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Global considerations on the Kuramoto model of sinusoidally coupled oscillators

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***Abstract*— In this article we study global stability properties of the Kuramoto model of sinusoidally coupled oscillators. We base our analysis on previous results by the control community that analyze local properties of the consensus point of different kinds of Kuramoto models. We prove that for the complete symmetric case, the consensus point is almost globally stable, that is, the set of trajectories that do not converge to it has zero measure. We present a counter-example of that when the completeness hypothesis is removed. We also show that the general non-symmetric case is more complex and we analyze the particular case of oscillators coupled in a ring structure, where we can establish some global stability properties.**

I. PRELIMINARIES

In the 1970s, Kuramoto proposes a model to describe a population of weakly coupled oscillators, following the works of A. T. Winfree on collective synchronization of biological systems [1],[2]. Each individual oscillator is described by its phase and the coupling between two individuals is a function of the phase difference. The general Kuramoto model takes the following form [3]:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i) \ , \ i = 1, \ldots, N$$

where Γ_{ij} are the *interaction* functions that model the coupling and N is the total number of oscillators. Since $\theta \in [0, 2\pi)$, the corresponding state space is the N-dimensional torus $\mathcal{T}^N$. This model has turned to be suitable for describing many different systems in biology, physics and engineering [2],[3],[4]. The key question is whether or not the system behavior reaches the *consensus*, i.e. the state where all the agents are locked or synchronized. Recently, this model has received the attention of control theorists interested in the coordination and consensus of multi-agent systems (see [5] and references there in).

In this paper we focus on the global properties of the consensus of many agents described by two different versions of the Kuramoto model. We begin with the classic symmetric sine model [2],[3] in which the mutual interaction between agents depends on the sine of the phase difference between them. This model was studied for an arbitrary interaction graph in [5], and La Salle's Invariance principle was invoked to show convergence to the equilibrium set. However, as we will show in Section II, the characterization of these equilibria in [5] is incomplete, so the resulting almost global stability claims are not valid in the general case. Indeed, we characterize situations where the system has other attractors in addition to consensus. Nevertheless, in the classical Kuramoto case of a complete graph, we are able to show there are no other attractors and hence obtain almost global stability. We also show in Section III how some of these results extend to non-symmetric graphs, where the coupling is unidirectional instead of symmetric. We focus on the well-studied special case of a ring of coupled oscillators [6], whose local stability was studied extensively in [7]. Finally we present some conclusions.

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II. THE SYMMETRIC KURAMOTO MODEL

A. Dynamics

The dynamics of a given agent depends on the sine of the phase difference of neighbors. As in [8], we can build a directed graph with the agents as nodes and the edges representing the relationships between agents. Along this paper, we assume that the graph is connected. Let e be the number of edges. We construct the incidence matrix $B_{N\times e}$ as follows:

$$B_{ij} = \begin{cases} 1 & \text{if edge } j \text{ reaches node } i \\ -1 & \text{if edge } j \text{ leaves node } i \\ 0 & \text{otherwise} \end{cases}$$

We will say that two agents are neighbors if they are related by a link. For $i = 1, \ldots, N$, the dynamic is given by

$$\dot{\theta}_i = \omega_i + \frac{K}{N}. \sum_{j\in\mathcal{N}_i} \sin(\theta_j - \theta_i) \qquad (1)$$

where $\mathcal{N}_i$ is the set of index of the neighbors of agent i. When $\mathcal{N}_i = \{1, \ldots, N\} \setminus \{i\}$ for every i, we have the *complete* or *all to all* case. In matrix notation, the Dynamics can be written as

$$\dot{\theta} = \omega_i - \frac{K}{N} \cdot B \sin\left(B^T\theta\right) \qquad (2)$$

Moreover, we will assume that all the oscillators have the same natural frequency. In this case, with a shift, the model can be reduced to

$$\dot{\theta}_i = \frac{K}{N}. \sum_{j\in\mathcal{N}_i} \sin(\theta_j - \theta_i) \qquad (3)$$

$i = 1, \ldots, N$. We can further simplify the notation by eliminating the factor $\frac{K}{N}$; this amounts to renormalizing time. In matrix notation, we get

$$\dot{\theta} = -B. \sin\left(B^T\theta\right) \qquad (4)$$

Remarks:

