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Input-Output Stability of Wireless Networked Control Systems†

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Abstract— **This paper provides a general framework for analyzing the stability of general nonlinear networked control systems (NCS) with disturbances in the setting of $\mathcal{L}_p$ stability. With general, weak sufficient conditions on the NCS, we are able to conclude much tighter performance bounds than [1] and [2] for a very wide class of NCS. We also define the notion of persistently exciting (PE) scheduling protocols, discuss how PE leads to $\mathcal{L}_p$ stability for high enough transmission rates and show that PE is a natural property to demand, especially in the design of wireless scheduling protocols. Our results are of wide applicability as the analyses require neither the existence nor explicit construction of Lyapunov functions and the performance and stability bounds are completely parameterized by properties of the "network-free" system and a single NCS parameter chosen in the design of the scheduling protocol.**

I. INTRODUCTION

Although networked control systems (NCS) are a relatively new arrival to the world of automatic control, an increasing number of practical implementations and their respective traffic scheduling protocols now exist, particularly in the automotive and aeronautical industries that are seeing high adoption-rates of drive-by-wire and fly-by-wire designs. Standards-based component connectivity offers lower implementation costs, greater interoperability and a wide range of choices in developing control systems. The price paid for these advantages is the added complexity in the initial design and analysis of NCS.

The fundamental difference between NCS and traditional control systems is that numerous point-to-point connections between system nodes are replaced with a single shared serial bus. In the most general scenario, disparate actuators and sensors vie for exclusive access to the network at transmission instants presenting the immediate problem of how such access is arbitrated. The arbitration of network access amongst links, or *scheduling* is of fundamental importance to NCS design and analysis. A survey of scheduling and scheduling protocols is provided in [3] and stability and performance results of wireline NCS have been examined in [1], [3], [4], [5], [6] and [7]. Although round-robin (RR) scheduling is used almost exclusively in practice, the aforementioned works present various alternative protocols that demonstrate a performance gain over RR in simulations and, in special cases, demonstrate the superiority of the alternative protocols analytically.

The NCS design approach adopted in [1], [2], [4], [6], [7], and this paper consists of the following steps: (i) design of a stabilizing controller ignoring the network; (ii) and analysis of robustness of stability with respect to effects that scheduling within a network introduce. The principal advantage of this approach is its simplicity – the designer of the NCS can exploit familiar tools for controller design and select an appropriate scheduling protocol and transmission rate such that the desired properties of the network-free system are preserved.

Our analysis framework analyzes the input-output $\mathcal{L}_p$ stability (IOS) of NCS, the essence of which is that outputs (or state) of an NCS verify a robustness property with respect to exogenous disturbances. This notion of stability is the cornerstone of modern robust and optimal control (see [8], for instance) and closely related to the notion of input-to-state stability (ISS). In fact, if the network-free system is $\mathcal{L}_p$ stable, we show that the NCS remains so with any scheduling protocol that always visits its links within every $T < \infty$ consecutive transmissions, whenever transmissions occur faster than once every τ^* secs, where[1]

$$\tau^* = \frac{\ln(z)}{|A|T}, \tag{1}$$

and satisfies $T\gamma|A|z^{1+1/T} + (|A| - \gamma T)z - 2|A| = 0, z \in [1, 2]$. The parameter γ captures robustness properties of the network-free system and $|A|$ captures the speed at which the NCS departs from it nominal network-free behavior in the absence of exogenous disturbances. In particular, when $\gamma = 0$, $\tau^* = \ln(2)/(|A|T)$. Under additional technical conditions, $\mathcal{L}_p$ stability specializes to uniform global exponential stability (UGES) in the absence of exogenous disturbances.

The condition that a scheduling protocol visits all its links is particularly important in our analysis and, for reasons that will become apparent, we say that a scheduling protocol is uniformly persistently exciting in time T, or just PE_T, if *there is a fixed number $T < \infty$ such that all links of the NCS have transmitted within every T consecutive transmissions.* The technological limitations of communications channels,

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[1]When there is no ambiguity, $|\cdot|$ denotes both the 2-norm on $\mathbb{R}^n$ and the induced matrix 2-norm.

especially wireless channels, leave us with few guarantees about what information is available to the scheduler for making its scheduling decisions or it may be the case that the communication channel is using a collision protocol, in which case we have very few guarantees at all. Save for RR, the protocols described and analyzed in the literature (see [1], [3] and [9], for instance) require more information (NCS state information) than is typically available, particularly in a wireless setting. The results presented in [2] partially address the issue, concluding UGES for linear systems on wireless networks when using the protocols described therein. The results in this paper address the issue more precisely and in greater generality through PE and $\mathcal{L}_p$ stability.

