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Perimeter Estimates for Attainable Sets in Control Theory

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***Abstract*— The reachable set at time $T > 0$ from a given closed set $\mathcal{K} \subset \mathbb{R}^n$, $\mathcal{A}(T;\mathcal{K})$, is a well known object in control theory. Here such a set is investigated for the symmetric system**

$$\dot{x}(t) = f(x(t))u(t)\,, \quad u(t) \in \overline{B}\,.$$

A recent result obtained by the first author in collaboration with Frankowska guarantees that, under suitable assumptions, $\mathcal{A}(\mathcal{K};T)$ satisfies a uniform interior sphere condition for $T > 0$. Using such a property we show that, for $f(x)$ smooth and nondegenerate, $\mathcal{A}(\mathcal{K};T)$ has finite perimeter, and we obtain sharp estimates for the time-dependence of the perimeter and volume of such a set.

I. INTRODUCTION

In this paper we investigate regularity properties of the reachable set of control systems of the form

$$x'(t) = f(x(t))u(t) \qquad u(t) \in \overline{B}\,, \tag{1}$$

where $\overline{B}$ denotes the closed unit ball of $\mathbb{R}^N$ centered at 0. We shall assume that $f : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is sufficiently smooth, and $f(x)$ is invertible for any $x \in \mathbb{R}^N$.

Given a nonempty compact subset $\mathcal{I}$ of $\mathbb{R}^N$ and a time $t \geq 0$, we will study the reachable set from $\mathcal{I}$ at time t, $\mathcal{R}(t)$, which is defined as the set of all points $x \in \mathbb{R}^N$ such that there exists a solution $x(\cdot)$ of (1) with $x(0) \in \mathcal{I}$ and $x(t) = x$.

Our main objective is to show that, for any $t > 0$, $\mathcal{R}(t)$ is a set of finite perimeter (in the sense of De Giorgi), and that its perimeter can only increase with time in a controlled way. More precisely, we will show that, for any $T > 0$, there are positive constants c_1 and c_2 such that, for every $0 < t_1 \leq t_2 \leq T$,

$$\int_{\partial^*\mathcal{R}(t_2)} |f^*(x)\nu_{\mathcal{R}(t_2)}(x)|\, d\mathcal{H}^{N-1}(x) \leq \left(\frac{t_2}{t_1}\right)^{c_2} e^{c_1(t_2-t_1)} \int_{\partial^*\mathcal{R}(t_1)} |f^*(x)\nu_{\mathcal{R}(t_1)}(x)|\, d\mathcal{H}^{N-1}(x) \tag{2}$$

where $\partial^*\mathcal{R}(t)$ denotes the reduced boundary of $\mathcal{R}(t)$, $\mathcal{H}^{N-1}$ is the standard $(N-1)-$dimensional Hausdorff measure, and $\nu_{\mathcal{R}(t)}(x)$ is the measure theoretic outward unit normal of $\mathcal{R}(t)$ at $x \in \partial^*\mathcal{R}(t)$. Consequently, we will obtain that the volume of $\mathcal{R}(t)$, denoted by $|\mathcal{R}(t)|$, is a locally Lipschitz continuous function of t on $(0, +\infty)$ and, for a.e. $t > 0$,

$$\frac{d}{dt}|\mathcal{R}(t)| = \int_{\partial^*\mathcal{R}(t)} \left|f^*(x)\nu_{\mathcal{R}(t)}(x)\right| d\mathcal{H}^{N-1}(x)\,. \tag{3}$$

This work is strongly motivated by [1], where estimates on volume and perimeter of the reachable sets for control systems of the form (1), with $f(x) = c(x)I_n$ (where $c : \mathbb{R}^N \to \mathbb{R}$), are crucial in order to prove existence and uniqueness of some first order front propagation problems. The estimates given here are much sharper than those of [1], and we think these improvement should be of some help for other problems.

Let us briefly explain our method of proof. The starting point is a joint paper [5] by Frankowska and the first author where, under more general stucture conditions on the control system, $\mathcal{R}(t)$ is shown to satisfy a uniform interior sphere condition with a radius proportional to t. This property entails that reachables sets are of finite perimeter in the sense of De Giorgi (see [1] or [8]). Moreover, the reduced boundary of $\mathcal{R}(t)$ can be characterized as follows: $\partial^*\mathcal{R}(t)$ coincides ($\mathcal{H}^{N-1}$ a.e.) with the set of points $x \in \partial\mathcal{R}(t)$ such that the contingent cone $T_{\mathcal{R}(t)}(x)$ of $\mathcal{R}(t)$ at x is exactly a half-space.

Another crucial notion for the proof is that of extremal solution (or boundary trajectory). Let us recall that an extremal solution $x(\cdot)$ of (1) on the time interval $[0, T]$ is a solution which remains on boundary of $\mathcal{R}(t)$ for every $t \in [0, T]$. Following Theorem 3.8 of [7], we note that, if $x(\cdot)$ is an extremal solution on $[0, T]$, then the contingent cone of $\mathcal{R}(t)$ at $x(t)$ is a half-space for all $t \in (0, T)$. Thus, loosely speaking, extremal solutions remain in the reduced boundary of $\mathcal{R}(t)$.

