be solved using a straightforward extension of the results presented in this paper.

D. Assumptions

The functions a and b: $\mathbb{R}^n \longrightarrow \mathbb{R}$ are smooth, that is, C^∞. In addition, the function a(ξ) $\neq 0$ for all ξ.

E. Control Design Procedure

1) Step 1: Consider the first scalar subsystem

$$\dot{\xi}_1 = \xi_2 \tag{5}$$

It is required that the propagation of the error in ξ_1 through to the next subsystem is bounded. The reason why this is necessary is explained at the end of this design step. Consequently, the growth of the control Lyapunov function (CLF) for (5) is to be restricted to a linear one. That is

$$V_1(\xi_1) = \mathbf{O}(\xi_1), \text{ as } |\xi_1| \to \infty \tag{6}$$

This is the first key condition of our design. A candidate CLF is

$$V_1(\xi_1) = k_1 \xi_1 \arctan(\xi_1) \tag{7}$$

$$\Longrightarrow \dot{V}_1 = k_1 \xi_2 \left[\arctan(\xi_1) + \frac{\xi_1}{1+\xi_1^2} \right], \tag{8}$$

where the gain $k_1 > 0$ is a design constant.

Since ξ_2 is required to be bounded, the stabilising function for (5), $\xi_{2_{ref}}$, must first be bounded. *This is the second key condition of our design procedure.* A choice for $\xi_{2_{ref}}$ is

$$\xi_{2_{ref}} = -c_1 \arctan(\xi_1) \tag{9}$$

$$z_1 = \xi_2 - \xi_{2_{ref}}, \tag{10}$$

where z_1 is the *error signal* variable, and $c_1 > 0$ is a design constant. The stabilizing function $\xi_{2_{ref}}$ is bounded by

$$|\xi_{2_{ref}}(t)| < \frac{\pi}{2} c_1 \tag{11}$$

Substituting (9) and (10) into (5) and (8) yields

$$\dot{\xi}_1 = -c_1 \arctan(\xi_1) + z_1 \tag{12}$$

$$\begin{aligned} \dot{V}_1 &= -k_1 c_1 \arctan(\xi_1) \left[\arctan(\xi_1) + \frac{\xi_1}{1+\xi_1^2} \right] \\ &\quad + k_1 z_1 \left[\arctan(\xi_1) + \frac{\xi_1}{1+\xi_1^2} \right] \\ &\leq -W_1(\xi_1) + k_1 z_1 \left[\arctan(\xi_1) + \frac{\xi_1}{1+\xi_1^2} \right], \end{aligned} \tag{13}$$

respectively, where

$$W_1(\xi_1) = k_1 c_1 \arctan(\xi_1) \left[\arctan(\xi_1) + \frac{\xi_1}{1+\xi_1^2} \right], \tag{14}$$

and is positive-definite in ξ_1. It is clear from (13) that $\dot{V}_1(\xi_1)$ becomes negative-definite once z_1 is driven to 0.

It is necessary to limit the growth of $V_1(\xi_1)$ in order to prevent the forward propagation of the error in ξ_1 through to the next subsystem via the cross-term in (13).

2) Step 2: Consider the augmented subsystem for (12)

$$\begin{aligned} \dot{z}_1 &= \dot{\xi}_2 - \dot{\xi}_{2_{ref}} \\ &= \xi_3 + \frac{c_1 \xi_2}{1+\xi_1^2} \end{aligned} \tag{15}$$

and define the *error signal* variable z_2 by

$$z_2 = \xi_3 - \xi_{3_{ref}} \tag{16}$$

With the stabilising function for (5), $\xi_{2_{ref}}$, already bounded in Step 1 as shown by (11), we are now only required to saturate z_1 in order to bound ξ_2, see (10). This is achieved by defining the CLF for (15) with a barrier function structure such that the growth of the CLF is governed by

$$|z_1| \to \Delta_{z_1} \quad \Longrightarrow \quad V_2(z_1) \to \infty, \tag{17}$$

where the constant Δ_{z_1} is the desired hard-bound on z_1. *This is the third and final key condition of our design.* A valid choice for $V_2(\xi_1, z_1)$ is