PE in the sense we have described is verified by many network technologies[2] and many protocols described in the aforementioned NCS literature. Building on the informal definition of PE, the primary contributions of this paper are: **1.)** that we (formally) introduce the fundamental notion of PE scheduling protocols and demonstrate the importance of PE in stability analysis; **2.)** we generalize the NCS model presented in [1] to handle a wider class of scheduling protocols with a focus on PE and protocols implementable on wireless channels – in particular, RR, every protocol presented in [2] and the hybrid-TOD protocol that we introduce are all PE and can be analyzed with our framework; **3.)** under general conditions we show that every PE scheduling protocol leads to the $\mathcal{L}_p$ stability of general nonlinear NCS with disturbances whenever the network-free system is $\mathcal{L}_p$ stable; **4.)** and we provide significantly better analytic performance bounds than [1] for RR scheduling and significantly better analytic performance bounds for every protocol considered in [2].

The paper is divided into three additional sections: Section II presents our model for NCS and outlines the limitations inherent in the design of scheduling protocols. The notion of PE is also described in detail and characterized. Section III presents our main result and several remarks on its use in examples and Section IV presents a case-study with applications of our results demonstrating large improvements over existing results in the literature.

Preliminaries: Consider the hybrid system $\Sigma_z : \dot{z} = f(t, z, w)$ with "jump" equation $z(t_i^+) = h(i, z(t_i))$, initialized at (t_0, z_0) with input w and a prescribed output $y = g(t, z)$.[3] We say that Σ_z is $\mathcal{L}_p$ stable from w to y with gain γ if[4]

$$\exists\, K \geq 0 : \|y[t_0, t]\|_p \leq K|z_0| + \gamma\|w[t_0, t]\|_p.$$

The state z of Σ_z is said to be $\mathcal{L}_p$ to $\mathcal{L}_q$ detectable from output y with gain γ if

$$\exists\, K \geq 0 : \|z[t_0, t]\|_q \leq K|z_0| + \gamma\|y[t_0, t]\|_p + \gamma\|w[t_0, t]\|_p.$$

An exposition of these ideas as they pertain to NCS can be found in [1, Section II]. We will also use the standard notions of uniform global exponential and asymptotic stability (UGES and UGAS, respectively) e.g., see [11, Chapter 4].

II. WIRED AND WIRELESS NETWORKED CONTROL SYSTEMS

Computer networks and communications systems present rich and sophisticated models of varying degrees of complexity, within stochastic and deterministic settings, and of various underlying physical communication media. The network model presented in this paper aims to capture the essential aspects of control over networks in a deterministic setting. In that vein, we assume:

Assumption 1: All NCS links are connected via a shared, serial bus on which such links either have exclusive access to the network medium or no access at all. ◁

Assumption 2: Transmitted frames are received error-free and network throughput is high enough to ignore delays due to transmission time.[5] ◁

A. A Model for NCS

We consider general nonlinear NCS with disturbances conceptually depicted in Figure 1, where x_P and x_C are, respectively, states of the plant and controller; y is the plant output and u is the controller output; $\hat{y}$ and $\hat{u}$ are the vectors of the most recently transmitted plant and controller output values via the network and e is the network-induced error defined as $e(t) := \begin{pmatrix} \hat{y}(t)-y(t) \\ \hat{y}(t)-y(t) \end{pmatrix}$. The key difference between this model and that presented in [1] is the addition of the *scheduler* and its corresponding decision-vector $\hat{e}$ that are represented by "dashed" elements in Figure 1. Special cases of $\hat{e}$-based scheduling were first considered in [2]. The model we introduce formalizes the $\hat{e}$-based scheduling that was considered in [2] and it generalizes the NCS models considered in [1]. Motivation for considering the class of NCS depicted in Figure 1 is given later in Assumption 3 and the ensuing discussion.

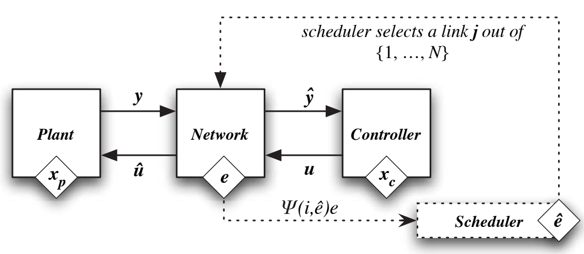

Fig. 1. Conceptual diagram of scheduled NCS.

[2]Ethernet and 802.11 are examples of CSMA/CD protocols where it is known (see [10], for instance) that for a finite number of users (links), the expected waiting time for a link is finite. Although our results are cast in a deterministic setting, we mention collision protocols to motivate how natural the notion of PE is in the context of real networks and to suggest that similar results maybe pursued in a stochastic setting.

