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Using Kolmogorov-Smirnov Tests to Detect Track-Loss in the Absence of Truth Data

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Abstract— **Of significant interest in the practical application of data association algorithms to target tracking in cluttered environments is how to determine track-loss in the absence of truth data. An approach is laid out for Kalman filter based data association algorithms where sample gated measurement count distributions are compared to theoretical measurement count distributions of the "tracking" or "track-lost" regimes to determine the regime of filter operation for single-target tracking applications. The comparisons are done via a pair of Kolmogorov-Smirnov tests. Among the advantages of this method are that confidence intervals are associated with the track regime tests, and that the number of samples required to discriminate between regimes can be determined adaptively. Simulation results for the method are provided.**

I. Introduction

Real-world sensors often report more than one measurement that may be from a given target. These may either be measurements of the desired target or "clutter" measurements. Clutter refers to detections or returns from nearby objects, clouds, electromagnetic interference, acoustic anomalies, false alarms, *etc*. Due to measurement origin uncertainty, the estimate will eventually diverge from the true track in data association algorithms. Although there is a continuum of estimator behavior, the operation of these filters is often divided into two regimes, non-divergent ("tracking") and divergent ("track-lost"). Typically the emphasis is more on detecting the transition between regimes ("track-loss") than on defining the regimes.

In simulation, two metrics that are commonly used to define the transition between these two regimes are [14]: 1) the normative distance of the estimate from the truth becoming repeatedly larger than some threshold, and 2) repeated failure of the truth measurement to "gate". Gating is a commonly used method of limiting the number of measurements processed by considering only those within some normative distance of the predicted measurement. In practice, neither of these metrics is applicable, as neither the true state nor truth measurement is known. However, there are differences between the regimes that can be exploited to build track-loss metrics that do not require knowing the true state or truth measurements [8], [10].

Among other methods, detection of track-loss in the absence of truth data has been done for the Probabilistic Data Association Filter (PDAF) by testing the normality of "effective" innovations in the state space [10], and by applying a Neyman-Pearson decision rule to the comparison of sample variances of "effective" innovations to theoretical predictions of these variances for the two track regimes [12]. However, the innovations covariance is a complex, non-linear, and time-varying stochastic function of the target dynamics, data association algorithm, clutter density, gate, *etc.*, so unique innovations-based tests must be developed for each data association algorithm.

Many common data association algorithms share a Kalman Filter (KF) structure (where the truth measurements are assumed to be Gaussian distributed about the predicted measurement) and the assumption of uniformly distributed clutter measurements. In single target tracking, before divergence of the filter the gated measurements then are a mixture of the truth and clutter measurements, while after divergence the gated measurements are solely clutter measurements. In this paper, the empirical (cumulative) distribution function of the gated (sample) measurement count is compared to a theoretical cumulative density function of the measurement count in each track regime, via Kolmogorov-Smirnov (KS) tests of fit. The KS tests use a single measure of deviation to make a determination of whether the sample is consistent with the hypothesis of a given theoretical distribution. In concert, these tests comprise a track-loss detector. Using this measure of deviance to determine the tracking regime has several advantages beyond an indication of when track-loss may be imminent, or has occurred. It provides an intuitive metric for assessing the difficulty of the tracking problem, and confidence intervals are associated with the assignments of the data to a given regime. Additionally, the number of samples required for good test performance can be determined adaptively.

This paper is organized as follows. In Section II, Kalman filters for linear time invariant (LTI) systems, as well as data association and gating, are reviewed. In Section III, the cumulative distribution functions of the gated measurement count, required to use the KS test, are developed. In Section IV, the Kolmogorov-Smirnov tests of fit are discussed. In Section V, the implementation of the two-sided Kolmogorov-Smirnov test of fit as a track-loss detector is detailed, and an adaptive method of determining the number of samples to use for good track regime determination is outlined. Simulation results and conclusions are presented in Sections VI and VII.

II. Data Association

In this section, a linear time invariant (LTI) model of the system to be tracked and the discrete-time KF are reviewed.