$$V_2(\xi_1, z_1) = V_1(\xi_1) + \frac{1}{2} k_3 \log\left(\frac{k_2^2}{k_2^2 - z_1^2}\right), \tag{18}$$

where the gain $k_3 > 0$ is a design constant. Such a choice for V_2 yields

$$|z_1(t)| < k_2, \quad \forall t \geq 0 \tag{19}$$

Differentiating (18) with respect to time gives

$$\dot{V}_2 = -W_1(\xi_1) + z_1 \left[k_1 \arctan(\xi_1) + \frac{k_1 \xi_1}{1 + \xi_1^2} + \frac{k_3 \dot{z}_1}{k_2^2 - z_1^2}\right] \tag{20}$$

whenever $V_2(\xi_1, z_1)$ is well-defined, and bounded at every $t \geq 0$. To simultaneously make $\dot{V}_2$ negative-definite and bound the stabilizing function for (15), $\xi_{3_{ref}}$, we choose $\xi_{3_{ref}}$ to be

$$\begin{aligned} \xi_{3_{ref}} = &-c_2 z_1 - \frac{c_1 \xi_2}{1 + \xi_1^2} \\ &- \frac{\left(k_2^2 - z_1^2\right)}{k_3} \left[k_1 \arctan(\xi_1) + \frac{k_1 \xi_1}{1 + \xi_1^2}\right], \end{aligned} \tag{21}$$

where $c_2 > 0$ is a design constant. Since every term on the right hand side (RHS) of (21) is bounded, $\xi_{3_{ref}}$ is consequently bounded.

Substituting (16) and (21) into (15) and (20) yields

$$\dot{z}_1 = -c_2 z_1 - \frac{\left(k_2^2 - z_1^2\right)}{k_3} \left[k_1 \arctan(\xi_1) + \frac{k_1 \xi_1}{1 + \xi_1^2}\right] + z_2 \tag{22}$$

and

$$\dot{V}_2 = -W_1(\xi_1) - \frac{k_3 c_2 z_1^2}{k_2^2 - z_1^2} + \frac{k_3}{k_2^2 - z_1^2} z_1 z_2, \tag{23}$$

respectively, where $W_1(\xi_1)$ is as defined in (14) and is positive-definite in ξ_1. Thus, from (23), it follows that $\dot{V}_2(\xi_1, z_1)$ becomes negative-definite once z_2 is driven to 0.

Now that z_1 is bounded, and with the stabilising function $\xi_{2_{ref}}$ also bounded, ξ_2 is consequently bounded as a direct result of (10).

The procedure to bound each of the remaining states is iterative and is analogous to Step 2. The exception of Step 1 is due to the desire to have the signal ξ_1 unbounded. The generic algorithm to bounding each of the remaining states of system (3) is detailed below.

3) Step 3: Consider the augmented subsystem for (22)

$$\begin{aligned} \dot{z}_2 &= \dot{\xi}_3 - \dot{\xi}_{3_{ref}} \\ &= \xi_4 - \dot{\xi}_{3_{ref}} \end{aligned} \tag{24}$$

and define the *error signal* variable

$$z_3 = \xi_4 - \xi_{4_{ref}} \tag{25}$$

As the stabilising function $\xi_{3_{ref}}$ has already been bounded in Step 2, as shown by (21), we are now only required to saturate the error signal z_2 in order to bound ξ_3, see (16). To achieve this we again impose a barrier function structure on the CLF for (24) with a growth condition defined by

$$|z_2| \to \Delta_{z_2} \Longrightarrow V_3(z_2) \to \infty, \tag{26}$$

where the constant Δ_{z_2} is the desired hard-bound on the *error signal* z_2. A valid CLF choice is

$$V_3(\xi_1, z_1, z_2) = V_2(\xi_1, z_1) + \frac{1}{2} k_5 \log\left(\frac{k_4^2}{k_4^2 - z_2^2}\right) \tag{27}$$