The key remark of the paper is an estimate that bounds the distance between points reached by two extremal solutions at the same time: if $x_1(\cdot)$ and $x_2(\cdot)$ are extremal solutions, then we show that

$$|f^{-1}(x_1(t_2))(x_1(t_2) - x_2(t_2))| \leq \left(\frac{t_2}{t_1}\right)^{C_2} e^{C_1(t_2-t_1)} |f^{-1}(x_1(t_1))(x_1(t_1) - x_2(t_1))|$$

for any t_1, t_2 with $0 < t_1 < t_2 \leq T$. Note that the above estimate measures the difference $x_1(t_2) - x_2(t_2)$ in terms of $x_1(t_1) - x_2(t_1)$, with respect to a position-depending norm. The reason why such an inequality is true is the fact that reachable sets have the interior sphere property, and so outward normals at different points cannot point at "too opposite directions". Then, the proof of (2) follows introducing a Hausdorff-like measure, $\mathcal{H}_f^{N-1}$, that reduces to the above expression for sets of finite perimeter.

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The paper is organized in the following way. In section 1, we recall basic facts concerning the sets of finite perimeter and extremal solutions of controlled system (1). Section 2 is devoted to the study of sets satisfying a uniform interior sphere condition. In section 3 we define the measure $\mathcal{H}_f^{N-1}$ and give an equivalent expression for such a measure. Finally, in section 4 we give our perimeter estimate.

We complete this introduction by stating the assumption we need throughout this paper:

$$\begin{array}{lll} (a) & f : \mathbb{R}^N \to \mathbb{R}^{N\times N} \quad \text{is of class } \mathcal{C}^2\,; & \\ (b) & f(x) \quad \text{is invertible for any } x \in \mathbb{R}^N\,; & (4)\\ (c) & f \quad \text{and} \quad f^{-1} \text{ are globally bounded.} & \end{array}$$

II. INNER BALL PROPERTY

A. Notation

We start this section by collecting standard notations used throughout the paper. We denote by $\langle\cdot,\cdot\rangle$ and by $|\cdot|$ the scalar product and the euclidean norm of $\mathbb{R}^N$, by $B(x,r)$ the closed ball of radius r centered at the point x. Recall that $\overline{B} = B(0,1)$. If K is a subset of $\mathbb{R}^N$, $d_K(x)$ denotes the distance of the point x to the set K: $d_K(x) = \inf_{y\in K}|y-x|$. For $r>0$, we denote by $K+rB$ the set of points $x\in\mathbb{R}^N$ such that $d_K(x)\le r$.

For a vector $v\in\mathbb{R}^N$ we denote by $(v)^-$ the negative polar cone of v, i.e., the set of vectors $w\in\mathbb{R}^N$ such that $\langle v,w\rangle\le 0$.

Finally, if K is a subset of $\mathbb{R}^N$, $\mathbf{1}_K$ is the function equal to 1 in K and to 0 outside.

We now recall some well-known results on functions with bounded variation. For a general presentation and proofs, see for instance [2], [9]. A function $u\in L^1(\mathbb{R}^N,\mathbb{R})$ whose gradient Du in the sense of distribution is a vector-valued Radon measure with finite total variation is called a function of bounded variation. The total variation of Du in an open set U, denoted by $\|Du\|(U)$, is given by

$$\sup\left\{\int u\,\mathrm{div}\,\phi dx \mid \phi\in\mathcal{C}_c^1(U),\ |\phi|\le 1\right\}\,.$$

If B is a Borel set, then the total variation of u on B is defined as:

$$\inf\{\|Du\|(U) \mid U \text{ open and } B\subset U\}\,.$$

A measurable set $E\subset\mathbb{R}^N$ is said to be of finite perimeter if the function $\mathbf{1}_E$ has bounded variation. The perimeter of E in a Borel set B is then given by $P(E,B):=\|D\mathbf{1}_E\|(B)$. For sets of finite perimeter, one can define the essential boundary ∂^*E of E, which is countably $(N-1)$ rectifiable with finite $\mathcal{H}^{N-1}$ measure. The outer unit normal $\nu_E(x)$ is then defined at for all points x of ∂^*E. If we set $E_t=\{y\in\mathbb{R}^N \mid x+t(y-x)\in E\}$, then the function $\mathbf{1}_{E_t}$ converges to $\mathbf{1}_{H^-(x)}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t\to 0^+$, where $H^-(x)=\{y\in\mathbb{R}^N \mid \langle\nu_E(x),y-x\rangle\le 0\}$. Moreover, the measure $P(E,\cdot)$ coincides with the restriction of $\mathcal{H}^{N-1}$ to ∂^*E. The measure theoretic boundary of E, denoted by ∂_*E, is the set of points $x\in\mathbb{R}^N$ such that

$$\limsup_{t\to 0^+}\frac{|B(x,t)\cap E|}{t^N}>0 \quad\text{and}\quad \limsup_{t\to 0^+}\frac{|B(x,t)\backslash E|}{t^N}>0\,,$$

where $|A|$ denotes the Lebesgue measure of a set A. It is known that

$$\partial^*E\subset\partial_*E \quad\text{and}\quad \mathcal{H}^{N-1}(\partial_*E\backslash\partial^*E)=0\,.$$