[3]By $z(t_i^+)$ we mean "evaluated just after the jump": $z(t_i^+) = \lim_{s \to t_i, s > t_i} z(s)$, where t_i belongs to a sequence of jump times.

[4]Specifically, $\|y[t_0, t]\|_p := \left(\int_{t_0}^t |y(s)|^p ds\right)^{1/p}$ for $p \in [1, \infty)$ and $\|y[t_0, t]\|_\infty = \text{ess.sup}\{|y(s)| : s \in [t_0, t]\}$.

[5]This is especially true for wireline channels but some wireless channels may exhibit notable delays. As the focus of this paper is scheduling, we defer the treatment of delays and appeal to the robustness properties verified by the class of systems considered to assert that the results in this paper remain true for sufficiently small delays.

We model the NCS in Figure 1 as a so-called jump-continuous (hybrid) system. Node data (controller and sensor values) are transmitted at transmission instants at times $\{t_0, t_1, \ldots, t_i\}, i \in \mathbb{N}$ satisfying $\epsilon < t_{j+1} - t_j \leq \tau$ for all $j \geq 0$ where $\tau > 0$ and $\epsilon > 0$.[6] The constant τ is the *maximum allowable transmission interval* (MATI). The system is initialized at time t_s with $t_s \in [0, t_0], t_0 - t_s < \tau$.

Our NCS model is prescribed by the following dynamical and jump equations. In particular, for all $t \in [t_{i-1}, t_i]$:

$$\dot{x}_P = f_P(t, x_P, \hat{u}, w) \tag{2}$$

$$\dot{x}_C = f_C(t, x_C, \hat{y}, w) \tag{3}$$

$$u = g_C(t, x_C) \quad y = g_P(t, x_P) \tag{4}$$

$$\dot{\hat{y}} = 0 \quad \dot{\hat{u}} = 0 \quad \dot{\hat{e}} = 0, \tag{5}$$

and at each transmission instant t_i,

$$e(t_i^+) = (I - \Psi(i, \hat{e}(t_i)))e(t_i) \tag{6}$$

$$\hat{e}(t_i^+) = \Lambda(i, \Psi(i, \hat{e}(t_i))e(t_i), \hat{e}(t_i)) \tag{7}$$

and we refer to Ψ as the scheduling function and Λ as the decision-update function. The effect of the protocol on the error is such that if the mth to nth nodes are scheduled[7] at transmission instant t_i the corresponding components of error, $e_n, \ldots, e_m$, experience a "jump". For transmission of nodes mth to nth nodes, though it is not necessary, we will always assume that $e_n(t_i^+), \ldots, e_m(t_i^+) = 0$ and, hence, $\Psi(i, \hat{e}(t_i))$ is a diagonal matrix consisting of entries $[a_{ij}]$, where $a_{ii} = 1$ for $n \leq i \leq m$ and 0 elsewhere. We group the nodes that are scheduled together into logical links, associating a partition of size N_i, denoted by $\mathbf{e}_i = (e_{i1}, e_{i2}, \ldots, e_{iN_i})$, of the error vector e such that we can write $e = (\mathbf{e}_1, \ldots, \mathbf{e}_N)$. We say that the NCS has N links and $\sum_{i=1}^{N} N_i$ nodes. Note that this is purely a notational convenience and simplifies the description of scheduling protocols and the NCS itself. We combine the controller and plant states into a vector $x = (x_P, x_C)$ and similarly to [1, pp. 1653], we can rewrite (2)-(5):

$$\dot{x} = f(t, x, e, w) \quad t \in [t_{i-1}, t_i] \tag{8}$$

$$\dot{e} = g(t, x, e, w) \quad t \in [t_{i-1}, t_i] \tag{9}$$

$$\dot{\hat{e}} = 0 \quad t \in [t_{i-1}, t_i] \tag{10}$$

$$e(t_i^+) = (I - \Psi(i, \hat{e}(t_i)))e(t_i) \tag{11}$$

$$\hat{e}(t_i^+) = \Lambda(i, \Psi(i, \hat{e}(t_i))e(t_i), \hat{e}(t_i)) \tag{12}$$

where $x \in \mathbb{R}^{n_x}, e \in \mathbb{R}^{n_e}, w \in \mathbb{R}^{n_w}, \hat{e} \in \mathbb{R}^{n_{\hat{e}}}$. Note that implicit to our model is the following assumption:

Assumption 3: Only the transmitted value $\Psi(i, \hat{e}(t_i))e(t_i)$ is accessible by the scheduler – the full vector $e(t_i)$ is unavailable. ◁

[6]This ensures that Zeno solutions cannot occur.