which yields

$$|z_2(t)| < k_4, \quad \forall t \geq 0, \tag{28}$$

where the gain $k_5 > 0$ is a design constant. The time derivative of (27) is given by

$$\begin{aligned} \dot{V}_3 = &-W_1(\xi_1) - \frac{k_3 c_2}{k_2^2 - z_1^2} z_1^2 \\ &+ z_2 \left[\frac{k_3}{k_2^2 - z_1^2} z_1 + \frac{k_5}{k_4^2 - z_2^2} \dot{z}_2\right], \end{aligned} \tag{29}$$

whenever $V_3(\xi_1, z_1, z_2)$ is well-defined and bounded at every t ≥ 0. To bound the stabilising function for (24), $\xi_{4_{ref}}$, we choose $\xi_{4_{ref}}$ to be

$$\xi_{4_{ref}} = -c_3 z_2 + \dot{\xi}_{3_{ref}}, \tag{30}$$

where $c_3 > 0$ is a design constant. As in Step 2, it is straight forward to verify that $\dot{\xi}_{3_{ref}}$ is bounded. The bound on $\xi_{4_{ref}}$ is

$$|\xi_{4_{ref}}(t)| < c_3 k_4 + |\dot{\xi}_{3_{ref}}(t)| \tag{31}$$

Substituting (30) and (25) into (24) and (29) yields

$$\dot{z}_2 = -c_3 z_2 + z_3 \tag{32}$$

$$\begin{aligned} \dot{V}_3 = &-W_1(\xi_1) - \frac{k_3 c_2}{k_2^2 - z_1^2} z_1^2 + \frac{k_3}{k_2^2 - z_1^2} z_1 z_2 - \frac{k_5 c_3}{k_4^2 - z_2^2} z_2^2 \\ &+ \frac{k_5}{k_4^2 - z_2^2} z_2 z_3, \end{aligned} \tag{33}$$

respectively, where $W_1(\xi_2)$ is as defined in (14) and is positive definite in ξ_1.

We choose to not explicitly cancel the cross-term $\frac{k_3}{k_2^2 - z_1^2} z_1 z_2$ in (29) because as $|z_1| \to k_2$, $\frac{k_3}{k_2^2 - z_1^2} z_1 z_2 \to \infty$. This means that the reference signal $\xi_{4_{ref}}$ is not bounded if direct cancelation of the cross-term is employed. To avoid this problem and still render $\dot{V}_3(\xi_1, z_1, z_2)$ negative definite, we dominate the cross-term by appropriately tuning the design constants. This is achieved by first manipulating (33) into the form

$$\begin{aligned}\dot{V}_3 = &-W_1(\xi_1) - \frac{k_3}{2\left(k_2^2 - z_1^2\right)}\left(z_1 - z_2\right)^2 - \frac{k_3 c_2}{k_2^2 - z_1^2} z_1^2 \\ &+ \frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_1^2 + \frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_2^2 - \frac{1}{2}\frac{k_5 c_3}{k_4^2 - z_2^2} z_2^2 \\ &- \frac{1}{2}\frac{k_5 c_3}{k_4^2 - z_2^2} z_2^2 + \frac{k_5}{k_4^2 - z_2^2} z_2 z_3 \end{aligned} \tag{34}$$

What we have done is complete the squares for the cross-term $\frac{k_3}{k_2^2 - z_1^2} z_1 z_2$, and add the terms $\frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_1^2$ and $\frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_2^2$ which come from the completion of the squares. These are the fourth and fifth terms on the RHS of (34), respectively. We then split the term $\frac{k_5 c_3 z_2^2}{k_4^2 - z_2^2}$ in two parts to clearly indicate our intention to use one part of the term to dominate the additional terms coming from the completion of the squares, and other part in the next step of the backstepping design procedure.

