Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Summation-type conditions for uniform asymptotic convergence in discrete-time systems: applications in identification

Antonio Loría Françoise Lamnabhi-Lagarrigue Dragan Nešić

***Abstract*— We establish summation-type conditions to ensure the uniform convergence of discrete-time systems, parameterized in the sampling time. The main results are analogous to previous results obtained in the domain of continuous-time systems and that we have referred to as “integral conditions”. The sufficient conditions that we present here can also be interpreted as conditions for convergence of series. Our main results are also useful for design of sampled-data controllers via approximate models; for illustration, we present two results on control design reminiscent of common problems arising in identification and adaptive control.**

I. INTRODUCTION

In general the analysis tools that are tailored for purely continuous-time models and which aid in (continuous-time) control design fail to guarantee the stability of the computer controlled system that is, involving the sampler and holders which introduce hybrid dynamics.

A prescriptive framework for control design for sampled-data control systems has been introduced in [16], [17] and, recently, extended to the case of systems of difference inclusions in [14]. At the basis of this framework is the formulation of *parameterized* discrete-time systems that is, whose dynamics depend on the sampling period. Considering parameterized systems is fundamental for different reasons: from a practical viewpoint, approximate models are easily computed while exact models are rarely available for nonlinear systems; besides, parameterizing the models in the sampling period leads to more general representations than models relying on fixed sampling periods; from a theoretical viewpoint, relying on the framework of [16], [17], [14] this allows to lay the conditions on the *approximate* discrete-time models in order to conclude uniform asymptotic stability (in a semi-global practical sense) of the sampled-data system without knowledge of exact models.

In this paper we consider discrete-time systems parameterized in the sampling time T that is, systems of the form:

$$x_{k+1} = F_T(k, x_k) \tag{1}$$

where $T \in (0, T^*)$ for some $T^* > 0$. Our results establish uniform global asymptotic stability of the origin based on so-called “summability” conditions. Inscribed in the mentioned framework, our main results are useful in design of sampled-data control systems.

Such conditions are alternative to the well known Lyapunov conditions, involving difference equations. Summability conditions are the discrete-time counterpart of the integral conditions originally presented in [19, Appendix] for continuous time systems and which establish uniform asymptotic stability. An advantage of the summability conditions over classical Lyapunov theory is that the appropriate conditions may be easily inferred in particular cases where finding a strict Lyapunov function may become considerably hard (e.g., in adaptive control, tracking, time-varying stabilization, etc.). The summability conditions presented here can also be regarded as conditions for convergence of series.

We demonstrate the utility of our main lemmas by addressing the problem of studying the adaptive stabilization of a discrete-time adaptive systems. The examples that we present are reminiscent of the so-called speed-gradient system (cf. [12], [2]) and the closed-loop system appearing in Model Reference Adaptive Control (MRAC) (cf. [5], [4]). However, as it will be clear from our analysis the results for the continuous time case, which are well known, may not be directly “transcripted” into the parameterized discrete-time context. In particular the important property of strict positivity of the continuous-time system is lost when applying the Euler discretization.

The rest of the paper is organized as follows. In the following section we introduce some notations and definitions that we use throughout. In Section III we present our main results, both for asymptotic and exponential stability. In Section IV we present their application into the MRAC problem mentioned above and we conclude with some remarks in Section V.

II. PRELIMINARIES

Throughout the paper we denote by $\mathbb{Z}$ the set of integer numbers and by $\mathbb{R}$ the set of reals. $|\cdot|$ stands for the 1-norm of vectors, *i.e.* $|x| := \sum_i |x_i|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is continuous, strictly increasing and zero at zero; $\alpha \in \mathcal{K}_\infty$ if, in addition, it is unbounded. For an arbitrary $r \in \mathbb{R}$ we use the notation $\lfloor r \rfloor := \max_{z \in \mathbb{Z}, z \leq r} z$. Given strictly positive real numbers L, T we use the following notation:

$$\ell_{L,T} := \left\lfloor \frac{L}{T} \right\rfloor . \tag{2}$$

The solution of system (1) at time k, starting at initial time $k_\circ$ and emanating from the initial condition $x_\circ = x(k_\circ)$, is denoted as $\phi_T^x(k, k_\circ, x_\circ)$ or ϕ_T^x if $k_\circ, x_\circ$ are clear from the context.

In what follows, the qualifier “uniformly” refers to the initial states and the initial $continuous$ times $t_\circ := k_\circ T$.

Definition 1 We say that system (1) is uniformly forward complete (UFC), if there exist $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ and $T^*, c > 0$ such that for all $k_\circ \geq 0$, $x(k_\circ) = x_\circ$, with $x_\circ \in \mathbb{R}^n$, and

Antonio Loría and Françoise Lamnabhi-Lagarrigue are with CNRS–LSS, Supélec, 3, Rue Joliot Curie, 91192 Gif s/Yvette, France. E-mail: loria@lss.supelec.fr, lamnabhi@lss.supelec.fr.
Dragan Nešić is with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3010, Victoria, Australia. E-mail: d.nesic@ee.mu.oz.au. This work was supported by the Australian Research Council under the Discovery Grants and Australian Professorial Fellow schemes.