- The dynamics depend only on the phase difference of the oscillators.
- we will name by **consensus** or **synchronization** the state where all the phase differences are zero. In this case, we say that the oscillators are locked. Every consensus state is of the form $\theta = c.\mathbf{1}_N$, with $c \in [0, 2\pi)$ ($\mathbf{1}_N$ denotes the column vector with all the elements equal to one). We have a curve of consensus points.
- if $\bar{\theta}$ is an equilibrium point, so is $\bar{\theta} + c.\mathbf{1}$ for every $c \in [0, 2\pi)$. The stability properties of $\bar{\theta}$ refer to this whole set of equilibrium points.
- As was done by Kuramoto [2], we associate the individual oscillator phases to points running around the circle of radius 1 in the plane. Then, each oscillator can be described by a unitary phasor $V_i = e^{j\theta_i}$. At a consensus point, all the phasors coincide.

Local stability of the consensus point was studied in [5] using La Salle's invariance principle [9]. The function

$$U(\theta) = e - \mathbf{1}_e^T \cos(B^T\theta) \qquad (5)$$

(with $\mathbf{1}_e$ is the e-dimensional column vector of all ones) is non-negative, and such that the system can be written in the gradient form

$$\dot{\theta} = -\nabla U;$$

In particular this implies that

$$\dot{U}(\theta) = -\|\dot{\theta}\|^2,$$

hence the function is non-increasing along trajectories. Since the state space is compact, every trajectory has a non-empty ω-limit set. La Salle's result ensures that every trajectory goes to the set $W = \left\{\theta \mid \dot{U}(\theta) = 0\right\}$, i.e., goes to an equilibrium point. In order to establish almost global stability of the consensus point, it must be true that the consensus point is the only attractor. The local analysis of all the equilibrium points of (4) done in [5] is incomplete, and therefore the conclusion of almost global stability is in general not true, as is shown in the next Example.

Example 2.1: Consider the case with $N = 6$ in which the dynamics of the agents are as follows:

$$\dot{\theta}_i = [\sin(\theta_{i-1} - \theta_i) + \sin(\theta_{i+1} - \theta_i)]$$

Here the configuration is circular; we identify Θ_1 with Θ_7. Consider the equilibrium point showed in Fig. 1. The characteristic polynomial of the linear approximation has the roots 0 and -2 (simple), and $-\frac{1}{2}$ and

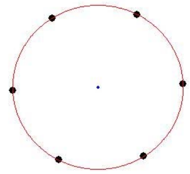

Fig. 1. *Stable non-consensus equilibrium for Kuramoto model with $N = 6$ (Example 2.1).*

$-\frac{3}{2}$ (double). Therefore, this configuration is locally attractive.

△

We thus see that guaranteeing asymptotic consensus is more involved; in the following subsections we provide some theory that may help classify these other equilibria, and also show that in the complete graph case, there is indeed almost global stability of the consensus point.

B. Equilibrium points

To obtain the equilibrium points we must annihilate the vector field. It is clear that the consensus point $\theta = \theta_0.\mathbf{1}$ is an equilibrium point, with arbitrary $\theta_0 \in \mathcal{R}$ and being $\mathbf{1}$ the vector of all ones. But there are other equilibrium points as well.

Lemma 2.1: At any equilibrium point $\bar{\theta}$ of (3), it must be true that the phasors

$$\sum_{h \in \mathcal{N}_i} V_h \quad , \quad V_i$$

are parallel in the complex plane, for every i.[1].

Proof:
For every $i = 1, \ldots, N$, consider the number

$$\alpha_i = \sum_{h \in \mathcal{N}_i} \frac{V_h}{V_i} = \sum_{h \in \mathcal{N}_i} e^{j(\bar{\theta}_h - \bar{\theta}_i)} =$$

$$= \sum_{j \in \mathcal{N}_i} \cos(\bar{\theta}_h - \bar{\theta}_i) + j. \sum_{j \in \mathcal{N}_i} \sin(\bar{\theta}_h - \bar{\theta}_i)$$

Since $\bar{\theta}$ is an equilibrium point, α_i is a real number and $\sum_{h \in \mathcal{N}_i} V_h = \alpha_i.V_i$.