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Additional measurement origin uncertainties beyond those considered by the KF are then discussed.

A. System Dynamics

Consider the discrete-time LTI system with target state dynamics $\mathbf{x}(k)$ and measurements $\mathbf{z}(k)$ governed by

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{F}\,\mathbf{x}(k)+\mathbf{G}\,\mathbf{w}(k)\\ \mathbf{y}(k) &= \mathbf{H}\,\mathbf{x}(k)\\ \mathbf{z}(k) &= \mathbf{H}\,\mathbf{x}(k)+\mathbf{J}\,\mathbf{v}(k)\\ &= \mathbf{y}(k)+\mathbf{J}\,\mathbf{v}(k)\end{aligned}$$

where $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, and $\mathbf{J}$ are assumed known, and $\mathbf{w}$ and $\mathbf{v}$ are zero-mean Gaussian white noise vectors with known covariances $\mathbf{Q}=E[\mathbf{w}\mathbf{w}']$ and $\mathbf{R}=E[\mathbf{v}\mathbf{v}']$, respectively.

B. Kalman Filter

For the LTI system, the discrete-time KF equations are

Predicted covariance:

$$\mathbf{P}(k|k-1)=\mathbf{F}\,\mathbf{P}(k-1|k-1)\,\mathbf{F}'+\mathbf{G}\,\mathbf{Q}\,\mathbf{G}'$$

Predicted state:

$$\hat{\mathbf{x}}(k|k-1)=\mathbf{F}\,\hat{\mathbf{x}}(k-1|k-1)$$

Covariance of the innovations:

$$\mathbf{S}(k)=\mathbf{H}\,\mathbf{P}(k|k-1)\,\mathbf{H}'+\mathbf{R}$$

Kalman gain:

$$\mathbf{K}(k)=\mathbf{P}(k|k-1)\,\mathbf{H}'\,\mathbf{S}(k)^{-1}$$

State estimate:

$$\hat{\mathbf{x}}(k|k)=\hat{\mathbf{x}}(k|k-1)+\mathbf{K}(k)\,\nu(k)$$

where $\nu(k)=\{\mathbf{z}(k)-\mathbf{H}\,\hat{\mathbf{x}}(k|k-1)\}$ is the innovation.

Estimate error covariance:

$$\begin{aligned}\mathbf{P}(k|k) &= \mathbf{P}(k|k-1)-\mathbf{K}(k)\,\mathbf{S}(k)\,\mathbf{K}(k)'\\ &= \{\mathbf{I}-\mathbf{K}(k)\,\mathbf{H}\}\,\mathbf{P}(k|k-1).\end{aligned}$$

C. Handling Additional Measurement Uncertainties

In the data association problem, additional measurement uncertainties are considered beyond those in the Kalman filter (which are due strictly to Gaussian white noise in the measurements and states). The PDAF [1], Mixture Reduction [14], and Mean-Field Event-Averaged Maximum Likelihood Estimator [6] (among others) use the structure of the Kalman filter with the following common changes to accommodate data association: i) there is a probability P_D, $P_D \le 1$, that the measurement for a given target will be detected, ii) there are clutter measurements with density λ uniformly distributed throughout the measurement field, and iii) in order to limit the number of measurements considered, all available measurements are *gated*, keeping only those measurements that fall into a validation volume (gate) about the predicted measurement $\hat{\mathbf{z}}(k)=\mathbf{H}\,\hat{\mathbf{x}}(k|k-1)$. The χ^2 gating test for a measurement $\mathbf{z}_i(k)$ is

$$\nu_i(k)'\;\mathbf{S}(k)^{-1}\,\nu_i(k)\le\gamma$$

where $\nu_i(k)=\{\mathbf{z}_i(k)-\mathbf{H}\,\hat{\mathbf{x}}(k|k-1)\}$, and γ is the χ^2 gate sizing parameter measured in standard deviations squared. Under the assumption that the truth measurement is Gaussian distributed about the predicted measurement, the probability of gating the truth measurement, P_G, is a function of γ:

$$P_G=\int_0^{\gamma}\frac{\gamma^{n_z/2-1}\exp\{-\gamma/2\}}{2^{n_z/2}\,\Gamma(n_z/2)}d\gamma\,,\qquad \gamma\ge 0$$

where $\Gamma(\cdot)$ is the gamma function. All measurements validated by this test are inside a hyper-ellipsoid with volume

$$V_k \;=\; c_{n_z}\,\gamma^{n_z/2}\,|\mathbf{S}(k)|^{1/2} \tag{1}$$

where n_z is the dimension of the measurement vector and c_{n_z} is the volume of a n_z-dimensional unit hyper-sphere,

$$c_{n_z} \;=\; \pi^{n_z/2}/\,\Gamma(n_z/2+1)\,.$$

The data association problem is then to use information from all the gated measurements to update the target track.

III. Gated Measurement Count Distributions

Defining track-loss as the transient between non-divergent and divergent filter operation, it can be detected by noting when the regime of operation switches from tracking to track-lost. Mathematical definitions of the two regimes can be constructed in terms of cumulative distribution functions ($\mathbf{cdf}$s) of the gated measurement counts. With the appropriate assumptions, the regime $\mathbf{cdf}$s can be determined exactly.

A. Distribution for the Truth Measurements

In the tracking regime it is commonly assumed that the actual covariance of the truth innovations are well approximated by the predicted innovations covariance, and are thus Gaussian distributed with covariance $\mathbf{S}(k)$. Under this assumption, the number of truth measurements m_k^{tru} in a volume as in (1) is Bernoulli distributed, with prior probability density function ($\mathbf{pdf}$):

$$\begin{aligned}f(0) &= 1-P_DP_G\\ f(1) &= P_DP_G\,.\end{aligned}$$

B. Distribution for the Track-Lost Regime

Under the assumption of uniform clutter distribution, the number of clutter measurements m_k^{cl} in the volume V_k conditioned on the track-lost regime is Poisson distributed [1] with prior $\mathbf{pdf}$

$$f_{\mathrm{L}}(m_k^{\mathrm{cl}})=\frac{(\lambda V_k)^{m_k^{\mathrm{cl}}}\exp(-\lambda V_k)}{m_k^{\mathrm{cl}}!}\,,\; m_k^{\mathrm{cl}}=0,1,2,\ldots$$

and prior $\mathbf{cdf}$

$$F_{\mathrm{L}}(m_k^{\mathrm{cl}})=\sum_{i=0}^{m_k^{\mathrm{cl}}}\frac{(\lambda V_k)^{i}\exp(-\lambda V_k)}{i!}\,,\; m_k^{\mathrm{cl}}=0,1,2,\ldots\,. \tag{2}$$

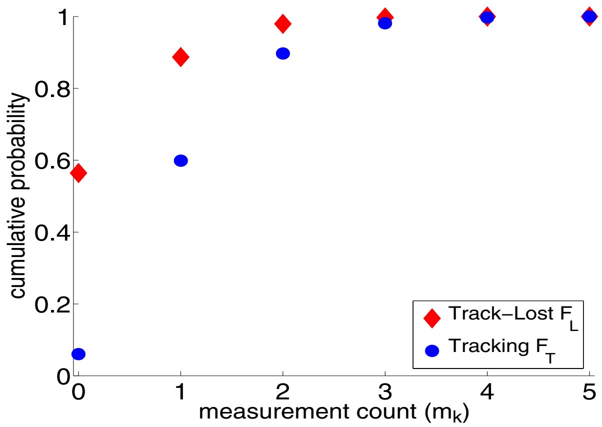

Fig. 1. Regime $\mathbf{cdf}$s for $P_D = 1$, $P_G = 0.8931$, $\lambda V_k = 0.2$.