$T \in (0, T^*)$ we have

$$|\phi_T^x(k, k_\circ, x_\circ)| \leq \sigma_1(|x_\circ|) + \sigma_2(T(k - k_\circ)) + c \qquad (3)$$

for all $k \geq k_\circ$. □

Definition 2 The system (1) is uniformly semiglobally bounded, *i.e.* USB, (resp. uniformly globally bounded UGB), if there exist $\kappa \in \mathcal{K}_\infty$ and c, such that for any $\Delta > 0$ there exists $T^* > 0$ (there exists $T^* > 0$) such that $k_\circ \geq 0$, $x(k_\circ) = x_\circ$ with $|x_\circ| \leq \Delta$ and $T \in (0, T^*)$ ($x_\circ \in \mathbb{R}^n$ and $T \in (0, T^*)$) implies

$$|\phi_T^x(k, k_\circ, y_\circ)| \leq \kappa(|x_\circ|) + c\,, \qquad (4)$$

for all $k \geq k_\circ$. □

Definition 3 The parameterized time-varying system (1) is:
(i) uniformly semiglobally stable, *i.e.* USS, (resp. uniformly globally stable) if the bound in (4) holds with $c = 0$;
(ii) semiglobally practically uniformly asymptotically stable (SP-UAS) if there exist $\kappa \in \mathcal{K}_\infty$ and for any pair of positive numbers (Δ, ν) there exists $T^* > 0$ such that:

(a) the system is USS;

(b) for each $\sigma > 0$, there exists $L > 0$ such that

$$|\phi_T^x(k, k_\circ, x_\circ)| \leq \max\{\sigma,\, \nu\} \qquad (5)$$

for all $k \geq k_\circ + \ell_{L,T}$, $k_\circ \geq 0$, all $|x_\circ| \leq \Delta$ and all $T \in (0, T^*)$;

(iii) uniformly globally asymptotically stable (UGAS) if there exists $T^* > 0$ such that:

(a) the system is UGS;

(b) for each $\sigma > 0$, there exists $L > 0$ such that

$$|\phi_T^x(k, k_\circ, x_\circ)| \leq \sigma \qquad \forall k \geq k_\circ + \ell_{L,T} \qquad (6)$$

for all $k_\circ \geq 0$, all $x_\circ \in \mathbb{R}^n$ and all $T \in (0, T^*)$;

(iv) semiglobally practically *uniformly* exponentially stable (SP-UES) if for any pair of strictly positive real numbers (Δ, ν), there exist $T^* > 0$, κ, λ such that

$$|x_\circ| \leq \Delta \quad \Longrightarrow \quad |\phi_T^x(k)| \leq \max\{\kappa\,|x_\circ|\,e^{-\lambda T(k-k_\circ)},\, \nu\} \qquad (7)$$

for all $k \geq k_\circ \geq 0$, and all $T \in (0, T^*)$;
(v) uniformly globally exponentially stable (UGES) if there exist $T^* > 0$, κ, λ such that for all $x_\circ \in \mathbb{R}^n$ and all $k \geq k_\circ \geq 0$ it holds that $|\phi_T^x(k)| \leq \kappa\,|x_\circ|\,e^{-\lambda T(k-k_\circ)}$. □

III. Main results

A. Conditions for UGAS

Our first lemma establishes uniform asymptotic stability for the case when UGS can be established by other means (cf. inequality (9)).

Lemma 1 If system (1) is UGS and there exist: a constant $T^* > 0$, a positive definite continuous function $\gamma(\cdot)$ and, for each $r, \nu > 0$ there exist $\beta_{r\nu} > 0$ such that

$$T \sum_{k=k_\circ}^{\infty} [\gamma(|\phi_T^x(k)|) - \nu] \;\leq\; \beta_{r\nu} \qquad (8)$$

for all $k_\circ \geq 0$, $x_\circ \in B_r$ and $T \in (0, T^*)$ then, the origin is UGAS. □

Proof. Since the system is UGS there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\max_{k \geq k_\circ} |\phi_T^x(k)| \leq \alpha(|x_\circ|) \qquad (9)$$

for all $k_\circ \geq 0$, $x_\circ \in \mathbb{R}^n$ and $T \in (0, T^*)$. We only need to prove uniform global attractivity. This follows from the following.

Claim 1 For any $\delta > 0$ there exists $L > 0$ and $k' \in [k_\circ, k_\circ + \ell_{L,T}]$ such that $|\phi_T^x(k')| \leq \delta$.

Claim 2 Inequality (9) implies that for each σ there exists $\delta > 0$ such that

$$|\phi_T^x(k')| \leq \delta \quad \Longrightarrow \quad |\phi_T^x(k)| \leq \sigma \qquad \forall k \geq k'\,.$$

So we see that Claims 1 and 2 imply that for any $r, \sigma > 0$ there exists $L > 0$ such that (5) holds.

Proof of Claim 1. Suppose the claim does not hold. Then, there exists $\delta > 0$ such that for all $L > 0$ and $k' \in [k_\circ, k_\circ + \ell_{L,T}]$ we have $|\phi_T^x(k')| > \delta$. In other words, for all $k \in \mathbb{Z}_{\geq k_\circ}$, $|\phi_T^x(k)| > \delta$.