Next, we go back to system (1) and reachable sets. If $T>0$ and $x\in\partial\mathcal{R}(T)$, then it is well-known that there is a solution $x(\cdot)$ to (1) such that

$$x(0)\in\mathcal{I},\ x(T)=x \text{ and } \qquad x(s)\in\partial\mathcal{R}(s)\,,\ \forall s\in[0,T]\,.$$

Such a solution is called an extremal (or a boundary) solution on the time interval $[0,T]$. Let us now introduce the Hamiltonian of the problem, i.e.,

$$H(x,p)=\sup_{u\in\overline{B}}\langle f(x)u,p\rangle=|f^*(x)p|\,,$$

and the polar of H given by

$$H^0(x,q)=|f^{-1}(x)q|\,. \tag{5}$$

We note for later use that

$$\frac{\partial H}{\partial p}(x,p)=\frac{f(x)f^*(x)p}{|f^*(x)p|} \qquad \forall (x,p)\in\mathbb{R}^N\times\mathbb{R}^N_*\,.$$

B. Pontryagin Maximum Principle

The Pontryagin maximum principle states that, if $x(\cdot)$ is an extremal solution on the time interval $[0,T]$, then there is an absolutely continuous map $p:[0,T]\to\mathbb{R}^N\backslash\{0\}$ such that

$$\left\{\begin{array}{l} x'=\dfrac{\partial H}{\partial p}(x,p)\\[2ex] p'=-\dfrac{\partial H}{\partial x}(x,p)\end{array}\right. \tag{6}$$

The map $p(\cdot)$ is called the adjoint state of the extremal solution $x(\cdot)$.

C. Inner ball property of reachable sets

We say that a closed set $K\subset\mathbb{R}^N$ has the inner ball property of radius $r>0$ if

$$\forall x\in\partial K,\ \exists p\in\mathbb{R}^N,\ |p|=1, \text{ such that } B(x-rp,r)\subset K\,.$$

Such a class of sets (or, more precisely, their complements) was introduced as sets of positive reach by Federer [10].

Reachable sets have the inner ball property. Moreover, Frankowska and the first author proved that, under assumption (4), for any $T>0$, there is a constant c_T such that $\mathcal{R}(t)$ has the inner ball property of radius c_Tt for any $t\in(0,T]$.

More precisely, if $x(\cdot)$ is an extremal solution on some time interval $[0,T]$ (with $T>0$), and if we denote by $p(\cdot)$ its adjoint, then

$$B\left(x(t)-c_Tt\frac{p(t)}{|p(t)|},c_Tt\right)\subset\mathcal{R}(t) \qquad \forall t\in(0,T]\,.$$

D. Consequences of the inner ball property

Let us denote by $T_K(x)$ the contingent cone to K at $x \in \partial K$, i.e., the set of vectors $v \in \mathbb{R}^N$ for which there exist $h_n \to 0^+$, $v_n \to v$ with $x + h_n v_n \in K$ (see [3]).

It is a fact that, if a set $K \subset \mathbb{R}^N$ has the inner ball property of radius $r > 0$, then there is some closed subset K_0 of K such that $K = K_0 + rB$.

We give the proof for the reader's convenience. Let us set $K_0 = \{x \in K \mid d_{\partial K}(x) \geq r\}$. Then $K_0 + rB \subset K$, from the definition of K_0. To obtain the converse inclusion, let $x \in K$. If $d_{\partial K}(x) \geq r$, then $x \in K_0 \subset K_0 + rB$. If $x \in \partial K$, then, from the interior ball property, there is some $p \in \mathbb{R}^N$, with $|p| = 1$, such that $B(x - rp, r) \subset K$. Hence $d_{\partial K}(x - rp) = r$, which proves that $x - rp \in K_0$ and therefore that $x \in K_0 + rp$.

Let us finally assume that $x \notin \partial K$ and $d_{\partial K}(x) < r$. Let y belong to the projection of x onto ∂K. Let us set $\rho = d_{\partial K}(x)$. Recall that, from [10], $d_{\partial K}$ is $\mathcal{C}^{1,1}$ in $\mathcal{O} := \{0 < d_{\partial K} < r\}$. Let $x(\cdot)$ be the solution to $x'(t) = Dd_{\partial K}(x(t))$. Then $x(\cdot)$ is well defined for $t \geq 0$ as long as $d_{\partial K}(x(t)) < r$. Since $|Dd_{\partial K}| = 1$ in $\mathcal{O}$, we have

$$\frac{d}{dt} d_{\partial K}(x(t)) = 1 ,$$

whence $d_{\partial K}(x(t)) = d_{\partial K}(x) + t$ on $t \in [0, r - \rho]$. Futhermore, $x(\cdot)$ being $1-$Lipschitz continuous and $d_{\partial K}$ being also $1-$Lipschitz continuous, $t \to x(t)$ is necessarily affine on $[0, r - \rho]$, and therefore $Dd_{\partial K}(x(t))$ is constant. Noting finally that $Dd_{\partial K}(x) = (x-y)/|x-y|$, we have proved that the point $z = x+(r-\rho)(x-y)/|x-y|$ satisfies $d_{\partial K}(z) = r$. In particular, $z \in K_0$ and $x \in B(z, r) \subset K_0 + rB$.