C. Distribution for the Tracking Regime

Since the distributions of clutter and truth measurements are independent, the prior $\mathbf{pdf}$ for the mixture of truth and clutter measurements $m_k^{\text{mix}} = m_k^{\text{tru}} + m_k^{\text{cl}}$ inside the gate conditioned on the tracking regime is

$$f_{\text{T}}(m_k^{\text{mix}}) = \begin{cases} f(m_k^{\text{tru}}=0)\, f(m_k^{\text{cl}}=m_k^{\text{mix}}) \\ +\, f(m_k^{\text{tru}}=1)\, f(m_k^{\text{cl}}=m_k^{\text{mix}}-1)\,, \\ \hfill m_k^{\text{mix}}=1,2,\ldots \\ f(m_k^{\text{tru}}=0)\, f(m_k^{\text{cl}}=0)\,, \quad m_k^{\text{mix}}=0\,. \end{cases}$$

The prior $\mathbf{cdf}$ is then

$$F_{\text{T}}(m_k^{\text{mix}}) = \begin{cases} \sum_{i=0}^{m_k^{\text{mix}}-1} \frac{(\lambda V_k)^i \exp(-\lambda V_k)}{i!} \\ +\, (1-P_D P_G)\, \dfrac{(\lambda V_k)^{m_k^{\text{mix}}} \exp(-\lambda V_k)}{m_k^{\text{mix}}!}\,, \\ \hfill m_k^{\text{mix}} = 1,2,\ldots \\ (1-P_D P_G)\, \exp(-\lambda V_k)\,, \quad m_k^{\text{mix}} = 0\,. \end{cases} \tag{3}$$

Figure 1 shows tracking (F_{T}) and track-lost (F_{L}) regime measurement count $\mathbf{cdf}$s for $P_D = 1$, $P_G = 0.8931$, and $\lambda V_k = 0.2$.

IV. Kolmogorov-Smirnov Tests of Fit

A. Overview

If $x_1, x_2, \ldots, x_n$ are independent observations of a random variable with cumulative distribution function $F(x)$ which is unknown, and the null hypothesis is

$$H_0 : F(x) = F_0(x)$$

then any test of the hypothesis is a goodness-of-fit test [16]. If the hypothesis is completely specified (such as being normal with known mean and variance), then it is a simple hypothesis. The many variants of the Kolmogorov-Smirnov (KS) test are simple goodness-of-fit tests. KS tests determine whether the sample is from the hypothesized distribution with a specified confidence interval using the maximum deviation between the hypothesis and sample distribution functions as the metric. While originally developed for continuous distributions, KS tests can be extended for use with discontinuous distributions, and are known to be conservative if the function $F(x)$ is discrete [3].

B. One-Sided and Two-Sided KS Tests for Continuous $F(x)$

Given n ordered observations of x such that

$$x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)},$$

the empirical sample $\mathbf{cdf}$ is defined [16]

$$S_n(x) = \begin{cases} 0, & x < x_{(1)} \\ \frac{r}{n}, & x_{(r)} \le x < x_{(r+1)} \\ 1, & x_{(n)} \le x \end{cases} .$$

If $F_0(x)$ is the true, fully specified cumulative distribution function from which the observations in the sample are drawn, then from the strong law of large numbers

$$\lim_{n\to\infty} P\left\{S_n(x) = F_0(x)\right\} = 1\,.$$

The positive one-sided measure of deviation is defined [7]

$$D_n^+ = \sup_x \left\{S_n(x) - F_0(x)\right\}\,.$$

Similarly, the negative one-sided measure of deviation is

$$D_n^- = \sup_x \left\{F_0(x) - S_n(x)\right\}$$

and the two-sided measure of deviation is

$$D_n = \sup_x |S_n(x) - F_0(x)| = \max\left\{D_n^+, D_n^-\right\}\,. \tag{4}$$

The distributions of D_n^+, D_n^-, and D_n can be calculated both in the asymptotic limit as $n \to \infty$ and for finite values of n [2], [5], [16]. Amazingly, these distributions are independent of the distribution $F_0(x)$ for continuous $\mathbf{cdf}$s [5]. Denoting D_n^* as any of the D_n^+, D_n^-, or D_n measures of deviance and $d_n^*(\alpha)$ any of the corresponding $d_n^+(\alpha)$, $d_n^-(\alpha)$, or $d_n(\alpha)$ critical values associated with significance level α (probability of incorrectly rejecting the hypothesis), then if $S_n(x)$ is drawn from $F_0(x)$ [16]