Let us now consider the sum of the third, fourth, fifth, and sixth term on the RHS of (34) separately

$$\begin{aligned}Y_1 = &-\frac{k_3 c_2}{k_2^2 - z_1^2} z_1^2 + \frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_1^2 + \frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_2^2 \\ &- \frac{1}{2}\frac{k_5 c_3}{k_4^2 - z_2^2} z_2^2 \end{aligned} \tag{35}$$

Our goal is to tune the design constants such that Y_1 is rendered negative-definite. If we choose

$$c_2 \geq \frac{k_4^2}{[k_2 \alpha_{z_1}]^2} + \frac{1}{2}, \tag{36}$$

where $\alpha_{z_1} \in (0, \sqrt{2})$ is a constant, then the following is obtained

$$\begin{aligned}Y_1 &= -\frac{k_3}{k_2^2 - z_1^2}\left[\frac{k_4^2}{[k_2\alpha_{z_1}]^2} + \frac{1}{2}\right] z_1^2 + \frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_1^2 \\ &\quad + \frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_2^2 - \frac{1}{2}\frac{k_5 c_3}{k_4^2 - z_2^2} z_2^2 \qquad (37) \\ &= -\frac{k_3}{\left(k_2^2 - z_1^2\right)}\frac{k_4^2}{[k_2\alpha_{z_1}]^2} z_1^2 + \frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_2^2 \\ &\quad - \frac{k_5 c_3}{2\left(k_4^2 - z_2^2\right)} z_2^2 \end{aligned} \tag{38}$$

From (28), it follows that

$$\frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_2^2 < \frac{k_3}{2\left(k_2^2 - z_1^2\right)} k_4^2 \tag{39}$$

Thus, when $|z_1| \geq k_2\alpha_{z_1}$, Y_1 is negative-definite if the design constant c_2 is tuned in accordance with (36).

When $|z_1| < k_2\alpha_{z_1}$, we employ the term $-\frac{1}{2}\frac{k_5 c_3}{k_4^2 - z_2^2} z_2^2$ to dominate $\frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_2^2$. By simply examining $\frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_2^2$, we can deduce that

$$-\frac{1}{2}\frac{k_5 c_3}{k_4^2 - z_2^2} z_2^2 \geq -\frac{1}{2}\frac{k_5 c_3}{k_4^2} z_2^2 \tag{40}$$

In addition, when $|z_1| < k_2\alpha_{z_1}$, the following is true for the second term on the RHS of (38)

$$\frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_2^2 < \frac{k_3}{2\left(k_2^2 - [k_2\alpha_{z_1}]^2\right)} z_2^2 \tag{41}$$

Thus, if we choose

$$\frac{k_5 c_3}{k_4^2} \geq \frac{k_3}{k_2^2 - [k_2\alpha_{z_1}]^2} \tag{42}$$

then it follows that

$$\frac{k_5 c_3}{2\left(k_4^2 - z_2^2\right)} z_2^2 \geq \frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_2^2, \quad \forall |z_1| < k_2\alpha_{z_1} \tag{43}$$

Consequently,

$$\begin{aligned}Y_1 = &-\frac{k_3 c_2}{k_2^2 - z_1^2} z_1^2 + \frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_1^2 + \frac{k_3}{2\left(k_2^2 - z_1^2\right)} z_2^2 \\ &- \frac{1}{2}\frac{k_5 c_3}{k_4^2 - z_2^2} z_2^2, \end{aligned} \tag{44}$$

and is negative-definite for $|z_1| < k_2$, $|z_2| < k_4$, if the design constants are tuned according to (36) and (42). If (44) is negative-definite, then from (34), we obtain

$$\dot{V}_3 \leq -W_2(\xi_1, z_1, z_2) - \frac{1}{2}\frac{k_5 c_3}{k_4^2 - z_2^2} z_2^2 + \frac{k_5}{k_4^2 - z_2^2} z_2 z_3, \tag{45}$$

which is negative-definite inside the set S = $\{\xi_1 \in \mathbb{R}, |z_1| < k_2, |z_2| < k_4\}$, once z_3 is driven to 0. The function W_2 is defined as

$$W_2 = W_1(\xi_1) + \frac{k_3 c_2}{k_2^2 - z_1^2} z_1^2 - \frac{k_3}{k_2^2 - z_1^2} z_1 z_2 + \frac{1}{2}\frac{k_5 c_3}{k_4^2 - z_2^2} z_2^2, \tag{46}$$

which is positive-definite inside the same set if (36) and (42) are satisfied.