E. Perimeters of sets with the inner ball property

Let $K \subset \mathbb{R}^N$ be a set with the interior ball property of radius $r > 0$, K_0 be such that $K = K_0 + rB$, and denote by Π the projection onto K_0. Then,

$$T_K(x) = \bigcup_{y \in \Pi(x)} (x - y)^- \qquad \forall x \in \partial K .$$

Moreover, if a closed set K has the interior ball property, then K is a set of finite perimeter and the following inclusions holds

$$\partial^* K \subset \{x \in \partial K \mid T_K(x) \text{ is a half-space}\} \subset \partial_* K , \quad (7)$$

where $\partial^* K$ is the reduced boundary of K and $\partial_* K$ its measure theoretic boundary. Furthermore, if $x \in \partial^* K$, then $T_K(x) = (\nu_K(x))^-$.

The fact that K has a finite perimeter is proved in [1] or [8]. We reproduce here the proof of [8] for the reader's convenience. Since K has the interior ball property, it is well-known that the distance function $d_{\partial K}$ is $\mathcal{C}^{1,1}$ in the set $\mathcal{O} = \{x \in Int(K) \mid d_{\partial K}(x) < 2\epsilon\}$, for some $\epsilon > 0$. Let $K_\epsilon = \{x \in K \mid d_{\partial K}(x) \geq \epsilon\}$. Then the measure $\mathcal{H}^{N-1}(\partial K_\epsilon)$ is finite, because K_ϵ has a $\mathcal{C}^{1,1}$ boundary and is bounded. Therefore $\mathcal{H}^{N-1}(\partial K)$ is also finite, because it is the image of ∂K_ϵ by the Lipschitz map $x \to x - \epsilon Dd_{\partial K}(x)$. This completes the proof thanks to Theorem 1, p. 222 of [9].

Next, we proceed to show the two inclusions in (7). Let

$$x \in S := \{x \in \partial K \mid T_K(x) \text{ is a half-space}\} .$$

Let us show that x belongs to the measure theoretic boundary $\partial_* K$ of K, i.e.,

$$\limsup_{t \to 0^+} \frac{|B(x,t) \cap K|}{t^N} > 0 \quad \text{and} \quad \limsup_{t \to 0^+} \frac{|B(x,t) \backslash K|}{t^N} > 0 .$$

The first assertion is obvious, because, from the interior ball condition, there is some $p \in \mathbb{R}^N$, $|p| = 1$, with $B(x - rp, r) \subset K$ and so

$$\limsup_{t \to 0^+} \frac{|B(x,t) \cap K|}{t^N} \geq \limsup_{t \to 0^+} \frac{|B(x,t) \cap B(x - rp, r)|}{t^N} = \frac{1}{2}\omega_N .$$

As for the second, since $T_K(x)$ is a half-space, there is some $v \in \mathbb{R}^N \backslash T_K(x)$. From the definition of $T_K(x)$, there is some $\epsilon > 0$ such that the troncated cone $x + (0, \epsilon)(v + \epsilon B)$ does not intersect K. Then

$$\limsup_{t \to 0^+} \frac{|B(x,t) \backslash K|}{t^N} \geq \limsup_{t \to 0^+} \frac{|B(x,t) \cap [x + (0,\epsilon)(v + \epsilon B)]\,|}{t^N} > 0 .$$

So, x belongs to the measure theoretic boundary of K.

Let now $x \in \partial^* K$. Let us set $K_t = \{y \in \mathbb{R}^N \mid x + t(y - x) \in K\}$. As recalled in the previous section, the function $\mathbf{1}_{K_t}$ converges to $\mathbf{1}_{H^-(x)}$ in $L^1_{\text{loc}}(\mathbb{R}^N)$, where $H^-(x) = \{y \in \mathbb{R}^N \mid \langle \nu_K(x), (y - x) \rangle \leq 0\}$. We claim that $\Pi(x)$ reduces to the singleton $\{x - r\nu_K(x)\}$. For let $y \in \Pi(x)$. Then, $B(y, r) \subset K$, and so, after scaling, $B(x + (y - x)/t, r/t) \subset K_t$. Therefore,

$$\lim_{t \to 0+} \mathbf{1}_{K_t}(z) = 1 \quad \text{for any } z \text{ with } \langle z - x, x - y \rangle < 0 .$$

Since $\lim_{t \to 0+} \mathbf{1}_{K_t}(z) = \mathbf{1}_{H^-(x)}$, this proves that $(x - y)/r = \nu_K(x)$, i.e., $y = x - r\nu_K(x)$. In particular, x has a unique projection onto K_0. Then, $T_K(x)$ is a half-space and that $T_K(x) = (x - y)^- = (\nu_K(x))^-$.