Let $\gamma_m := \gamma(\delta)$. Let $\nu := \frac{\gamma_m}{2}$ generate $\beta_{r\nu}$ such that (8) holds and let $L := \frac{\beta_{r\nu}}{\nu} + T^*$. We obtain that $\beta_{r\nu} = (L - T^*)\nu \leq (L - T)\nu$. We also have that

$$|\phi_T^x(k)| > \delta\,, \quad k \geq k_\circ \qquad \Longrightarrow$$

$$\sum_{k=k_\circ}^{\infty} \gamma(|\phi_T^x(k)|) > \sum_{k=k_\circ}^{k_\circ+\ell_{L,T}} \gamma(|\phi_T^x(k)|) \geq \left(\sum_{k=k_\circ}^{k_\circ+\ell_{L,T}} \gamma_m\right) = 2\nu\ell_{L,T}\,.$$

On the other hand,

$$\sum_{k=k_\circ}^{k_\circ+\ell_{L,T}} \gamma(|\phi_T^x(k)|) = \sum_{k=k_\circ}^{k_\circ+\ell_{L,T}} [\gamma(|\phi_T^x(k)|) - \nu] + \sum_{k=k_\circ}^{k_\circ+\ell_{L,T}} \nu \leq \frac{\beta_{r\nu}}{T} + \ell_{L,T}\nu\,.$$

So from the above we conclude that

$$\begin{aligned} 2\ell_{L,T}\nu \leq \sum_{k=k_\circ}^{k_\circ+\ell_{L,T}} \gamma(|\phi_T^x(k)|) \quad &\leq \quad \frac{\beta_{r\nu}}{T} + \ell_{L,T}\nu \\ &\leq \quad \frac{\nu(L-T)}{T} + \ell_{L,T}\nu \\ &< 2\ell_{L,T}\nu \end{aligned}$$

which is a contradiction. △

Proof of Claim 2. Inequality (9) implies that, for all $k' \geq k_\circ$,

$$\max_{k \geq k'} |\phi_T^x(k)| \geq |\phi_T^x(k)| \qquad \forall k \geq k'\,.$$

Given $\sigma > 0$ define $\delta := \alpha^{-1}(\sigma)$. If $|\phi_T^x(k')| \leq \delta$ then it follows, again from inequality (9), that

$$\max_{k \geq k'} |\phi_T^x(k)| \leq \sigma$$

which implies that $|\phi_T^x(k)| \leq \sigma$ for all $k \geq k'$. △

This completes the proof of the Lemma. ■

B. Conditions for UGES

We show next that the conditions of Lemma 1 maybe strengthened to guarantee exponential stability. Roughly, we show that if the bound in (9) is assumed to hold with linear gain and the bound in (8) holds with $\nu = 0$ and $\gamma(s) = s^p$, exponential convergence follows.

Lemma 2 If for system (1) there exist: $p \geq 1$, $T^* > 0$, $\eta > 0$ and c such that for all $k \geq k_\circ$, all $x_\circ$ and all $T \in (0, T^*)$,

$$\max_{k \geq k_\circ} |\phi_T^x(k)| \quad \leq \quad \eta\, |x_\circ| \tag{10}$$

$$\left(T \sum_{k=k_\circ}^{\infty} |\phi_T^x(k)|^p \right)^{1/p} \quad \leq \quad c\, |x_\circ| \tag{11}$$

then,

$$|\phi_T^x(k)| \leq \kappa\, |x_\circ|\, \mathrm{e}^{-\lambda T(k-k_\circ)} \quad \forall\, k \geq k_\circ$$

where $\kappa := \max\{\varepsilon,\, \sigma_*\}$, $\varepsilon \geq \eta \exp\left(\dfrac{L}{c^p p}\right)$, $\lambda = 1/c^p p$ and

$$\sigma_* := \frac{\eta c}{(L-1)^{1/p}} \exp\left(\frac{L}{c^p}\right) < \infty\, .$$

□

Proof. For each $k \geq k_\circ$ define $w_k := \sum_{i=k}^{\infty} |\phi_T^x(i)|^p$. The bounds in (10), (11) imply respectively, that for any $k \geq k_\circ \geq 0$ we have

$$\max_{i \geq k} |\phi_T^x(i)|^p \quad \leq \quad \eta^p\, |\phi_T^x(k)|^p \tag{12}$$

$$w_k \quad \leq \quad \frac{c^p}{T}\, |\phi_T^x(k)|^p\, . \tag{13}$$

From (13) we have

$$w_{k+1} - w_k \leq -\frac{T w_k}{c^p} \qquad \forall\, k \geq 0 \tag{14}$$

hence w_k tends exponentially fast to zero (see e.g. [16]) moreover, defining $\lambda_w := 1/c^p$ we have

$$w_k \leq w_\circ \mathrm{e}^{-\lambda_w T(k-k_\circ)}\, , \quad w_\circ := w_{k_\circ}\, . \tag{15}$$

Next, we observe that for any integer $\Delta > 0$ we have

$$\Delta\, |\phi_T^x(k+\Delta)|^p \leq \sum_{\ell=k}^{k+\Delta} \max_{i \geq \ell} |\phi_T^x(i)|^p\, . \tag{16}$$

Let $L > 0$ in (2) be such that $L \geq c^p + 1$, hence $\ell_{L,T} \geq \dfrac{c^p}{T}$. Define $\Delta := \ell_{L,T}$ then, using (12) and (16) we obtain

$$\ell_{L,T}\, |\phi_T^x(k+\ell_{L,T})|^p \leq \eta^p \sum_{i=k}^{k+\ell_{L,T}} |\phi_T^x(i)|^p \leq \eta^p w_k\, . \tag{17}$$

Using (15) in the inequality above and then, (13) with $k = k_\circ$, we get that

$$\begin{aligned} |\phi_T^x(k+\ell_{L,T})|^p &\leq \frac{\eta^p c^p}{T \ell_{L,T}}\, |x_\circ|^p\ \mathrm{e}^{-\lambda_w T(k-k_\circ)} \\ &\leq \frac{\eta^p c^p}{L-1}\, |x_\circ|^p\ \mathrm{e}^{-\lambda_w T(k-k_\circ)}\, . \end{aligned} \tag{18}$$