[7]It may be the case that a single logical node (a "link") consists of several sensors or several actuators or both with the scheduling of that node having the effect of setting multiple components of e to zero. It may also be the case that the network allows the scheduling of more than one node at each transmission and our model allows for this extra degree of freedom.

The above assumption is quite realistic in a range of wireline NCS and archetypical in wireless NCS. It was the main motivation for introducing $\hat{e}$-based scheduling in [2]. Moreover, Assumption 3 is not satisfied by the protocols considered in [1], excepting RR, that otherwise assume that $e(t_i)$ is always accessible by the scheduler, where (11)-(12) are replaced with the single e jump equation:

$$e(t_i^+) = (I - \Psi(i, e(t_i)))e(t_i) \ . \tag{13}$$

The maximum-error-first try-once-discard (MEF-TOD or just TOD) protocol presented in [6] provides part of the rationale for Assumption 3 and operates as follows:

Example 2.1 (MEF-TOD): At a transmission instant t_j, the scheduler transmits link i with the largest error $|\mathbf{e}_i(t_j)|$ so that the instantaneous error is minimized after jumps. TOD can be written in the form (13), where $\Psi(i, e) = \Psi(e) = \mathrm{diag}\{d_j(e)I_{n_j}\},\ j = [1, \ldots, N]$, I_{n_j} are $n_j \times n_j$ identity matrices and

$$d_j(\hat{e}) = \begin{cases} 1 & \text{if } j = \min\left(\arg\max_{1\leq k\leq N} |\mathbf{e}_k|\right) \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Clearly, TOD does not satisfy Assumption 3 since it requires access to the full error vector e. It is easy to see that this is equivalent to the scheduler having access to, for example, each *remote* sensor's measurements *prior to transmission* which is quite unreasonable for most NCS.[8] ◁

B. Persistently Exciting Scheduling Protocols

Motivated by the results in [1], we consider the following auxiliary discrete-time system induced by the protocol:

$$e^+ = (I - \Psi(i, \hat{e}))e \tag{15}$$

$$\hat{e}^+ = \Lambda(i, \Psi(i, \hat{e})e, \hat{e}) \tag{16}$$

Henceforth, we will "disconnect" the protocol from the NCS and refer to (15)-(16) as the protocol. It was shown in [1] that stability properties of the auxiliary system $e^+ = (I - \Psi(i, e))e$ induced by (13) are instrumental in proving $\mathcal{L}_p$ stability properties for a class of NCS for which Assumption 3 does not hold. For the class of NCS we consider, partial stability (only in e) of the system (15)-(16) is essential for stability. To motivate our PE definition, let $V(i, e, \hat{e}) = |e|$ in (15)-(16), we have $\Delta V = -|\Psi(i, \hat{e})e|$ and if there exists a $T < \infty$ such that, for every $k \in \mathbb{N}$, $\hat{e}(k) = \hat{e} \in \mathbb{R}^{n_{\hat{e}}}$, $e(k) = e \in \mathbb{R}^{n_e}$:

$$\sum_{i=k}^{k+T-1} |\Psi(i, \phi_{\hat{e}}(i))e| \geq |e| \ , \tag{17}$$

where $\phi_{\hat{e}}(i) := \phi_{\hat{e}}(i, e, \hat{e})$, we can conclude partial stability (in e) of the system (15)-(16) using persistency of excitation (PE) arguments (see [13], for instance). We can rephrase (17) by saying that each link is visited within T transmissions. It is tempting to conjecture that (17) is enough to prove the stability of (8)-(12) but, notwithstanding the results presented in [1], this is not true in general.

[8]TOD is implementable through binary countdown in CAN-like medium access protocols – see [12] and [3] for details.

Indeed, it can be shown that if we integrate the equations (9)-(10) on the interval $[t_{i-1}^{+}, t_i]$ and then use (11), (12) at t_i, the NCS induces the following discrete-time system:

$$e^{+} = (I - \Psi(i, \hat{e}))(e + d) \tag{18}$$
$$\hat{e}^{+} = \Lambda(i, \Psi(i, \hat{e})(e + d), \hat{e}) \ , \tag{19}$$

where d captures the inter-sample behavior of $e(\cdot)$. For specific initializations $(k, e(k), \hat{e}(k))$ and specific (bounded) values of $d(j), j \geq k$ the solution of the system (18)-(19) coincides with that of (8)-(12) at time instants $t_j^{+}, j \geq k$. In contrast, while the system (15)-(16) may satisfy the PE condition (17) and, hence, is partially stable in e, there may exist a bounded disturbance d for its perturbed counterpart (18)-(19) that destroys the PE property along with partial stability in e. Hence, our formal definition of PE needs to possess an appropriate robustness property and will be stated using (18)-(19):

Definition 2.2: The protocol (15)-(16) is said to be (robustly) persistently exciting in T or PE_T if there exists $T \in (0, \infty)$ such that (17) holds for all $k \in \mathbb{N}$, $\hat{e}(k) = \hat{e} \in \mathbb{R}^{n_{\hat{e}}}$, $e(k) = e \in \mathbb{R}^{n_e}$ and all $d \in \ell_\infty$, where $\phi_{\hat{e}}(i) := \phi_{\hat{e}}(i, e, \hat{e}, d_{[k,i]})$ is the $\hat{e}$ component of the solution of the system (18)-(19). ◁

Two PE_T protocols that satisfy Assumption 3 are presented next.