□

[1]As noted below, this result extends to the non-symmetric case.

Lemma 2.2: Let $\bar{\theta}$ be an equilibrium point of (3) **for the complete case**. Then, we have three different types of equilibrium points:

- *synchronization*: $\sum_{i=1}^{N} V_i = N.V_1$;
- *partial consensus*: non synchronized state with all V_i parallel;
- *balanced* (non-parallel): $\sum_{i=1}^{N} V_i = 0$.

Proof:
Let us denote by α the sum of all the phasors. It is clear that for each $i = 1, \ldots, N$,

$$\alpha = \sum_{i=1}^{N} V_i = V_i. \left[1 + \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{V_k}{V_i} \right]$$

Note that

$$\sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{V_k}{V_i} = \sum_{k \in \mathcal{N}_i} \frac{V_k}{V_i} = \alpha_i$$

with α_i is a real number defined as in Lemma 2.1. We can write $\alpha = V_i. [1 + \alpha_i]$, $i = 1, \ldots, N$. At a synchronization point, all the V_i coincide and then $\sum_{i=1}^{N} V_i = N.V_1$ and $\alpha_i = N - 1$ for $i = 1, \ldots, N$.

At partial consensus point, all V_i are parallel and we can take the reference such that there are m agents with $V_i = -1$, ($1 \leq 2m \leq N$), and $N - m$ agents with $V_i = 1$. The first group contains the *unsynchronized* variables (an unsynchronized variable θ_h agrees with $m-1$ variables and disagree with the other $N - m$.). It this case $\sum_{i=1}^{N} V_i = N - 2m$.

Finally, consider an equilibrium point with V_i and V_k non parallel. Then

$$\alpha = (1 + \alpha_i).V_i = (1 + \alpha_k).V_k$$

It follows that

$$\alpha = \sum_{i=1}^{N} V_i = 0 \ , \ \alpha_i = -1 \ , \ i = 1, \ldots, N$$

□

C. Stability analysis

We will analyze the stability of the equilibrium points using Jacobian linearization. Again, we begin with the case of a general connected graph, and later focus on the complete case.

A first order approximation of the system at an equilibrium point $\bar{\theta}$ takes the form $\dot{\delta\theta} = A.\delta\theta$, with $\delta\theta = (\theta - \bar{\theta})$ and A the symmetric matrix $N \times N$ with entries

$$\begin{cases} a_{ii} = & -\sum_{k \in \mathcal{N}_i} \cos(\bar{\theta}_k - \bar{\theta}_i) = -\alpha_i \\ a_{hi} = & \begin{cases} \cos(\bar{\theta}_h - \bar{\theta}_i) & , \quad h \in \mathcal{N}_i \\ 0 & , \quad h \notin \mathcal{N}_i \end{cases} \end{cases}$$

with α_i defined as in Lemma 2.1.

Remarks: The matrix A is symmetric, reflecting the bidirectional influence of the agents[2]. It always has the eigenvector $\mathbf{1}$ with zero eigenvalue, due to the invariance of field under translations parallel to $\mathbf{1}$ [10].

The following results can help classify locally the equilibrium points.

Lemma 2.3: Let $\bar{\theta}$ be an equilibrium point of (3), such that there is at least one $\alpha_i < 0$. Then, $\bar{\theta}$ is unstable.

Proof:
As we mentioned at the beginning of this Section, the numbers $-\alpha_i$ appear at the diagonal of the symmetric matrix A. Then, A can not be negative definite nor semi-definite and so $\bar{\theta}$ is unstable.

□

Lemma 2.4: Let $\bar{\theta}$ be an equilibrium point of (3), such that $\cos(\bar{\theta}_k - \bar{\theta}_i) > 0$ for every $k \in \mathcal{N}_i$. Then, $\bar{\theta}$ is stable.