F. A technical inequality

Next, we need a technical inequality linking two points x_1 and x_2 of the boundary of a set K, at which this set has an interior ball, and the two normals to this set at these points.

Let K be a closed subset of $\mathbb{R}^N$, $r > 0$, and fix $x_1, x_2 \in \partial K$ such that there are $p_1, p_2 \in \mathbb{R}^N$ with $|p_1| = |p_2| = 1$ and $B(x_i - rp_i, r) \subset K$ for $i = 1, 2$. Then, for any invertible matrix A, we have

$$\langle x_1 - x_2, \frac{p_1}{|A^{-*}p_1|} - \frac{p_2}{|A^{-*}p_2|} \rangle \leq \frac{1}{r\lambda} |A(x_1 - x_2)|^2 ,$$

where $A^{-*} = (A^{-1})^*$ and where λ is the smallest singular value of A.

To prove the above result, let $K' = AK$, $x'_i = Ax_i$, $q_i = A^{-*}p_i/|A^{-*}p_i|$ for $i = 1, 2$. We claim that

$$A^{-1} B(x'_i - r' q_i, r) \subset B(x_i - rp_i, r) , \quad (8)$$

where $r' = \lambda r$. Indeed, let $z \in B(x_i' - r'q_i, r)$. Then $|z - x_i' + r'q_i|^2 \leq (r')^2$, which is equivalent to saying that $|z - x_i'|^2 + 2r'\langle z - x_i', q_i\rangle \leq 0$. Therefore

$$|A(A^{-1}z - x_i)|^2 + 2r'\langle A^{-1}z - x_i, p_i\rangle \leq 0 \,. \qquad (9)$$

Let us now check that this inequality implies that $A^{-1}z$ belongs to the ball $B(x_i - rp_i, r)$, i.e., $|A^{-1}z - x_i|^2 + 2r\langle A^{-1}z - x_i, p_i\rangle \leq 0$. Since λ is the smallest singular value of A, we have

$$\begin{aligned} & |A^{-1}z - x_i|^2 + 2r\langle A^{-1}z - x_i, p_i\rangle \\ \leq\ & \frac{1}{\lambda}|A(A^{-1}z - x_i)|^2 + 2r\langle A^{-1}z - x_i, p_i\rangle \\ \leq\ & \frac{1}{\lambda}\left(|A(A^{-1}z - x_i)|^2 + 2r'\langle A^{-1}z - x_i, p_i\rangle\right) \ \leq\ 0 \,, \end{aligned}$$

thanks to (9). So our claim (8) is proved.

Since $B(x_i - rp_i, r) \subset K$ for $i = 1, 2$, (8) implies that $B(x_i' - r'q_i, r') \subset K'$. We now mimic the proof of Lemma 2.1 of [1] to show that

$$\langle x_1' - x_2', q_1 - q_2\rangle \leq \frac{1}{r\lambda}|x_1' - x_2'|^2 \,. \qquad (10)$$

Since x_2' does not belong to the open ball centered at $x_1' - r'q_1$ and of radius r', we have

$$|x_2' - (x_1' - r'q_1)|^2 \geq (r')^2 \,,$$

whence

$$|x_2' - x_1'|^2 + 2r'\langle q_1, x_2' - x_1'\rangle \geq 0 \,.$$

In the same way, since x_1' does not belong to the open ball centered at $x_2' - r'q_2'$ and of radius r', we have $|x_2' - x_1'|^2 + 2r'\langle q_2, x_1' - x_2'\rangle \geq 0$. Putting the two inequalities together gives (10).

Using inequality (10) and writing explicitly what are x_i' and q_i we obtain the desired result.

III. HAUSDORFF MEASURES

A. On some Hausdorff measure

We now introduce the Hausdorff measure $\mathcal{H}_f^{N-1}$ adapted to our framework. For any set $E \subset \mathbb{R}^N$ and $\delta > 0$, we set

$$\mathcal{H}_{f,\delta}^{N-1}(E) = \inf\left\{\alpha_{N-1}(\partial B)\sum_{i=1}^{\infty}\left(\frac{\operatorname{diam}_f(K_i)}{2}\right)^{N-1}\right\}$$

where α_{N-1} is the volume of the unit ball of $\mathbb{R}^{N-1}$, and where the infimum is taken over the families (K_i) of compact subsets of $\mathbb{R}^N$ such that

$$E \subset \bigcup_{i=1}^{\infty} K_i \quad \text{and} \quad \operatorname{diam}_f(K_i) \leq \delta \,,$$

with, for any set K,

$$\operatorname{diam}_f(K) = \sup_{x,y\in K} |\det(f(x))|^{\frac{1}{N-1}}\left|f^{-1}(x)(x-y)\right| \,.$$

Then, we set

$$\mathcal{H}_f^{N-1}(E) = \lim_{\delta\to 0^+} \mathcal{H}_{f,\delta}^{N-1}(E) \,.$$

We note that $\mathcal{H}_f^{N-1}$ can be easily estimated by the usual Hausdorff measure: under assumption (4) on f, there is a constant $C > 1$ such that

$$\frac{1}{C}\,\mathcal{H}^{N-1}(E) \ \leq\ \mathcal{H}_f^{N-1}(E) \ \leq\ C\,\mathcal{H}^{N-1}(E)$$

for any set $E \subset \mathbb{R}^N$.