$$P\left\{D_n^* \ge d_n^*(\alpha)\right\} = \alpha \tag{5}$$

where $100(1-\alpha)$ is the confidence interval ($\mathbf{CI}$), so that

$$P\{F_0(x) - d_n^*(\alpha) \le S_n(x) \le F_0(x) + d_n^*(\alpha), \text{ for all } x\} = 1-\alpha\,.$$

C. Critical Value $d_n^*(\alpha)$

For every fixed value of $d^*(\alpha) \ge 0$, in the limit [5], [7]

$$\lim_{n\to\infty} P\left\{n^{1/2}\, D_n^* \ge d^*(\alpha)\right\} = \alpha \tag{6}$$

where for the two-sided test

$$\alpha = 1 - 2\sum_{j=1}^{\infty} (-1)^{j-1} \exp\{-2j^2 d(\alpha)^2\}$$

and for the positive and negative one-sided tests

$$\alpha = 1 - 2\sum_{j=1}^{\infty}(-1)^{j-1}\exp\{-2j^2 d^+(\alpha)^2\}\,.$$

because the distribution of D_n^- is (by symmetry) identical to the distribution of D_n^+.

There exist methods [4] and tables [9], [16] for determining $d_n^*(\alpha)$. Equivalent to (5) is [16]

$$P\left\{n^{1/2}\,D_n^* \geq n^{1/2}\,d_n^*(\alpha)\right\} = \alpha\,,$$

so noting that in the limit $d^*(\alpha) = n^{1/2}\,d_n^*(\alpha)$ provides an asymptotic approximation for $d_n^*(\alpha)$. Then for the two-sided test some important example asymptotic approximations of $d_n(\alpha)$ are [16]

$$\begin{aligned} d_{n,n\to\infty}(\alpha) &= 1.3581n^{-1/2} \quad \text{for} \quad \alpha = 0.05 \\ d_{n,n\to\infty}(\alpha) &= 1.6276n^{-1/2} \quad \text{for} \quad \alpha = 0.01\,. \end{aligned} \tag{7}$$

There are also closed-form solutions for finite values of n, but it is more convenient to tabulate the critical values via one of several common recursion relations [4]. For the two-sided test, let $c > 0$ be an integer. Then from [2], [7]

$$P\left\{D_n < \frac{c}{n}\right\} = \frac{n!}{n^n}\,\exp(n)\,R_{0,n}(c)$$

where $R_{i,k}(c)$ is defined for all integers i, all non-negative integers k, and integers $c = 1, 2, \ldots, n$ as

$$\begin{array}{ll} R_{0,0}(c) = 1 & \\ R_{i,0}(c) = 0 & \text{for } i \neq 0 \\ R_{i,k}(c) = 0 & \text{for } |i| \geq c \\ R_{i,k+1}(c) = \exp(-1)\sum_{s=0}^{2c-1} R_{i+1-s,k}(c)\frac{1}{s!} & \text{for } |i| \leq c-1 \end{array}$$

Using this recursion, the critical value $d_n(\alpha) = \frac{c}{n}$ of the distribution of D_n can be determined [2] for a given confidence probability $(1-\alpha) = \frac{n!}{n^n}\,\exp(n)\,R_{0,n}(c)$.

It should be noted that critical values determined with the asymptotic approximations are always conservative, that is

$$P\left\{D_n \geq d_{n,n\to\infty}(\alpha)\right\} \leq \alpha\,,$$

and this is true also for one-sided critical values determined in this way. Thus critical values determined with recursion relations are preferable, especially for small n where the approximation is most conservative. The asymptotic and recursive distributions are generally in good agreement for $n \geq 80$ [16].