Now that z_2 and $\xi_{3_{ref}}$ are both bounded, ξ_3 is therefore bounded as a direct result of (16).

The procedure to bound each of the remaining states is recursive and analogous to Step 3. The recursion terminates when system (3) is stabilized by the actual control u, which is at the n^{th} step of our control design.

4) Step n: Consider the final augmented subsystem

$$\begin{aligned}\dot{z}_{n-1} &= \dot{\xi}_n - \dot{\xi}_{n_{ref}} \\ &= b(\xi) + a(\xi)u - \dot{\xi}_{n_{ref}} \end{aligned} \tag{47}$$

To bound ξ_n we need to have the *error signal* variable for this step, z_{n-1}, bounded. This is achieved by imposing a barrier function structure on the CLF for (47) with a growth condition governed by

$$|z_{n-1}| \rightarrow \Delta_{z_{n-1}} \Longrightarrow V_n(z_{n-1}) \rightarrow \infty \tag{48}$$

A candidate CLF for system (3) is

$$V_n = V_{n-1} + \frac{1}{2}k_{2n-1}\log\left(\frac{k_{2n-2}^2}{k_{2n-2}^2 - z_{n-1}^2}\right), \tag{49}$$

where $k_{2n-1} > 0$ is a design constant, and k_{2n-2} is the desired bound on z_{n-1}. That is

$$|z_{n-1}(t)| < k_{2n-2}, \quad \forall t \geq 0 \tag{50}$$

The time derivative of (49) is given by

$$\dot{V}_n = -W_{n-1} - \frac{1}{2}\frac{k_{2n-3}c_{n-1}z_{n-2}^2}{k_{2n-4}^2 - z_{n-2}^2} + z_{n-1}\left[\frac{k_{2n-3}}{k_{2n-4}^2 - z_{n-2}^2}z_{n-2} + \frac{k_{2n-1}\dot{z}_{n-1}}{k_{2n-2}^2 - z_{n-1}^2}\right] \tag{51}$$

whenever V_n is well-defined and bounded for every $t \geq 0$. As there is no prescribed constraint on the control u, one choice for u is

$$u = \frac{1}{a(\xi)}\left\{-c_n z_{n-1} - b(\xi) + \dot{\xi}_{n_{ref}}\right\}, \tag{52}$$

where the gain $c_n > 0$ is a design constant. Again we choose to not directly cancel the cross-term $\frac{k_{2n-3}}{k_{2n-4}^2 - z_{n-2}^2}z_{n-2}$ in (51). This is because the presence of this term in the final control law produces extremely large actuator commands as $|z_{n-2}| \to k_{2n-4}$. Such a choice for u yields

$$\dot{V}_n = -W_{n-1} - \frac{1}{2}\frac{k_{2n-3}c_{n-1}z_{n-2}^2}{k_{2n-4}^2 - z_{n-2}^2} + \frac{k_{2n-3}}{k_{2n-4}^2 - z_{n-2}^2}z_{n-2}z_{n-1} - \frac{k_{2n-1}c_n z_{n-1}^2}{k_{2n-2}^2 - z_{n-1}^2} \tag{53}$$

The same trick to dominate the cross-term, as detailed in Step 3, is again needed here in order to make $\dot{V}_n$ negative-definite. As this is the final design step, there is no need to split the term $-\frac{k_{2n-1}c_n z_{n-1}^2}{k_{2n-2}^2 - z_{n-1}^2}$. Thus, if we choose

$$c_{n-1} \geq \frac{k_{2n-2}^2}{[k_{2n-4}\alpha_{z_{n-2}}]^2} + \frac{1}{2}, \tag{54}$$

$$\frac{k_{2n-1}c_n}{k_{2n-2}^2} \geq \frac{1}{2}\frac{k_{2n-3}}{k_{2n-4}^2 - [k_{2n-4}\alpha_{z_{n-2}}]^2}, \tag{55}$$