For sets of finite perimeter, $\mathcal{H}_f^{N-1}$ can also be defined in the following way: let K be a set of finite perimeter. Then

$$\mathcal{H}_f^{N-1}(\partial^* K) = \int_{\partial^* K} |f^*(x)\nu_K(x)|\, d\mathcal{H}^{N-1}(x) \,.$$

where $\partial^* K$ is the reduced boundary of K.

IV. PERIMETER ESTIMATE

A. Properties of extremal trajectories

Let us start noting that the reachable set is rather regular along extremal solutions: let x be an extremal trajectory on the time interval $[0, T]$. Then $T_{\mathcal{R}(t)}(x(t))$ is a half-space for any $t \in (0, T)$ and

$$x(t) \in \partial_*\mathcal{R}(t) \qquad \forall t \in (0, T) \,,$$

where $\partial_*\mathcal{R}(t)$ is the measure theoretic boundary of $\mathcal{R}(t)$.

To prove the above property, let us introduce the minimal time function given by:

$$\tau(x) = \min\{t \geq 0 \mid x \in \mathcal{R}(t)\} \qquad \forall x \in \mathbb{R}^N \,.$$

We note for later use that

$$\mathcal{R}(t) = \{x \in \mathbb{R}^N \mid \tau(x) \leq t\} \qquad \forall t > 0 \,. \qquad (11)$$

We now claim that τ is differentiable at $x(t)$ for any $t \in (0, T)$. For proving this claim, we note that $\mathcal{R}(\epsilon)$ has the inner ball property for any $\epsilon \in (0, T)$. Furthermore, from dynamic programming, $\tau - \epsilon$ is equal to the minimal time with the target $\mathcal{R}(\epsilon)$ and $y(t) = x(T - t)$ is an optimal trajectory on $(0, T - \epsilon)$ for this minimal time. Using Theorem 8.4.6 of [6], we deduce that $\tau - \epsilon$ is differentiable at $y(t)$ for any $t \in (0, T - \epsilon)$. Since $\epsilon > 0$ is arbitrary, this completes the proof of our claim.

We also note that $H(x(t), D\tau(x(t))) = 1$ for $t \in (0, T)$, because τ is a viscosity solution of this Hamilton-Jacobi equation. Thus $D\tau(x(t)) \neq 0$ for $t \in (0, T)$. Using (11), this shows that $T_{\mathcal{R}(t)}(x(t)) = (D\tau(x(t))^-$ is a half-space and the proof is complete.

Next, we investigate how two extremal trajectories depart from each other. Recall that $H^0(x, p) = |f^{-1}(x)p|$ for any $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$.

Let $T > 0$ be fixed. There are constants $C_1 = C_1(T)$ and $C_2 = C_2(T)$ such that for any extremal solution x_1 and x_2 on the time interval $[0, T]$, we have

$$\begin{aligned} & H^0(x_1(t_2), x_1(t_2) - x_2(t_2)) \\ \leq\ & \left(\frac{t_2}{t_1}\right)^{C_2} e^{C_1(t_2 - t_1)} H^0(x_1(t_1), x_1(t_1) - x_2(t_1)) \,, \end{aligned}$$

for any t_1, t_2 with $0 < t_1 < t_2 \leq T$.

To prove the above property, let us first introduce some notations. Let p_1 and p_2 be the adjoints of the extremal solutions x_1 and x_2, with $|p_1(0)| = |p_2(0)| = 1$. Let $R > 0$ be a constant such that any solution starting from $\mathcal{I}$ remains in $B(0,R)$ on the time interval $[0,T]$. Then, for some constant C,

$$\left|\frac{\partial H^0}{\partial x}(x,q)\right| \le C|q|, \ \left|\frac{\partial H^0}{\partial q}(x,q)\right| \le C,$$

and

$$\left|\frac{\partial H}{\partial x}(x,p)\right| \le C|p|, \ \left|\frac{\partial H}{\partial p}(x,p)\right| \le C,$$

for all $(x,p,q) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, with $|x| \le R$, $p \neq 0$, $q \neq 0$. We also choose C in such a way that

$$|q| \le C\, H^0(x,q)$$

for all $(x,q) \in \mathbb{R}^N \times (\mathbb{R}^N \backslash \{0\}$ with $|x| \le R$. Since, for $i = 1,2$, and $t \in (0,T)$,

$$\frac{d}{dt}\frac{1}{2}|p_i(t)|^2 = -\langle \frac{\partial H}{\partial x}(x_i(t),p_i(t)), p_i(t)\rangle \ge -C|p_i(t)|^2$$

we have

$$|p_i(t)| \ge e^{-CT} \qquad \forall t \in [0,T] \ .$$

Finally, we denote by C' a constant such that

$$\left|\frac{\partial^2 H}{\partial x \partial p}(x,p)\right| \le C'$$

for all $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$, with $|x| \le R$ and $|p| \ge e^{-CT}$.