D. One-Sided and Two-Sided KS Tests for Discrete $F(x)$

Technically, the relation (5) only applies to continuous functions, and the measurement count $\mathbf{cdf}$s are discrete. However, the KS test can be applied to discrete distributions using (either exact or asymptotic) critical values d_n^* derived for continuous distributions, and the results are known to be always conservative [3]. Alternatively, critical values can be determined with discrete function recursion relations [3].

E. Sample Sizes and Confidence Intervals

Choosing the $\mathbf{CI}$ and the sample size n fixes the critical value $d_n(\alpha)$. From (7), a sample size of 100 would have a $\mathbf{CI}$ of 95% (probability of 0.95) that the sample cumulative distribution function would everywhere be within $d_n(\alpha) = 0.13581$ of the true distribution function.

V. Implementing the Two-Sided KS Test for Track-Loss Detection

Once $\mathbf{cdf}$s for the total number of gated measurements for both the tracking and track-lost regimes are determined, it is possible to use either as the null hypothesis to perform a KS test. Measurement counts can then be sampled at several contiguous timesteps and these samples used to construct the necessary empirical $\mathbf{cdf}$. It is desirable to test the empirical $\mathbf{cdf}$ using KS tests for both regimes, since higher confidence can be placed in an overall track regime detector ($\mathbf{TRD}$) if the tests assign the empirical sample $\mathbf{cdf}$ as being from exactly one regime. While there are advantages to using one-sided KS tests, the $\mathbf{TRD}$ is implemented here using the two-sided KS test to reduce sensitivity to parameter errors in the theoretical regime $\mathbf{cdf}$s. Special notice can be taken of possible or impending track-loss in cases where either the tracking regime KS test rejects the null hypothesis or the track-lost regime KS test accepts the null hypothesis. Track-loss is detected when the paired tracking and track-lost KS regime tests respectively agree that the filter is no longer in the tracking regime and is in the track-lost regime. However, there are issues that must be addressed in order to construct a regime test.

A. Modeling Issues

It is clearly not possible to sample measurement counts at several timesteps and achieve the exact $\mathbf{cdf}$ for the tracking regime. If each sample is drawn from an identical volume, then the assumption about P_G is violated. On the other hand, if each sample is drawn from a volume such that P_G is constant, then the assumption of identical, Poisson distributed clutter samples is violated. Further, the filter estimates $\hat{\mathbf{x}}(k+1|k)$ and $\mathbf{S}(k)$ have data dependent errors which guarantee that the desired P_G will not match the true P_G regardless.

On the other hand, provided that the clutter density model proposed is accurate (λ is known and constant), then by using a constant-volume gate in the track-lost regime test, the samples can be drawn according to the track-lost $\mathbf{cdf}$. However, these conditions on λ may not be met in practice. The robustness of the $\mathbf{TRD}$ to modeling errors is a subject of on-going and future work. Some simulation results evaluating robustness will be presented in Section VII.B.

B. Choosing the Track Regime Detector Gating Volumes

Gating for the $\mathbf{TRD}$ should not be confused with gating for the data association algorithm. There are advantages to sampling using different gating strategies for the KS test. One strategy is as follows. For the track-lost regime, determine the average gate volume over the previous n timesteps $k-n+1,\, k-n+2,\, \ldots,\, k$ to be used in the detector,

$$V_c \;=\; \sum_{j=k-n+1}^{k} \frac{V_j}{n} \;=\; c_{n_z}\, \gamma_{\text{best}}^{n_z/2}\, \mathbf{\Sigma}_n\,, \tag{8}$$

where

$$\begin{aligned} V_j &= c_{n_z}\, \gamma_{\text{best}}^{n_z/2}\, |\mathbf{S}(j)|^{1/2} \\ \mathbf{\Sigma}_n &= \sum_{j=k-n+1}^{k} \frac{|\mathbf{S}(j)|^{1/2}}{n}\,, \end{aligned}$$