Notice that the last inequality in (18) is equivalent to (this may be more clear by replacing k with $k' = k + \ell_{L,T}$)

$$|\phi_T^x(k)| \leq \frac{\eta c\, |x_\circ|}{(L-1)^{1/p}} \exp\left(\frac{\ell_{L,T} \lambda_w T}{p}\right) \exp\left(-\frac{\lambda_w T}{p}(k-k_\circ)\right)$$

for all $k \geq k_\circ + \ell_{L,T}$ and since $T \ell_{L,T} \lambda_w \leq L/c^p$ we have $|\phi_T^x(k)| \leq \sigma_*\, |x_\circ|\, \mathrm{e}^{-\lambda T(k-k_\circ)}$ for all $T \in (0, T^*)$. On the other hand, it follows, using (10) and the definition of ε that

$$|\phi_T^x(k)| \leq \varepsilon\, |x_\circ|\, \mathrm{e}^{-\lambda T(k-k_\circ)} \tag{19}$$

for all $k \in [k_\circ, k_\circ + \ell_{L,T}]$.

■

C. Corollaries for non-parameterized systems

Two corollaries follow for discrete-time systems with fixed sampling rate, *i.e.*

$$x_{k+1} = F(k, x_k)\, . \tag{20}$$

Corollary 1 If for system (20) there exist a function $\alpha \in \mathcal{K}_\infty$, a positive definite continuous function $\gamma(\cdot)$ and, for each r, $\nu > 0$ there exist $\beta_{r\nu} > 0$ such that (8) and (9) hold with fixed $T = T^* = 1$ and for all $k_\circ \geq 0$, $x_\circ \in B_r$ then the origin is UGAS. □

Corollary 2 If for system (20) there exist $p \geq 1$, $\eta > 0$ and $c > 0$ such that for all $k \geq k_\circ$ and all $x_\circ$, (11) holds with $T = T^* = 1$. Then, there exist κ and $\lambda > 0$ such that

$$|\phi^x(k)| \leq \kappa\, |x_\circ|\, \mathrm{e}^{-\lambda(k-k_\circ)} \quad \forall\, k \geq k_\circ\, .$$

□

The proofs follow by fixing, in the proofs of Lemma 1 and Lemma 2 $\ell_{L,T} = L$ and, for the former, $\beta_{r,\nu} = (L-1)\nu$. For Corollary 2 notice that w_k defines a Lyapunov function so the proof is direct.

IV. Applications to design of sampled-data control systems

We present two case-studies in controlled sampled-data systems. Based on [17], [16] the control approach consists in designing a discrete-time controller for an approximate discrete-time model of the plant. Stability of the sampled-data system follows, under appropriate conditions, from the stability of the approximate discrete-time system (cf. [17]). In the case-studies below we use our main results to show uniform asymptotic stability for the Euler approximate model in closed-loop.

A. Persistency of excitation revisited

The examples that we address are reminiscent of problems arising in adaptive control and identification. The main conditions are stated in terms of the so-called property of *persistency of excitation* (PE) which, for the locally integrable function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$, means that there exist μ and $\tau > 0$ such that

$$\int_t^{t+\tau} \psi(s)^2 ds \geq \mu \qquad \forall\, t \geq 0\, .$$

While persistency of excitation was introduced in the identification of *discrete-time systems* (cf. [1]), for the purposes of the present paper we reformulate the PE definition to the case of *parameterized* systems and introduce another property tailored for nonlinear systems. The latter is a discrete-time counter part of the so-called uniform δ persistency of excitation (Uδ-PE) *along trajectories* introduced in[2] [19]. See also [15] for a similar property in the discrete-time context.

[2]The definition of Uδ-PE has evolved from its original form introduced in [10] into different versions but without loosing its original meaning. Here we refer to the definition given in [19] even though in that reference the terminology "along trajectories" is not employed. See the more recent paper [9] for an account of different definitions.

Consider a parameterized discrete-time system $x_{k+1} = F_T(k, x_k)$ with solutions $\phi_T^x(k)$ and a function $\varphi : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$.

Definition 4 (Discrete-time Uδ-PE along trajectories) Let $T^* > 0$. The function φ is said to be uniformly δ-persistently exciting along the trajectories $\phi_T^x(k)$ if for each $\delta > 0$ there exist positive numbers μ and L such that for all $T \in (0, T^*)$ and all $j \in \mathbb{Z}_{\geq 0}$,

$$\min_{j \in [k, k+\ell_{L,T}]} |\phi_T^x(j)| \geq \delta \implies T \sum_{k=j}^{j+\ell_{L,T}} \varphi(k, \phi_T^x(k)) \geq \mu . \tag{21}$$

□

For the case when φ does not depend on the state, we use the following stronger property (cf. [13]).

Definition 5 (Discrete-time persistency of excitation) Let $\varphi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a function produced by sampling a function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ at rate T. The function φ is said to be persistently exciting (PE) if there exist positive numbers μ, L and T^* such that for all $T \in (0, T^*)$ and all $j \in \mathbb{Z}_{\geq 0}$,

$$T \sum_{k=j}^{j+\ell_{L,T}} \varphi_T(k) \geq \mu . \tag{22}$$

□

The following properties of PE functions are useful. Firstly, notice that the Cauchy-Schwartz inequality implies that

$$\frac{\mu}{T} \leq \sum_{k=j}^{j+\ell_{L,T}} \varphi_T(k) \leq \left(\sum_{k=j}^{j+\ell_{L,T}} \varphi_T(k)^2 \right)^{1/2} \ell_{L,T}^{1/2} .$$

Hence, defining $\mu' := \frac{\mu^2}{L}$, we have

$$\sum_{k=j}^{j+\ell_{L,T}} \varphi_T(k)^2 \geq \frac{\mu'}{T} .$$

Secondly, for any $i \geq 0$, define

$$I_{L,T} := \left\{ k \in [i,\, i + \ell_{L,T}] \,:\, \varphi_T(k)^2 \geq \frac{\mu'}{2T\ell_{L,T}} \right\} \tag{23}$$

and let card$(I_{L,T})$ denote the cardinal of $I_{L,T}$ so, in general card$(I_{L,T})$ is an integer which depends on T but in the specific way imposed by (2). In particular, for each fixed L, card$(I_{L,T})$ grows at most linearly as T decreases. On the other hand, card$(I_{L,T}) \geq$ card(I_{L,T^*}) for all $T \in (0, T^*)$. The following claim follows closely [7, Lemma 2].