where $\alpha_{z_{n-2}} \in (0, \sqrt{2})$ is a constant. Then by following similar arguments detailed in Step 3, we have

$$\dot{V}_n(\xi_1, z_1, ..., z_{n-1}) \leq -W_n(\xi_1, z_1, ..., z_{n-1}), \tag{56}$$

which is negative-definite in the set S = $\{\xi_1 \in \mathbb{R}, |z_1| < k_2, ..., |z_{n-1}| < k_{2n-2}\}$. The function W_{n-1} is defined as

$$W_{n-1} = W_{n-2} + \frac{1}{2}\frac{k_{2n-3}c_{n-1}z_{n-2}^2}{k_{2n-4}^2 - z_{n-2}^2} - \frac{k_{2n-3}}{k_{2n-4}^2 - z_{n-2}^2}z_{n-2}z_{n-1} + \frac{k_{2n-1}c_n z_{n-1}^2}{k_{2n-2}^2 - z_{n-1}^2}, \tag{57}$$

and is positive-definite inside the same set S.

The last state in the cascade, ξ_n, is now bounded by virtue of (25) since both z_{n-1} and $\xi_{n_{ref}}$ are bounded.

Remark 2.1: The constraints on the design constants, that is (36), (42), ..., (54), and (55), are analytically derived from the worst case scenario. Numerical determination of those constants, based on the system's CLF and its derivative, reveals that the actual constraints on them are much less stringent. This can be observed in the simulation results later on.

Remark 2.2: It is interesting to note that if the functions $\frac{1}{a(\xi)}$ and $b(\xi)$ are bounded with bounded first derivatives for all ξ, then the control u, as defined in (52), is bounded in both magnitude and rate. This means that the algorithm detailed above is potentially applicable to systems with bounded controls and control rates as well as those with state constraints, or both.

5) The Closed-loop System: The closed-loop system with control given by (52) expressed in the error co-ordinates is as follows

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-2} \\ \dot{z}_{n-1} \end{bmatrix} = \begin{bmatrix} -A_{11} & z_1 & 0 & \cdots & 0 & 0 \\ -A_{21} & -c_2 z_1 & z_2 & \cdots & 0 & 0 \\ 0 & 0 & -c_3 z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -c_{n-1}z_{n-2} & z_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -c_n z_{n-1} \end{bmatrix} \tag{58}$$

where

$$A_{11} = c_1 \arctan(\xi_1)$$
$$A_{21} = \frac{\left(k_2^2 - z_1^2\right)}{k_3}\left[k_1 \arctan(\xi_1) + \frac{k_1\xi_1}{1 + \xi_1^2}\right]$$

Theorem 2.3: Consider system (3) in domain D defined by

$$D = \{\xi_1 \in \mathbb{R}, z_i \in \mathbb{R} : |z_i| < k_{2i}, i = 1, ..., n-1\} \tag{59}$$

The closed-loop system (58) with static state feedback control u given by (52), and the design constants tuned in accordance with (36), (42), ..., (54), and (55), is:

i) *Domain globally asymptotically stable in U.*
ii) *The control u is continuous in D.*

Proof:

Conclusion i) The control Lyapunov function for the closed-loop system (58), $V_n(\xi_1, z_1, ..., z_{n-1})$, is continuous, and positive definite, see (49), in the domain $D = \{\xi_1 \in \mathbb{R}, z_i \in \mathbb{R} : |z_i| \leq \epsilon_{z_i} < k_{2i}, i = 1, ..., n-1\}$. In addition, $V_n \to \infty$ as $|\xi_1| \to \infty, |z_{j-1}| \to k_{2j-2}$. Its derivative, $\dot{V}_n(\xi_1, z_1, ..., z_{n-1})$, is negative-definite in the same domain D when (36), (42), ..., (54), and (55) are satisfied. Thus, any trajectory starting from inside domain D will asymptotically converge to the origin.