and $\mathbf{S}_j$ are the n innovations variances in the sampling interval. Then re-gate all the measurements $z_i(\cdot)$ from those n timesteps with constant gate volume V_c. Re-gating using

$$\nu_i(j)' \left(\frac{V_j}{V_c}\right)^2 \mathbf{S}(j)^{-1}\, \nu_i(j) \leq \gamma_{\text{best}} \tag{9}$$

preserves the shapes of the gating volumes due to the individual innovations covariances, while achieving constant-volume gates V_c. The track-lost regime **cdf** in (2) can then be calculated using $V_k = V_c$. This gating strategy will be referred to hereafter as the constant-volume gate (**CG**).

Nothing constrains the gating parameter γ_{best} in (8) and (9) to be the same as the gating parameter for the data association algorithm (where γ is likely to be as large as processing power will allow). As will be seen, the choice of the gating parameter γ_{best} (and thus the gating volume V_c) here is a degree of freedom used to maximize the power of the regime tests to reject the alternative hypothesis.

As noted before, it is not possible to sample across several timesteps and achieve the exact **cdf** for the tracking regime. However, provided that both $\lambda(k)$ and $|\mathbf{S}(k)|$ are "well-behaved" (*i.e.*, have means that vary slowly and variances that are a small fraction of the mean), then the conflicting requirements for the gate volume can be handled by gating with a time-varying volume at each timestep using

$$\nu_i(j)'\; \mathbf{S}(j)^{-1}\, \nu_i(j) \leq \gamma_{\text{best}} \tag{10}$$

(where once again, nothing constrains the gating parameter γ_{best} in (10) to be the same as the gating parameter for the data association algorithm), but then assuming a constant volume gate in the tracking regime **cdf** (3) by again using $V_k = V_c$. This gating strategy will be referred to hereafter as the non-constant volume gate (**NG**). The **NG** should only be used to test the tracking regime hypothesis, while the **CG** should be used to test the track-lost regime hypothesis.

C. Power of the Test

When one regime **cdf** $F_0(x)$ is used as the null hypothesis H_0, the power of the KS test for H_0 with respect to the alternative H_1 is the probability that samples drawn from $F_1(x)$ will be rejected [16]. If β is the probability that samples from $F_1(x)$ will be accepted, then the power is $(1-\beta)$. If the samples are drawn from the alternative distribution $F_1(x)$, then

$$P\left\{D_n^* \geq d_n^*(\alpha)\right\} = 1 - \beta$$

defines the power of the test (to reject the hypothesis that the samples are from $F_0(x)$).

D. Maximizing the Asymptotic Power of the Test

There are methods of calculating the exact power of the test with respect to the alternative [4], [15], but due to the discrete **cdf**s involved it is not clear how to use these methods to devise an algorithm to search for the "optimum" value of γ_{best} (for given sample size n and significance α) that will yield the global maximum power.

The following heuristic argument is provided to maximize the power for the two-sided KS test with respect to the alternative in an asymptotic sense (similar arguments can also be made for the positive and negative one-sided tests). Those interested in a rigorous discussion of the concepts should consult [13]. First, propose two continuous cumulative distribution functions, and choose one of them as the null hypothesis. If the samples are drawn from the alternative $F_1(x)$, then from the strong law of large numbers

$$\lim_{n\to\infty} P\left\{S_n(x) = F_1(x)\right\} = 1\,,$$

and thus from (4) define

$$\Delta_F = \lim_{n\to\infty} D_n = \sup_x |F_1(x) - F_0(x)|\;.$$

Since the power is $P\left\{D_n \geq d_n(\alpha)\right\}$, and $d_n(\alpha)$ is independent of D_n, $F_0(x)$, and $F_1(x)$, in an asymptotic sense the power is maximized when Δ_F is maximized.