Example 2.3 (Hybrid RR-TOD Scheduling Protocol): The hybrid RR-TOD scheduling protocol enforces PE in a time-periodic manner. For a prescribed $M \in \mathbb{N}$, the protocol takes the form:

$$e^{+} = (I - \Omega(i, \hat{e}))(e + d) \tag{20}$$
$$\hat{e}^{+} = (I - \Omega(i, \hat{e}))\hat{e} + \Omega(i, \hat{e})(e + d) \ , \tag{21}$$

$$\Omega(i, \hat{e}) := \begin{cases} \text{diag}\{p_1(i) I_{n_1}, \ldots, p_N(i) I_{n_N}\}, \ \text{mod}(i, M) = 0 \\ \text{diag}\{d_1(\hat{e}) I_{n_1}, \ldots, d_N(\hat{e}) I_{n_N}\}, \ \text{otherwise,} \end{cases}$$

where, $p_n(i) = 1$ when $\text{mod}(i/M, N) = n - 1$ and $p_n(i) = 0$ otherwise with d_j defined in (14). The hybrid RR-TOD protocol is PE_T with $T := MN$. In particular, when $M = 1$, we obtain the simplest PE_T protocol: "classical" RR.

Example 2.4 (Constant-Penalty TOD): The protocols in [2] use the mechanism of "silent-time" to enforce PE: each link j has a counter r_j that is incremented at every transmission instant that it is *not* scheduled and reset to zero when it *is* scheduled. Irrespective of the underlying scheduling protocol, when a link's counter reaches a predetermined threshold, say M, it will be transmitted. This ensures that every link is scheduled within $M + N - 1$ transmission instants, hence, $T := M+N-1$. The underlying scheduler in this example is TOD and corresponds to the constant-penalty TOD scheme in [2] with a penalty (vector) of Θ:

$$e^{+} = (I - \Phi(r, \zeta))(e + d) \tag{22}$$
$$\zeta^{+} = (I - \Phi(r, \zeta))(\zeta + \Theta) + \Phi(r, \zeta)(e + d) \tag{23}$$
$$r^{+} = (I - \Phi(r, \zeta))(r + \mathbf{1}), \tag{24}$$

where $\mathbf{1} = [1 \ \ldots \ 1]^T$, the scheduling function Φ is given by $\Phi(r, \zeta) = \text{diag}\{\varphi_j(r, \zeta) I_{n_j}\}, j \in [1, \ldots, N]$ and

$$\varphi_n(r, \zeta) = \begin{cases} 1 & \text{if } [n = \min\{m : r_m \geq M\}) \\ & \vee \left(n = \min\left(\arg\max_{1 \leq j \leq N} |\boldsymbol{\zeta}_j|\right)\right) \\ & \wedge (\forall\, m \in \{1, \ldots, N\})(r_m < M)] \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

III. STABILITY ANALYSIS OF NCS

The framework of $\mathcal{L}_p$ stability and detectability lies at the core of the following theorem, the main result of the paper, and we confirm that PE protocols lead to $\mathcal{L}_p$ stability of NCS when the network-free system is $\mathcal{L}_p$ stable from w to x.

Theorem 3.1: Consider NCS (8)-(12) and suppose that:

1) there exists a $0 \leq T < \infty$ such that the NCS scheduling protocol (8)-(12) is PE_T;
2) there exists a symmetric positive semi-definite $n_e \times n_e$ matrix with nonnegative entries A and a continuous $\tilde{y} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_e} := G(x) + H(w)$ so that, for all t, x, e, w, the error dynamics (9) satisfy[9]
$$\overline{g}(t, x, e, w) \preceq A\overline{e} + \tilde{y}(x, w) \ ; \tag{26}$$
3) system (8) is $\mathcal{L}_p$ stable from $(e, H(w))$ to $G(x)$ with gain γ for some $p \in [1, \infty]$;
4) and x in (8) is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from $H(x)$
5) and MATI satisfies $\tau \in (\epsilon, \tau^*)$, $\epsilon \in (0, \tau^*)$, where
$$\tau^* = \frac{\ln(z)}{|A|T}, \tag{27}$$
where $z \in [1, 2]$ solves[10]
$$T\gamma|A|z^{1+1/T} + (|A| - \gamma T)z - 2|A| = 0 \tag{28}$$
and $|A|$ comes from (26).