Proof:
We can apply Gershgorin' s theorem to the matrix A described above. Since the diagonal elements are $-\alpha_i$, and the off-diagonal terms in the row add up to α_i, all eigenvalues lie in disks centered at $-\alpha_i$ and with radius α_i. Therefore the eigenvalues of A lie in $Re[\lambda] < 0$ or at $\lambda = 0$; if the graph is connected, the latter eigenvalue is simple, corresponding to perturbations in the direction of consensus.

□

Remark: The above lemma covers the example given in the previous section. Indeed, for sparse graphs, there can be equilibrium configurations where all neighboring angles are less than $\pi/2$, and thus provide attractors other than the consensus point.

In the case of complete (full mesh) graph, angles larger than $\pi/2$ are bound to occur. Indeed, we are now ready to prove that the consensus point is the only attractor for the complete symmetric case.

Theorem 2.1: Let $\bar{\theta}$ be an equilibrium point of (3) for the complete case. Then, the consensus point the only asymptotically stable equilibrium.

Proof:
As we have seen in Lemma 2.2, for the complete case we have three different types of equilibrium points. Next we study their local stability properties.

- **Synchronization:** at the consensus point, the matrix A takes a very particular form:
$$A = \left[-N.I + \mathbf{1}^T\mathbf{1}\right]$$
which is symmetric and circulant. It is straightforward to show that its characteristic polynomial is
$$p(\lambda) = \lambda.(\lambda + N)^{N-1}$$
and A has 0 as a single eigenvalue and $-N$ as an eigenvalue with multiplicity $(N-1)$ (see, for example, [10]). Then, the consensus point is a local attractor.
- **Partial consensus:** consider a partial consensus point $\bar{\theta}$, with its corresponding m, $1 \le 2m \le N$. All the phase differences are 0 or $\pm\pi$. The numbers
$$\alpha_h = (m-1).\cos(0) + (N-m).\cos(\pi)$$
$$= m - 1 - N + m = -N + 2m - 1$$
are the same for every unsynchronized variable and we denote it by α_U. In the same way, the number
$$\alpha_S = (N-m-1).\cos(0) + m.\cos(\pi)$$

[2] This hypothesis will be removed below.

$$= N - 2m - 1$$

corresponds to every synchronized variable. Since α_U is always negative, by Lemma 2.3, $\bar{\theta}$ is unstable.

- **Non-consensus:** let $\bar{\theta}$ be a balanced equilibrium point. Since by Lemma 2.2, $\alpha_1 = -1$, Lemma 2.3 implies that $\bar{\theta}$ is unstable.

□

Corollary 1.1: For the complete case, the synchronized or consensus state is almost globally stable.

Proof:
As we have mentioned in Subsection II-A, following [5], we can apply La Salle's Invariance result using the function U introduced in (5). From compactness of the state space, all the trajectories must converge to the largest invariant set contained in $\left\{\theta \mid \dot{U}(\theta) = 0\right\}$, i.e., must go to an equilibrium. From Theorem 2.1, the consensus point is the only attractor. Then, the only trajectories that are not attracted by the consensus point are the stable manifolds of the saddle equilibrium points, which are a zero measure set.

□

Converse results for almost global stability state the existence of a density function for the system [11],[12],[13]. For the cases $N = 2$ and 3, we were able to find a density function for the system, just inverting the function U introduced in (5) for the complete case:

$$\begin{aligned} \rho_2(\theta) &= \frac{1}{1-\cos(\theta_2-\theta_1)} \\ \rho_3(\theta) &= \frac{1}{3-\sum_{i=1}^{3}\cos(\theta_{i+1}-\theta_i)} \end{aligned} \tag{6}$$

We still haven't found a density function for higher dimensions.

III. An example of non-symmetric graphs

Previous results do not directly extend to the general case of non-symmetric graphs (i.e., where $k \in \mathcal{N}_i$ does not imply that $i \in \mathcal{N}_k$).

In this regard, we mention the following:

- The Jacobian linearization is not symmetric, which implies that this is no longer a gradient system. In particular, one can no longer give a La Salle-type theorem saying that the only attractors are equilibrium points. As shown in [7] for the ring example, there can be other periodic orbits in the system, where the phase differences converge but not the angles themselves.
- The characterization of other stable equilibria as in Lemma 2.4 is still valid.