From the way how C was chosen and equation (6) we have

$$\begin{aligned} &\frac{d}{dt}H^0(x_1(t), x_1(t)-x_2(t)) \\ &= \langle \frac{\partial H^0}{\partial x}(x_1, x_1-x_2), \frac{\partial H}{\partial p}(x_1,p_1)\rangle \\ &+\langle \frac{\partial H^0}{\partial q}(x_1,x_1-x_2), \frac{\partial H}{\partial p}(x_1,p_1) - \frac{\partial H}{\partial p}(x_2,p_2)\rangle \\ &\le\ C^2|x_1-x_2| \\ &+\langle \frac{\partial H^0}{\partial q}(x_1,x_1-x_2), \frac{\partial H}{\partial p}(x_1,p_1) - \frac{\partial H}{\partial p}(x_1,p_2)\rangle \\ &+\langle \frac{\partial H^0}{\partial q}(x_1,x_1-x_2), \frac{\partial H}{\partial p}(x_1,p_2) - \frac{\partial H}{\partial p}(x_2,p_2)\rangle \end{aligned}$$

We note that (omitting the x_1 argument in f and f^{-1})

$$\begin{aligned} &\langle \frac{\partial H^0}{\partial q}(x_1,x_1-x_2), \frac{\partial H}{\partial p}(x_1,p_1) - \frac{\partial H}{\partial p}(x_1,p_2)\rangle \\ &= \langle \frac{f^{-*}f^{-1}(x_1-x_2)}{|f^{-1}(x_1-x_2)|}, \frac{ff^*p_1}{|f^*p_1|} - \frac{ff^*p_2}{|f^*p_2|}\rangle \\ &= \frac{1}{H^0(x_1,x_1-x_2)}\langle x_1-x_2, \frac{p_1}{H(x_1,p_1)} - \frac{p_2}{H(x_1,p_2)}\rangle \ . \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{d}{dt}\, H^0(x_1(t), x_1(t)-x_2(t)) \\ &\le\ \frac{1}{H^0(x_1,x_1-x_2)}\langle x_1-x_2, \frac{p_1}{H(x_1,p_1)} \\ &-\frac{p_2}{H(x_1,p_2)}\rangle + C(C+C')|x_1-x_2| \ . \end{aligned}$$

Owing to the above considerations applied to the matrix $A = f^{-1}(x_1(t))$, we obtain

$$\langle x_1 - x_2, \frac{p_1}{H(x_1,p_1)} - \frac{p_2}{H(x_1,p_2)}\rangle \ \le\ \frac{C_2}{t}\left(H^0(x_1,x_1-x_2)\right)^2$$

where $C_2 = \max_{|x|\le R} 1/(c_T\lambda(x))$, $\lambda(x)$ being the minimal singular value of $f^{-1}(x)$.

Thus, for $C_1 = C^2(C+C')$ we have

$$\begin{aligned} &\frac{d}{dt}\, H^0(x_1(t), x_1(t)-x_2(t)) \\ &\le \left(C_1 + \frac{C_2}{t}\right) H^0(x_1(t), x_1(t)-x_2(t)) \ . \end{aligned}$$

Then, Gronwall's Lemma now gives

$$\begin{aligned} &H^0(x_1(t_2), x_1(t_2)-x_2(t_2)) \\ &\le \left(\frac{t_2}{t_1}\right)^{C_2} e^{C_1(t_2-t_1)} H^0(x_1(t_1), x_1(t_1)-x_2(t_1)) \ . \end{aligned}$$

B. Main result

We are now ready to prove the main result of this paper: under assumption (4), the set $\mathcal{R}(t)$ is a set of finite perimeter for any $t > 0$. Moreover, for any $T > 0$, there are constant c_1 and c_2 such that for any t_1, t_2 with $0 < t_1 < t_2 \le T$ we have

$$\begin{aligned} &\int_{\partial^*\mathcal{R}(t_2)} |f^*(x)\nu_{\mathcal{R}(t_2)}(x)|\, d\mathcal{H}^{N-1}(x) \le \\ &\left(\frac{t_2}{t_1}\right)^{c_2} e^{c_1(t_2-t_1)} \int_{\partial^*\mathcal{R}(t_1)} |f^*(x)\nu_{\mathcal{R}(t_1)}(x)|\, d\mathcal{H}^{N-1}(x) \ . \end{aligned}$$

The above result is aconsequence of the following property:

$$\mathcal{H}_f^{N-1}(\partial\mathcal{R}(t_2)) \ \le\ \left(\frac{t_2}{t_1}\right)^{c_2} e^{c_1(t_2-t_1)}\ \mathcal{H}_f^{N-1}(\partial^*\mathcal{R}(t_1))$$