Claim 3 Let $\phi_M > 0$ be such that $|\varphi_T(k)| \leq \phi_M$ for all $k \geq 0$ and all $T \in (0, T^*)$; let φ_T be PE according to Definition 5. Then, for each $T^* > 0$ there exists $\sigma > 0$ such that card$(I_{L,T}) > \sigma$ for all $T \in (0, T^*)$. Moreover an estimate for σ is

$$\sigma := \frac{\mu' \ell_{L,T^*}}{2\phi_M^2 L - \mu'} .$$

B. A nonlinear example

The first case-study concerns the stabilization of

$$\dot{x} = -p(t, x)u \tag{24}$$

where $p : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ is continuous and $x \mapsto p(t, x)$ is locally Lipschitz uniformly in t. We address the problem of stabilizing (24) under sampled feedback.

Assuming that $p(t, x)$ is uniformly δ persistently exciting (Uδ-PE) along the trajectories of (24) (cf. [19]) it can be shown, following the latter reference, that the feedback $u(t, x) := p(t, x)x$ renders the closed-loop system with (24) uniformly globally asymptotically stable. Leaving aside that $x \in \mathbb{R}$, the uniform asymptotic stability problem for the system (24) is a generalization of the problem studied in [12] and it has been solved, in the continuous time context, in [9]. Based on the framework of [16], [17], we propose a discrete-time controller for the Euler discretization of (24) such that the closed-loop system be uniformly globally asymptotically stable. To show the latter property we rely on Lemma 1.

Proposition 1 Consider the Euler discretization of (24), *i.e.*

$$x_{k+1} = x_k - Tp(k, x_k)u_k \tag{25}$$

and let $u_k := p(k, x_k)x_k$. Assume that p is uniformly bounded, that is, there exists $p_M > 0$ such that $|p(k, x)| \leq p_M$ for all $k \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$. Let $T^* > 0$ be such that $p_M^2 T^* < 2$. Then, the closed-loop system (25) is uniformly globally asymptotically stable. □

Proof. The closed-loop system is

$$x_{k+1} = x_k - T\varphi_T(k, x_k)x_k \tag{26}$$

where we defined $\varphi_T(k, x_k) = p(k, x_k)^2$ and, correspondingly, $\phi_M := p_M^2$. Consider the Lyapunov function $V(x) := x^2$ and define, along the trajectories of (25), $v_k := V(\phi_T^x(k))$. The difference equation for v_k, using (25) yields

$$\begin{aligned} v_{k+1} - v_k \leq & -2T\varphi_T(k, \phi_T^x(k))\phi_T^x(k)^2 \\ & + T^2\varphi_T(k, \phi_T^x(k))^2\phi_T^x(k)^2 \end{aligned}$$

which, defining $\alpha := 2 - \phi_M T^* > 0$, implies that there exists $\alpha > 0$ such that

$$v_{k+1} - v_k \leq -\alpha T\varphi_T(k, \phi_T^x(k))\phi_T^x(k)^2 \leq 0 \tag{27}$$

for all $k \geq k_\circ \geq 0$ and all $T \in (0, T^*)$. This implies that $v_k \leq v_{k_\circ}$ for all $k \geq k_\circ$ and, for any $L > 0$, $T \in (0, T^*)$, $k \geq k_\circ$ and all $j \in [k, k + \ell_{L,T}]$, $|\phi_T^x(k + \ell_{L,T})| \leq |\phi_T^x(j)|$. Hence, evaluating the sum on both sides of (27), from k to $k + \ell_{L,T}$ we obtain that

$$v_{k+\ell_{L,T}+1} - v_k \leq -\alpha\, T \sum_{j=k}^{k+\ell_{L,T}} \varphi_T(j, \phi_T^x(j))\phi_T^x(k+\ell_{L,T})^2 . \tag{28}$$

Fix $\nu > 0$ arbitrarily. Let $\delta := \sqrt{\nu}$ generate, via the assumption on discrete-time Uδ-PE, μ and $L > 0$ such that (21) holds. Define $k^* := \min\{k \geq k_\circ : |\phi_T^x(k)| \leq \delta\}$, $k^* = \infty$ if $|\phi_T^x(k)| > \delta$ for all $k \geq k_\circ$ and $k^* = 0$ if $|\phi_T^x(k_\circ)| \leq \delta$. Then, we have

$$\sum_{k=k_\circ}^{\infty} \sum_{j=k}^{k+\ell_{L,T}} T\varphi_T(j, \phi_T^x(j))\phi_T^x(j)^2 =$$

$$\sum_{k=k_\circ}^{k^*-\ell_{L,T}} \sum_{j=k}^{k+\ell_{L,T}} T\varphi_T(j, \phi_T^x(j))\phi_T^x(j)^2$$

$$+\sum_{k=k^*-\ell_{L,T}+1}^{\infty}\sum_{j=k}^{k+\ell_{L,T}} T\varphi_T(j,\phi_T^x(j))\phi_T^x(j)^2$$
$$\geq \sum_{k=k_\circ}^{\infty}\Big(\phi_T^x(k+\ell_{L,T})^2-\nu\Big)\mu$$
$$-\sum_{k=k^*-\ell_{L,T}+1}^{\infty}\Big(\phi_T^x(k+\ell_{L,T})^2-\nu\Big)\mu\,. \qquad (29)$$