Conclusion ii) comes directly from our construction of the control signal u, see (52). ■

III. State Bounds and Control Tuning

The goal of the control design is to obtain a controller that exploits the maximum range of state variation that is allowed while guaranteeing the state constraints are respected. The controller derivation provided a set of constraints on the error coordinates that depend on the controller parameters $\{k_i, c_i\}$. In practice, it is desirable that system states approach their hard bounds during the evolution of the system to ensure best performance of the closed-loop system. An important aspect of the proposed design is that choosing $\{k_i, c_i\}$ parameters that are self-consistent is a straightforward matter of satisfying a set of mutual constraints (36), (42), ..., (54), and (55).

The remaining task required is to relate the constraints on error coordinates back to constraints on the state coordinates.

The domain constraints obtained as a function of the proposed control design yield a set of conditions on the error variables

$$|z_i| = |\xi_{i+1} - \xi_{(i+1)_{\text{ref}}}| < k_{2i} \tag{60}$$

The reference trajectories, $\xi_{(i+1)_{\text{ref}}}$, are defined as algebraic functions of the error coordinates, system states and controller parameters. It is a straight forward exercise to obtain worst case over-bounds for the norms of the reference trajectories. Following from these bounds it is possible to obtain a set of non-linear bounds $\{X_1, X_2, \ldots, X_n\}$ for the state evolution in terms of the control parameters

$$\begin{aligned} |\xi_2(t)| &\leq |z_1(t)| + |\xi_{2_{ref}}(t)| \\ &< k_2 + \frac{\pi}{2} c_1 =: X_1 && (61) \\ |\xi_3(t)| &\leq |z_2(t)| + |\xi_{3_{ref}}(t)| \\ &< k_4 + c_2 k_2 + c_1 X_1 + \frac{k_1 k_2^2 (\pi + 1)}{2 k_3} =: X_2 && (62) \\ &\vdots \\ |\xi_n(t)| &\leq |z_{n-1}(t)| + |\xi_{n_{ref}}(t)| \\ &< k_{2n-2} + |\xi_{n_{ref}}(t)| =: X_{n-1} && (63) \end{aligned}$$

These bounds are defined recursively in the sense that X_2 depends on X_1, X_3 on X_2 and X_1, etc. Thus, for a given set of parameters it is straight forward to compute worst case bounds on the system states. From the point of view of the control design we think of the worst case state constraints as a non-linear function of the controller parameters

$$\begin{aligned} &Z := (k_1, \ldots, k_{2i}, c_1, \ldots, c_n) \\ &X := X(Z), \qquad X = (X_1, \ldots, X_n) \end{aligned}$$

The goal of tuning the control parameters is a multi-criteria constrained optimisation problem:

Find controller parameters Z, subject to constraints (36), (42), ..., (54), and (55) that maximises the multi-criteria cost function $X(Z)$.

Although this constrained optimisation problem is difficult it is quite tractable using modern numerical optimisation algorithms. (Consider trying to solve the original constrained nonlinear control problem using optimal control techniques.) An important property of the proposed methodology is that it is easy to find feasible values for the controller parameters Z by choosing very small values for those parameters. The nice algebraic form of the constraints (36), (42), ..., (54), and (55) is crucial in this process.

Remark 3.1: To simplify the optimisation procedure, it is possible to consider a single cost criterion

$$\Phi(Z) := X^T W X, \tag{64}$$

where $W > 0$ is a positive-definite weight matrix.

The approach taken to tune the controller parameters guarantees that the achieved bounds are optimised to ensure that they are tight worst case bounds. It is clear that some of the arguments used to estimate the bounds on the reference trajectories, $\xi_{(i+1)_{ref}}$, lead to sub-optimal bounds on the system states. However, in general, it is expected that the achieved closed-loop state bounds will be reasonably close to those obtained using optimal control design. Thus, given a good solution to the optimisation problem, the domain of validity D in Theorem 2.3 is a large subset of the possible domain of validity. The aggressiveness of the barrier functions will govern how abruptly the state constraints are applied during the closed-loop evolution of the system.