Then, the NCS is $\mathcal{L}_p$-stable from w to (x, e) with linear gain.

Sketch of proof: It can be shown (see [14]) that conditions 1 and 2 ensure that the NCS error subsystem (9)-(10), (11)-(12) is $\mathcal{L}_p$ stable from $\tilde{y}$ to e for $p \in [1, \infty]$ with gain

$$\tilde{\gamma}(\tau) = \frac{T \exp(|A|T\tau)(\exp(|A|\tau) - 1)}{|A|(2 - \exp(|A|T\tau))}. \tag{29}$$

Solving for τ^* in $\tilde{\gamma}(\tau^*)\gamma = 1$ is equivalent to applying the transformation $z = \exp(|A|T\tau^*)$ to (29) and solving for z in (28). As $\tilde{\gamma}(0) = 0$, and by monotonicity of $\tilde{\gamma}$, we have that $\tau^* > 0$ for every $\gamma < \infty$ and $\tilde{\gamma}(\tau)\gamma < 1$ for any $\tau \in [0, \tau^*)$. Together with detectability assumptions, the proof is complete in light of the small-gain theorem for jump-discontinuous systems discussed in [1, Theorem 1] and [1, Theorem 6]. Note that we do not require that $\hat{e}$ be detectable from e as $\hat{e}$ is a scheduler-maintained variable with no direct physical significance. □

[9] Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The *vector* partial order $\preceq$ is given by $x \preceq y \iff (x_1 \leq y_1) \wedge \cdots \wedge (x_n \leq y_n)$ and $\overline{e}$ and $\overline{g}$ are given by $\overline{e} := (|e_1|, \ldots, |e_{n_e}|)^T$ and $t \overset{\overline{g}}{\mapsto} \overline{g(t)}$, respectively. That is, $\overline{e}$ is the vector that results from taking the absolute value of each scalar component of e and $\overline{g}$ does operates analogously on the image of g.

[10] It can be show (see [14]) that there exists a unique positive solution to (28).

Remark 1: Our results are derived under realistic assumptions on the network, except perhaps for propagation delays, which will be an important direction for future work. Moreover, we emphasize that every condition of Theorem 3.1 can be checked by direct inspection of (8)-(12); we do not require any re-design of any component of the control loop and the analysis framework we proposed is valid for RR, hybrid RR-TOD, CP-TOD, every protocol described in [6] and PE protocols at large. ◁

While IOS and ISS results were presented in [1] and [7], NCS literature often focusses on UGES or UGAS. It is important to note that both IOS and ISS-style results can be specialized to UGES and UGAS, respectively, when disturbances are not considered.

Corollary 3.2: Suppose that the hypotheses of Theorem 3.1 hold for any $p \in [1, \infty)$ and let $F(i, t, x, e, \hat{e}, w)$ denote the (stacked) right-hand side of (8)-(12). If there exists an $L > 0$ such that $F(i, t, x, e, \hat{e}, 0) \leq L(|x| + |e| + |\hat{e}|)$ then the NCS is UGES when $w \equiv 0$. The assumptions on F can be weakened as discussed in [1, Section II]. ◁

Remark 2: It is relatively straightforward to show (see [14]) that the $\mathcal{L}_\infty$ IOS asserted by Theorem 3.1 can be strengthened to input-to-state stability (ISS). In particular, it is possible to conclude ISS with an exp-$\mathcal{KL}$ function β and linear gains: $|x(t)| \leq \beta(|x_0|, t - t_0) + \gamma \|w\|_\infty$. Moreover, in analogy with [7], we can conclude semi-global-practical ISS of the NCS for sufficiently small MATI whenever the network-free system is ISS in the absence of disturbances for a much larger class of nonlinear systems than considered in this paper. Finally, we can also infer UGAS of the NCS in the absence of disturbances using similar hypotheses to those of Corollary 3.2. See [7] and [14] for details. ◁