On the other hand, using $v_k \le v_{k_\circ}$, we see that

$$\sum_{k=k_\circ}^{\infty}\big(v_{k+\ell_{L,T}+1}-v_k\big) \;=\; \sum_{k=k_\circ}^{k_\circ+\ell_{L,T}+1} v_k \;\le\; [\ell_{L,T}+1]v_{k_\circ} \qquad (30)$$

From (28), (29) and (30) we obtain that

$$\sum_{k=k_\circ}^{\infty}\Big(\phi_T^x(k+\ell_{L,T})^2-\nu\Big) \;\le\; \frac{1}{\mu}\left(\frac{\ell_{L,T}+1}{\alpha}+\ell_{L,T}\right)|x_\circ|^2 \qquad (31)$$

which, again by virtue of the fact that $\phi_T^x(k)^2 \le \phi_T^x(k_\circ)^2$, implies that

$$\sum_{k=k_\circ}^{\infty}\Big(\phi_T^x(k)^2-\nu\Big) \;\le\; \frac{1}{\mu}\left(\frac{\ell_{L,T}+1}{\alpha}+2\ell_{L,T}\right)|x_\circ|^2\,. \qquad (32)$$

Hence, defining $\gamma(s) := s^2$ and, for each $r>0$,

$$\beta_{r\nu} := \frac{L(1+2\alpha)+T^*}{\mu\alpha}r^2\,,$$

we see that (8) holds. The result follows invoking Lemma 1. ■

C. A linear example

The second example stems from Model Reference Adaptive Control (cf. [5], [4]). Consider the adaptive system in the input-output representation

$$\dot e = \varphi(t)^\top\theta + u \qquad (33)$$

where θ is a vector of unknown parameters and ψ is piecewise continuous. It is desired to stabilize (33) via a sampled adaptive feedback and an update adaptive law. To that end, we consider the Euler discretization of (33), with sampling rate T, *i.e.* defining $\varphi_k := \varphi_T(k)$,

$$e_{k+1} = e_k + T\varphi_k^\top\theta_k + Tu_k\,. \qquad (34)$$

with the certainty-equivalence feedback

$$u_k := -\varphi_k^\top\hat\theta_k - ae_k\,, \qquad a>0 \qquad (35)$$

and the "speed-gradient" adaptive law

$$\hat\theta_{k+1} = \theta_k - T\varphi_k x_k\,. \qquad (36)$$

The closed-loop system of (34), (35) and (36) yields

$$\begin{bmatrix} e_{k+1}\\ \theta_{k+1}\end{bmatrix} = \begin{bmatrix}(1-Ta) & T\varphi_k\\ -T\varphi_k & 1\end{bmatrix}\begin{bmatrix} e_k\\ \theta_k\end{bmatrix}, \quad z := \begin{bmatrix} e\\ \theta\end{bmatrix}. \qquad (37)$$

Proposition 2 There exist T^* sufficiently small and μ', L sufficiently large such that, if φ_T is PE with such T^*, μ' and L and $|\varphi_k-\varphi_{k-1}| \le T\phi_M$, the Euler approximate model (34) in closed loop with (34), (35) and (36), is UGES for all $T\in(0,T^*)$. □

In general, (see e.g. [11], [4] and, for the discrete-time case, [15]), the closed-loop system in MRAC is nonlinear and time-varying, even when the uncontrolled plant is linear. This is reflected in that, typically, φ_k is a function that depends both on k and x; see [15]. In the result that we present here, we consider that φ_T depends only on time, this can be justified by proceeding as in [18], [3] redefining the regressor function *along trajectories*. Consequently, the imposed PE condition must hold uniformly in the system trajectories; see [7] for a detailed discussion on this.

In [15] it has been investigated the stability of general MRAC under discrete-time delta persistency of excitation; the result from that reference is the counterpart to that presented in [8] (see also [19] where *U*GAS of *nonlinear* MRAC systems was shown for the first time). In this note, we contribute w.r.t. [15] by giving convergence rates for the linear 2-dimensional system.

Proof of Proposition 2. We proceed to show, via Lemma 2 that the origin of (37) is uniformly globally asymptotically stable. To that end, we first consider the function $V_T(k,z) := |z|^2 - \varepsilon\varphi_{k-1}e\theta$ with $\varepsilon := \alpha_\theta + T$ and $\alpha_\theta > 0$. Observe that this function is positive definite and radially unbounded for sufficiently small α_θ, T^* and w_M; indeed, we have

$$c_1|z|^2 \le V_T(k,z) \le c_2|z|^2 \qquad (38)$$

with $c_1 := (1-0.5(\alpha_\theta+T^*)\phi_M)$ and $c_2 := (1+0.5(\alpha_\theta+T^*)\phi_M)$ which are clearly independent of T. Note that $c_1>0$ for a proper choice of α_θ hence, In the sequel, we assume that this is the case.