IV. Simulations

Simulations for a simple 4^{th}-order integrator cascade are presented to support our results. The system's equations of motion are given by

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_4 &= u \end{aligned} \tag{65}$$

The constraints on the system's states are as follows

$$\xi_2 \leq 2.5, \quad \xi_3 \leq 12, \quad \xi_4 \leq 700 \tag{66}$$

The design constants are numerically tuned as follows

$$\begin{aligned} &c_1 = \frac{0.5}{\pi}, c_2 = 1, c_3 = 50, c_4 = 1, k_1 = 1, k_2 = 2, \\ &k_3 = 3, k_4 = 5.61, k_5 = 1, k_6 = 354.5, k_7 = 10 \end{aligned}$$

The closed-loop system is simulated in Matlab/Simulink using the fixed step Dormand-Prince solver option with a step size of 0.005.

Figures 1 and 2 clearly show that ξ_2, ξ_3, and ξ_4 all remain within the constraints expressed by (66), irrespective of the initial condition. Note the ramp-like response of the system which provides a perfect example of how a system with velocity constraints, such as that of non-acrobatic and non-fighter types of aircraft, should respond to a change-in-position command. Also note the near optimal velocity obtained, ξ_2, which can be pushed closer to its true bound by further tweaking the design constants. This is where backstepping is most proficient as it can provide the high gains required to push the system states as close to their maximum limits as possible for optimal reasons.

V. Conclusions

The problem of stabilization for a class of feedback linearizable systems with state constraints has been considered. Such systems are very common in practice due to physical/performance limitations. The main contribution of this paper is the extension of the backstepping methodology to asymptotically stabilise such systems. Future research will focus on extending the proposed approach to a more general class of nonlinear systems.

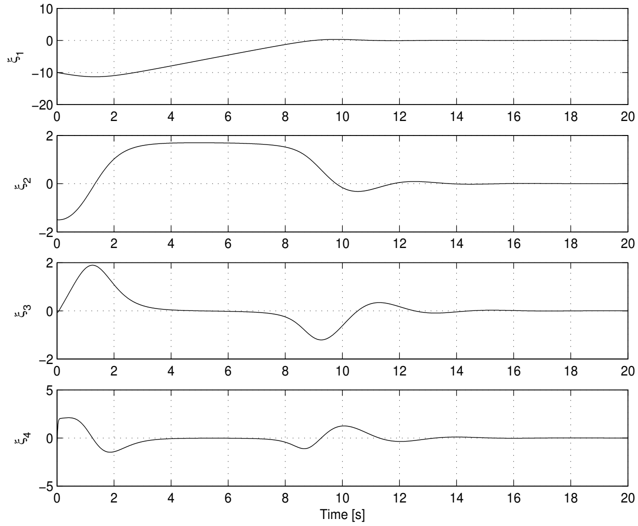

Fig. 1. *Closed-loop system response with initial condition:* $\xi_1(0) = -10, \xi_2(0) = -1.5, \xi_3(0) = -0.05, \xi_4(0) = -2.5$

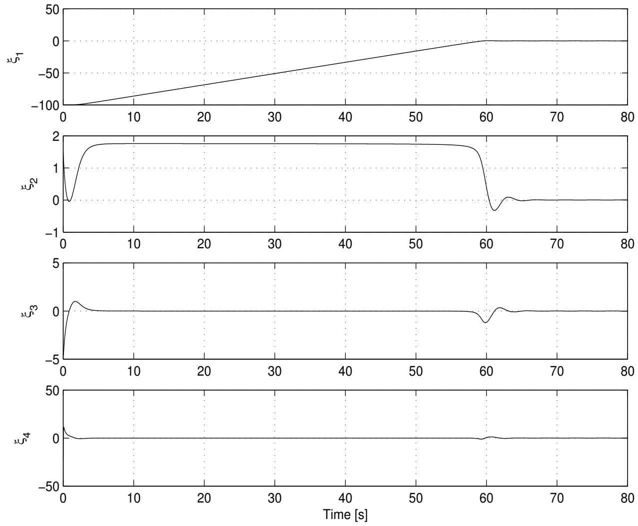

Fig. 2. *Closed-loop system response with initial condition:* $\xi_1(0) = -100, \xi_2(0) = 1.5, \xi_3(0) = -4, \xi_4(0) = -50$

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