Remark 3: Suppose that the simplified equations for an N-link NCS take the form

$$\dot{x} = A_{11}x + A_{12}e \quad \dot{e} = A_{21}x + A_{22}e \tag{30}$$

together with jump equations (11)-(12) and let σ_1 and σ_2 to be the largest and smallest singular value of P, respectively, that solve the Lyapunov matrix equation $A_{11}^T P + P A_{11} = -I$. We set $\tilde{y} = A_{21}x$, and let $k_h = \left| \begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix} \right|$. From the discussion in [1, Section VII], we have that the MATI bound stated therein, which we will refer to as $\tau^*_{[1]}$ is given by[11]

$$\tau^*_{[1]} = \frac{1}{k_h \sqrt{N}} \ln \left(\frac{k_h + \gamma}{k_h \sqrt{(N-1)/N} + \gamma} \right), \tag{31}$$

the MATI bound, $\tau^*_{[2]}$, obtained via [2][Theorem 1] is given by

$$\tau^*_{[2]} = \min \left\{ \frac{\ln(2)}{k_h T}, \frac{S}{8}, \frac{S}{16\sigma_2 \sqrt{\sigma_2/\sigma_1} k_h} \right\} \tag{32}$$

where $S = [k_h \sqrt{\sigma_2/\sigma_1} \sum_{i=1}^{N} (i + T - N)]^{-1}$, and that of this paper, τ^*_{new} is given by Theorem 3.1 – readily obtained by application of Remark 5 and solving (28) numerically. As a first step towards an analytic comparison, setting $\gamma = 0$ in (28) and solving yields a MATI bound of $\tau^*_0 := \ln(2)/(|A|T)$ and we note that $\lim_{T=N\to\infty} \frac{\tau^*_0}{N^{1/2}\tau^*_{[1]}} = 0$. In fact, we demonstrate in [14] that our (closed-loop) MATI bounds are analytically better than those provided in [1] for RR by a factor of $N^{1/2}$ as $N \to \infty$. The bounds in [1] are, in turn, analytically better than the bounds provided in [2] for every scheduling protocol considered therein. ◁

Remark 4: Suppose that the network-free system is $\mathcal{L}_p$ stable from w to x with gain γ and the NCS satisfies the hypotheses of Theorem 3.1. Then for any $\gamma^* > \gamma$, it is possible to show that there exists a MATI τ such that the NCS is $\mathcal{L}_p$ stable from w to x with gain γ^*. This corollary of Theorem 3.1 is particularly useful in the design of optimal/robust controllers. ◁

Remark 5: Suppose that $g(t, x, e, w) = Bx + Ce + Dw$ and let $A = \overline{[a_{ij}]}$, where $a_{ij} = \max\{|c_{ij}|, |c_{ji}|\}$ and $\tilde{y}(x, w) = \overline{Bx + Dw}$. We immediately have that A and $\tilde{y}(x, w)$ satisfy condition 2 of Theorem 3.1 and $\|\tilde{y}(x, w)\|_p = \|Bx + Dw\|_p \leq \|Bx\|_p + \|Dw\|_p$. Whenever g satisfies a linear growth bound of the form $|g(t, x, e, w)| \leq L(|x| + |e| + |w|)$, it is straightforward to construct an appropriate A and $\tilde{y}$. ◁

Remark 6: While frame dropouts and communication errors were not considered in our analysis, we conjecture that PE, as used in this paper, can be generalized to an "average-sense" PE condition for which we expect that an analogous NCS stability property can be verified. We regard the the development of the stochastic analogue to PE_T protocols and Theorem 3.1 as an important future problem. ◁

Remark 7: We believe that PE can be generalized in another direction to extend the notion of PE_T protocols to protocols where T is not fixed but dependent on e. For instance, [7, Example 2] describes a protocol that behaves in RR fashion whenever $|e| > 1/2$ but transmits less frequently as $|e|$ decreases. This behavior might be desirable when network traffic is at a premium, and asserting a property stronger than the UGAS concluded in [7] for specific systems would have important practical ramifications. ◁

IV. CASE STUDY: CH-47 TANDEM-ROTOR HELICOPTER

We consider the networked control of a CH-47 tandem-rotor helicopter in horizontal motion about a nominal airspeed of 40 knots as discussed in [15]:

$$\dot{x}_P = Ax_P + Bu; \quad y = Cx_P, \tag{33}$$

where $C_P = \left[\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57.3 \end{smallmatrix} \right]$,

$$A_P = \begin{bmatrix} -0.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B_P = \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0 & 0 \end{bmatrix}.$$

The outputs y_1, y_2 are vertical velocity (knots/hr) and pitch altitude (radians) respectively and the inputs u_1, u_2 are collective rotor thrust and differential collective rotor thrust respectively. Let the stabilizing output feedback be $u = Ky$ where K is given by $K = \left[\begin{smallmatrix} -12.7177 & -45.0824 \\ 63.5123 & 25.9144 \end{smallmatrix} \right]$. The design of

[11] In principle, different choices of Lyapunov function and "output" $\tilde{y}$ in the framework presented in [1] may lead to improved MATI bounds. We have followed [1, Section VII] in choices of Lyapunov function and $\tilde{y}$ – these were choices that lead to the best previously obtainable MATI bounds for RR.