Denoting the right hand side of equality (37) by $F_T(k,z)$ and evaluating the first difference equation of $V_T(k,z)$, we obtain that

$$\begin{aligned}\Delta V_T \;&:=\; V_T(k,F_T(k,z)) - V_T(k,z)\\ &=\; -T(2a-\varepsilon\varphi_k^2)e^2 - \varepsilon T\varphi_k^2\theta^2\\ &\quad + T^2 e\left([a^2+\varphi_k^2-\varepsilon a]e - [2a\varphi_k - \varepsilon\varphi_k^3]\theta\right)\\ &\quad + T^2\varphi_k^2\theta^2 - \varepsilon e\theta(\varphi_k-\varphi_{k-1}-\varphi_k aT)\,. \end{aligned} \qquad (39)$$

<u>Proof of UGB</u>: We first observe that under the assumptions of the proposition, $\varepsilon e\theta(\varphi_k-\varphi_{k-1}+\varphi_k aT) \le (1/2)[T^2\theta^2 + \varepsilon^2\phi_M^2(1+a)^2e^2]$. Define $\alpha_e := a - (\alpha_\theta+T^*)\phi_M^2 - (1/2)\varepsilon^2\phi_M^2(1+a)^2$ which is positive for sufficiently small values of ε and sufficiently large values of a. Let $\mathcal{O}(T^n)|z|^2$ upperbound the remaining terms of undefined or positive sign of order T^n with $n\ge 2$, in (39). Then,

$$\Delta V_T \le -T(\alpha_e e^2 + \alpha_\theta\varphi_k^2\theta^2) + \mathcal{O}(T^n)|z|^2\,. \qquad (40)$$

In particular, $\Delta V_T \le cV_T$ where $c \ge \mathcal{O}(T^n)/c_1$ for all $T\in(0,T^*)$. From [6, Proposition 5] we obtain that the system is uniformly forward complete, that is, there exist $\sigma_1,\sigma_2\in\mathcal{K}_\infty$, and $\sigma_3>0$ such that

$$|\phi_T^z(k)| \le \sigma_1(|z_\circ|) + \sigma_2(\,T(k-k_\circ)\,) + \sigma_3 \qquad (41)$$

for all $k\ge k_\circ\ge 0$.

Define $v_{k+1} := V_T(k, F_{1T}(k,\phi_T^z(k)))$ and $v_k := V_T(k,\phi_T^z(k))$. Let L, μ', $I_{L,T}$ be generated by the assumption of persistency of excitation of φ_k and let σ come from Claim

3. Define $\delta := \frac{\mu'}{2L}$, then,

$$\sum_{k=j}^{j+\ell_{L,T}} \Delta v_k \le -T\alpha_e \sum_{k=j}^{j+\ell_{L,T}} \left|\phi_T^e(k)\right|^2 - T\alpha_\theta \sum_{I_{L,T}} \left|\varphi_k \phi_T^\theta(k)\right|^2 + \mathcal{O}(T^n) \sum_{k=j}^{j+\ell_{L,T}} \left|\phi_T^z(k)\right|^2 . \quad (42)$$

Let $\ell_{L,T}$ be sufficiently large so that there exist $k_1^*, k_2^* \in I_{L,T}$ such that $|\phi_T^e(k_1^*)| > 0$ and $\left|\phi_T^\theta(k_2^*)\right| > 0$. Then, using the fact that $\delta \le \frac{\mu'}{2T\ell_{L,T}}$, we obtain that

$$v_{\ell_{L,T}+j+1} - v_j \le -T(\alpha_e \left|\phi_T^e(k_1^*)\right|^2 + \alpha_\theta \delta\sigma \left|\phi_T^\theta(k_2^*)\right|^2) + \ell_{L,T}\mathcal{O}(T^n) \max_{k\in[j,\, j+\ell_{L,T}]} \left|\phi_T^z(k)\right|^2 . \quad (43)$$

Notice that α_e, α_θ, δ and σ are independent of T.

We continue with the proof of UGB. To show contradiction, assume that the solutions grow unboundedly. Using (41) we obtain that $|\phi_T^z(k)| \le \gamma(|z_\circ|)$ with $\gamma(s) := \sigma_1(s) + \sigma_2(T^*\ell_{L,T}) + \sigma_3$ for all $k \in [j, j+\ell_{L,T}]$ and any $j \ge k_\circ$. Defining $k^* := \max\{k_1^*, k_2^*\}$ and $\alpha := \min\{\alpha_e,\, \alpha_\theta\delta\sigma\}$ it follows that

$$v_{\ell_{L,T}+j+1} - v_j \le -T\alpha \left|\phi_T^z(k^*)\right|^2 + \ell_{L,T}\mathcal{O}(T^n)\gamma(|z_\circ|)^2 . \quad (44)$$

From this and the fact that the solutions grow unboundedly as $j \to \infty$, there exist k^*, j sufficiently large such that

$$v_{\ell_{L,T}+j+1} - v_j \le -T\frac{\alpha}{2} \left|\phi_T^z(k^*)\right|^2 \le 0$$

which implies that v_j and therefore $|\phi_T^z(j)|$, cannot grow to infinity.

Proof of stability: ULS follows from (40) by restricting the set of initial conditions. This and UGB imply that the system is uniformly globally stable. Moreover, in view of (38) there exists $\eta > 0$ such that $\max_{k\ge k_\circ} |\phi_T^z(k)| \le \eta\, |z_{k_\circ}|$.