To derive the above inequality, we note that, since

$$\mathcal{H}^{N-1}(\partial_*\mathcal{R}(t_1)\backslash\partial^*\mathcal{R}(t_1)) = 0 \ .$$

and $\mathcal{H}_f^{N-1}$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$, we have

$$\mathcal{H}_f^{N-1}(\partial_*\mathcal{R}(t_1)\backslash\partial^*\mathcal{R}(t_1)) = 0 \ .$$

Since $\mathcal{R}(t_1)$ has the inner ball property, we already know that $\mathcal{H}_f^{N-1}(\partial_*\mathcal{R}(t_1)) < +\infty$. Let $R > 0$ be a constant such that any solution starting from $\mathcal{I}$ remains in $B(0,R)$ for every $t \in [0,T]$. We also denote by $k > 0$ a constant such that

$$\frac{|\det(f(y))|}{|\det(f(x))|} \le e^{k|y-x|} \tag{12}$$

for all $(x,y) \in B(0,R) \times B(0,R)$. Note that such a constant exists thanks to assumption (4). We also set $M = \|f\|_\infty$.

Let us fix $\epsilon > 0$, $\delta > 0$, K_i compact subsets of $\mathbb{R}^N$ such that, $\forall i \ge 1$,

$$0 < \mathrm{diam}_f(K_i) \ \le\ \delta(t_1/t_2)^{C_2} e^{-(C_1+kM/(N-1))(t_2-t_1)} \ ,$$

$$\partial_*\mathcal{R}(t_1) \subset \bigcup_{i=1}^{\infty} K_i$$

and

$$\mathcal{H}_f^{N-1}(\partial_*\mathcal{R}(t_1)) \geq \alpha_{N-1}\sum_{i=1}^{\infty}\left(\frac{\mathrm{diam}_f(K_i)}{2}\right)^{N-1} - \epsilon\ ,$$

where α_{N-1} is the volume of the unit ball in $\mathbb{R}^{N-1}$.

We denote by K_i' the subset of points z of $\partial\mathcal{R}(t_2)$ for which there is an extremal solution x on $[0,t_2]$ with $x(t_2) = z$ and $x(t_1) \in K_i$. Then, we know that

$$\partial\mathcal{R}(t_2) \subset \bigcup_{i=1}^{\infty} K_i'\ .$$

We now estimate the diameter $diam_f(K_i')$ of K_i'. Let z_1, z_2 belong to K_i', x_1, x_2 be extremal trajectories such that $x_j(t_2) = z_j$ and $x_j(t_1) \in K_i$ for $j = 1,2$. Then, by the definition of k in (12), we have

$$\begin{aligned}
&|\det(f(z_1))|^{\frac{1}{N-1}} H^0(z_1, z_1 - z_2) \\
&= |\det(f(z_1))|^{\frac{1}{N-1}} H^0(x_1(t_2), x_1(t_2) - x_2(t_2)) \\
&\leq |\det(f(x_1(t_1)))|^{\frac{1}{N-1}} e^{k|x_1(t_1)-z_1|/(N-1)} \\
&\left(\frac{t_2}{t_1}\right)^{C_2} e^{C_1(t_2-t_1)} H^0(x_1(t_1), x_1(t_1) - x_2(t_1)) \\
&\leq \left(\frac{t_2}{t_1}\right)^{C_2} e^{C_1+kM/(N-1))(t_2-t_1)} \mathrm{diam}_f(K_i)
\end{aligned}$$

because $|x_1(t_1) - z_1| \leq M(t_2 - t_1)$ since $\|f\|_\infty \leq M$. Hence,

$$\begin{aligned}
&\mathrm{diam}_f(K_i') \\
&\quad \leq \left(\frac{t_2}{t_1}\right)^{C_2} e^{(C_1+kM/(N-1))(t_2-t_1)} \mathrm{diam}_f(K_i) \leq \delta\ .
\end{aligned}$$

Therefore, setting $c_1 = (N-1)C_1$ and $c_2 = (N-1)C_2 + kM$, we get

$$\begin{aligned}
&\mathcal{H}_{f,\delta}^{N-1}(\partial\mathcal{R}(t_2)) \ \leq \alpha_{N-1}\sum_{i=0}^{\infty}\left(\frac{\mathrm{diam}_f(K_i')}{2}\right)^{N-1} \\
&\leq \left(\frac{t_2}{t_1}\right)^{c_2} e^{c_1(t_2-t_1)} \alpha_{N-1}\sum_{i=0}^{\infty}\left(\frac{\mathrm{diam}_f(K_i)}{2}\right)^{N-1} \\
&\quad \leq \left(\frac{t_2}{t_1}\right)^{c_2} e^{c_1(t_2-t_1)} (\mathcal{H}_f^{N-1}(\partial_*\mathcal{R}(t_1)) + \epsilon)\ .
\end{aligned}$$

Letting first $\delta \to 0^+$, then $\epsilon \to 0^+$, gives the result.

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