K is due to [16]. We assume that only outputs are transmitted via the network thus $e = \hat{y} - y$ with two links ($N = 2$) in the NCS and, hence, with respect to Remark 3, we have $A_{11} = A_P + B_P K C_P$, $A_{12} = B_P K$, $A_{21} = -C_P A_{11}$, and $A_{22} = -C_P A_{12}$. $k_h = 19482$. For the purposes of applying Remark 5, we select $A = \left[\begin{smallmatrix} 567.98 & 239.09 \\ 239.09 & 0 \end{smallmatrix}\right]$ with $|A| = 655.23$ and $A_{22} \preceq A$. We can readily compute the gain from $\tilde{y}$ to e and have $\gamma = 10397$.

We focus on comparing MATI bounds required to achieve UGES, as discussed in Remark 3.2, and compare the results of this paper, [1] and [2]: From the perspective of MATI bounds, neither the results of this paper nor those in [2] distinguish between different PE_T scheduling protocols. The comparison results are summarized below:

1.) The MATI bounds are shown in Table I with the bounds in this paper larger than those obtained using the results of [2] by a factor of 10^{13} and longer than the bound obtained by the results of [1] by factor of 10^4. The bounds τ^*_{new} and $\tau^*_{[2]}$ apply to any PE_T protocol for the original two-link system (30). The bound $\tau^*_{[1]}$ only applies to RR ($T = N = 2$).

2.) The improvements are significant and to put them into perspective, when using RR, τ^*_{new} that achieves UGES is equivalent to a network throughput of 24 Mbps (assuming 128 byte frames), achievable on current 802.11g wireless networks, while $\tau^*_{[1]}$ requires an effective network throughput of over 1003 Gbps. For completeness, the equivalent throughput for $\tau^*_{[2]}$ is over 300 Ebps (300×10^{18} bits per second).

3.) We formally fix σ_1, σ_2, γ, k_h and $|A|$ and plot $\tau^*_{[1]}$ and $\tau^*_{[2]}$ with $T = N \in [1, 1000]$ in Figure 2 to examine the behavior of the bounds as the number of links grow. We also fix $N = 2$ and allow $T \geq 2$ to vary for $\tau^*_{[2]}$ and τ^*_{new}.[12] The differences are marked on the $\log_{10}(T) \times \log_{10}(\tau^*)$ scale used in Figure 2.

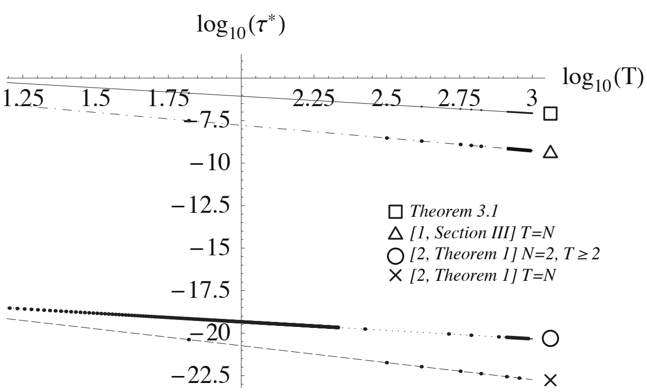

Fig. 2. MATI bounds comparison for PE_T protocols, $T \in [1, 1000]$.

V. Conclusion

This paper highlighted the importance of the PE property in scheduling protocols – a property naturally verified by many scheduling protocols presented in the literature and a property that is crucial in the design of wireless NCS. We presented results where the MATI bounds obtained are completely parameterized by T and properties of the network-free system for arbitrary PE_T protocols, as are the bounds that characterize the robustness of the NCS to exogenous disturbances. We believe that the results will be particularly useful in practical applications in light of the significant improvements over existing results, the generality of the model and the fact that every condition required to apply our results can be checked directly and easily from properties of the network-free system and the protocol alone.

[12] For the purposes of applying Theorem 3.1, it is immaterial as to whether N is fixed and $T \geq N$ is varied or whether $T = N$ is varied.

	$T = 2$	$T = 6$	$T = 50$
τ^*_{new}	4.23×10^{-5}	1.42×10^{-5}	1.71×10^{-6}
$\tau^*_{[1]}$	1.02×10^{-9}	N/A	N/A
$\tau^*_{[2]}$	3.08×10^{-18}	8.41×10^{-19}	9.35×10^{-20}
$\tau^*_{\text{new}}/\tau^*_{[2]}$	1.37×10^{13}	1.68×10^{13}	1.83×10^{13}
$\tau^*_{\text{new}}/\tau^*_{[1]}$	4.147×10^4	N/A	N/A

TABLE I

MATI bounds achieving UGES for the system in (30) with PE_T protocols.

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