A. Dynamics

We focus on the study of the dynamics of N oscillators coupled in a ring structure, in a way that the system is described by the equations

$$\dot{\theta}_i = K.\sin(\theta_{i+1} - \theta_i) \tag{7}$$

$i = 1, \ldots, N$, $N+1 = 1$ [6]. Besides the consensus point, we are also interested in the solution where all the oscillators are locked, in the sense that the phase differences between them remain constant in time; these are the *phase-locking solutions* [7] (these are equilibrium points or limit-cycles). So a particular phase-locking solution is characterized by a unique number α, $0 \le \alpha < 2\pi$, such that

$$\dot{\theta}_i = \sin(\alpha) \quad , \quad i = 1, \ldots, N$$

It follows that α or $\pi - \alpha$ represents the distance between two consecutive oscillators. So

$$\theta_i(t) = \sin(\alpha).t + \theta_{i0} \quad , \quad i = 1, \ldots, N$$

represents a limit cycle in the N-Torus (or an equilibrium point if α is 0 or π). Observe that the orbit of a phase-locking solution with non zero $\sin(\alpha)$ is invariant under translations with associated vector $c.\mathbf{1}$.

It is useful to re-write equations (7) in terms of the sequential phase difference

$$\Phi_i = \theta_{i+1} - \theta_i$$

The new description of the system is

$$\dot{\Phi}_i = K.\left[\sin(\Phi_{i+1}) - \sin(\Phi_i)\right] \tag{8}$$

$i = 1, \ldots, N$, $N + 1 = 1$. In this context, the phase-locking solutions are the equilibrium points of (8) and for a given phase-locking solution, the phase difference between consecutive oscillators can take only one of two possible values: an angle α or its complement $\pi - \alpha$; when $\alpha = \pi/2$, there is only one value. It is clear that $\sum_{i=1}^{N} \Phi_i = 2k\pi$, for some $k \in \mathcal{Z}$.

B. Stability analysis

A complete local analysis of the stability of equilibrium points and phase-locking solutions of (7) was done in [7], using Jacobian linearization techniques combined with Gershsgorin Theorem of localization of the eigenvalues of a given matrix. It follows that for the cases $N = 2$ and 3, the consensus point is the only attractor, but for higher dimensions, there are asymptotically stable limit cycles. So, we may try to establish almost global stability of the consensus point only in low dimensions. We use a function U similar to function V introduced in (5). In this case, we have that $U(\Phi) = N - \sum_{i=1}^{N} \cos(\Phi_i)$, where Φ is the vector of all the cyclic phase differences. Using (8) we have that

$$\dot{U}(\Phi) = \sum_{i=1}^{N} \sin(\Phi_i) \, . \dot{\Phi}_i =$$

$$-\sum_{i=1}^{N} \left[\sin^2(\Phi_i) - \sin(\Phi_{i+1}) \, . \sin(\Phi_i) \right]$$

Re-arranging terms we obtain

$$\dot{U}(\Phi) = -\sum_{i=1}^{N} \left[\sin(\Phi_{i+1}) - \sin(\Phi_i) \right]^2$$

Then $\dot{U} \leq 0$ in the torus and every trajectory goes to the set where $\dot{U}$ vanishes. But

$$C = \left\{ \dot{U} = 0 \right\} = \left\{ \sin(\Phi_{i+1}) = \sin(\Phi_i) \right\}$$

which contains the phase-locking solutions as its only invariants. So, we can affirm that almost all the trajectories in the torus converge to one of the stable phase-locking solutions. For the cases $N = 2$ and $N = 3$, we conclude the almost global stability of the consensus point. We remark that the functions we have introduced in (6) are also density functions for (8) for $N = 2$ and 3.

IV. Conclusions

In this work, we have presented some global considerations for the Kuramoto model with sinusoidal influence functions. We first deduced some results for the general symmetric case. For the symmetric case with complete associated graph, we proved the almost global attraction of the synchronized state. We also analyzed the non-symmetric case of coupled oscillators in a ring structure, where we have shown the almost global stability of the stable phase-locking solutions. For low dimensions, this implies the almost global stability of the consensus point.

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