Proof of uniform convergence: Since the system is UGS and $\max_{k\ge k_\circ} |\phi_T^z(k)| \le c\,|z_{k_\circ}|$ we may reconsider (43) with $k_1^* = k_2^* = j + \ell_{L,T}$ to write

$$\begin{aligned} v_{\ell_{L,T}+j+1} - v_j \quad &\le \quad -T\Big(\alpha_e \left|\phi_T^e(j+\ell_{L,T})\right|^2 \\ &\quad + \alpha_\theta\delta\sigma \left|\phi_T^\theta(j+\ell_{L,T})\right|^2\Big) + \\ &\quad \ell_{L,T}\mathcal{O}(T^n) \left|\phi_T^z(j+\ell_{L,T})\right|^2 . \end{aligned}$$

It follows that for sufficiently large α_e and σ (hence for large a, $\ell_{L,T}$ and μ') and sufficiently small T^*, there exists $a > 0$, independent of T such that for all $T \in (0, T^*)$, and all $j \ge 0$,

$$v_{\ell_{L,T}+j+1} - v_j \le -Ta \left|\phi_T^z(j+\ell_{L,T})\right|^2 .$$

The latter implies that

$$\begin{aligned} \sum_{k=k_\circ}^{\infty} \left|\phi_T^z(k)\right|^2 &\le \frac{c_2}{Ta} \left|z_\circ\right|^2 + \sum_{k=k_\circ}^{k_\circ+\ell_{L,T}} \left|\phi_T^z(k)\right|^2 \\ &\le \left[\frac{c_2}{Ta} + \ell_{L,T}c\right] \left|z_\circ\right|^2 \qquad (45) \end{aligned}$$

The result follows from the UGS property with linear gain, (45) and Lemma 2 with $p = 2$ and $c = \frac{1}{\sqrt{T}}\left[\frac{c_2}{a} + \eta L\right]^{1/2}$. ■

V. Conclusions

We have presented sufficient conditions for uniform global asymptotic and exponential stability for parameterized time-varying discrete-time systems. Our conditions are analogous to integral conditions for continuous-time systems and may be regarded as conditions for series convergence. We demonstrated the utility of our main results in the analysis of systems appearing in adaptive control and identification; in particular, the Euler discretization of a system appearing in model reference adaptive control.

References

[1] K. J. Åström and Bohn. Numerical identification of linear dynamic systems from normal operating records. In P. H. Hammond, editor, Proc. of the 2nd IFAC Symp. on Theory of Self-adaptive Control Systems, pages 96–111, Nat. Phys. Lab., Teddington, England, 1965.

[2] A. L. Fradkov. *Adaptive control in complex systems*. Phys. Math. Lit. Nauka, Moscow, 1990. (In Russian).

[3] M. Janković. Adaptive output feedback control of nonlinear feedback linearizable systems. *Int. J. Adapt. Contr. Sign. Process.*, 10(1):1–18, 1996.

[4] H. Khalil. *Nonlinear systems*. Macmillan Publishing Co., 2nd ed., New York, 1996.

[5] Y. D. Landau. *Adaptive Control: the model reference approach*, volume 8 of *Control and systems theory*. Dekker, 1979.

[6] A. Loría and D. Nešić . On uniform boundedness of parameterized discrete-time systems with decaying inputs: applications to cascades. *Syst. & Contr. Letters*, 49(3):163–174, 2003.

[7] A. Loría and E. Panteley. Uniform exponential stability of linear time-varying systems:revisited. *Syst. & Contr. Letters*, 47(1):13–24, 2002.

[8] A. Loría E. Panteley, D. Popović, and A. Teel. δ-persistency of excitation: a necessary and sufficient condition for uniform attractivity. In *Proc. 41st. IEEE Conf. Decision Contr.*, Las Vegas, CA, USA, 2002. Paper no. REG0623 .

[9] A. Loría E. Panteley, D. Popović, and A. Teel. A nested Matrosov theorem and persistency of excitation for uniform convergence in stable non-autonomous systems. *IEEE Trans. on Automat. Contr.*, 50(2):183–198, 2005.

[10] A. Loría E. Panteley, and A. Teel. A new persistency-of-excitation condition for UGAS of NLTV systems: Application to stabilization of nonholonomic systems. In *Proc. 5th. European Contr. Conf.*, 1999. Paper no. 500.

[11] R. Marino and P. Tomei. Global adaptive output feedback control of nonlinear systems. Part I : Linear parameterization. *IEEE Trans. on Automat. Contr.*, 38:17–32, 1993.

[12] A. P. Morgan and K. S. Narendra. On the stability of nonautonomous differential equations $\dot{x} = [A+B(t)]x$ with skew-symmetric matrix $B(t)$. *SIAM J. on Contr. and Opt.*, 15(1):163–176, 1977.

[13] D. Nešić and A. Loría. On uniform asymptotic stability of time-varying parameterized discrete-time cascades. *IEEE Trans. on Automat. Contr.*, 49(6):875–887, 2004.

[14] D. Nešić and A. Teel. A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Trans. on Automat. Contr.*, 49(7):1103–1123, 2004.

[15] D. Nešić and A. Teel. Matrosov theorem for parameterized families of discrete-time systems. *Automatica*, 40(6):1025–1034, 2004.

[16] D. Nešić , A. Teel, and P. Kokotović. Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations. *Syst. & Contr. Letters*, 38:259–270, 1999.

[17] D. Nešić , A. Teel, and E. Sontag. Formulas relating $\mathcal{KL}$ stability estimates of discrete-time and sampled-data nonlinear systems. *Syst. & Contr. Letters*, 38:49–60, 1999.

[18] R. Ortega and A. L. Fradkov. Asymptotic stability of a class of adaptive systems. *Int. J. Adapt. Contr. Sign. Process.*, 7:255–260, 1993.

[19] E. Panteley, A. Loría and A. Teel. Relaxed persistency of excitation for uniform asymptotic stability. *IEEE Trans. on Automat. Contr.*, 46(12):1874–1886, 2001.