Turning to the discrete track regime distributions, the expected number of gated measurements in the tracking regime is always greater than or equal to the number in the track-lost regime (see Figure 1), so a non-negative difference measure between regime **cdf**s can be defined

$$\Delta_F(m_k,\gamma) \;=\; F_{\mathrm{L}}(m_k) - F_{\mathrm{T}}(m_k) \;\geq\; 0\,, \qquad m_k = 0,\,1,\,2,\,\ldots\;. \tag{11}$$

Then from (2), (3), (8), and (11),

$$\Delta_F(m_k,\gamma) \;=\; P_D P_G \frac{(\lambda V_c)^{m_k} \exp(-\lambda V_c)}{m_k!} \;\geq\; 0\,, \qquad \gamma \geq 0,\; m_k = 0, 1, 2\,, \ldots \tag{12}$$

where V_c and P_G are both functions of the gating parameter. If the factorial in (12) is generalized to the Gamma function $\Gamma(m_k+1)$, then real values of m_k can be considered,

$$\Delta_F(m_k,\gamma) \;=\; P_D P_G \frac{(\lambda V_c)^{m_k} \exp(-\lambda V_c)}{\Gamma(m_k+1)} \;\geq\; 0\,, \qquad \gamma \geq 0\,, \quad m_k \geq 0\;.$$

For fixed values of γ, $\Delta_F(m_k,\gamma)$ is maximized for any given value of γ when $m_k = E[m_k] = \lambda V_c$. Further, defining

$$\begin{aligned} \Delta_{F_{\max}} &= \sup_\gamma\; \Delta_F(\lambda V_c, \gamma) \\ &= \sup_\gamma\; P_D P_G \frac{(\lambda V_c)^{\lambda V_c} \exp(-\lambda V_c)}{\Gamma(\lambda V_c + 1)} \;\geq\; 0 \end{aligned}$$

where $P_D P_G$ is monotonically increasing in γ and

$\dfrac{(\lambda V_c)^{\lambda V_c} \exp(-\lambda V_c)}{\Gamma(\lambda V_c + 1)}$ is monotonically decreasing in γ

means that there is exactly one global maximum, which occurs at $\gamma = \gamma_{\text{best}}$. Thus for a given set of system parameters, clutter density, gating probability, experimentally determined innovations covariance, *etc.*, it is possible to determine and use the gating parameter associated with the the most powerful test in an asymptotic sense.

E. Some Important Trends of the Track Regime Detector

$\Delta_F(\lambda V_c, \gamma)$ is a function of $\lambda \mathbf{\Sigma}_n$, n_z, and γ. The parameters λ and $\mathbf{\Sigma}_n$ are lumped together as a single parameter because $\mathbf{\Sigma}_n$ increases or decreases as λ does. A plot of $\Delta_F(\lambda V_c, \gamma)$ vs. γ for various values of n_z and $\lambda \mathbf{\Sigma}_n$ is shown in Figure 2. The general trends, as demonstrated by the four curves, are: i) as $\lambda \mathbf{\Sigma}_n$ decreases, $\Delta_{F_{\max}}$ and γ_{best} increase, and ii) as n_z increases, $\Delta_{F_{\max}}$ decreases while γ_{best} increases. The implications are that it is "easier" to determine the regime of operation when the clutter density is lower, as the maximum difference between the two regimes, $\Delta_{F_{\max}}$, is larger. Further, the optimum γ becomes larger both for lower clutter densities and higher measurement dimensions.

F. Adapting the Sample Size and Confidence Interval

The power of the test can also be increased by increasing the number of samples used to construct the empirical **cdf**, since if the samples are drawn from a distribution $F_1(x) \neq F_0(x)$, then from the strong law of large numbers

$$\lim_{n\to\infty} P\left\{S_n(x) = F_0(x)\right\} = 0.$$

There are, however, a series of trade-offs involved in the selection of n. Increasing n increases the likelihood that an alternative regime hypothesis will be rejected, but it will also increase the time necessary to detect track-loss as well as increase the likelihood of incorrectly rejecting the correct hypothesis due to modeling errors. Hence, using an arbitrarily large value of n is undesirable. Testing the empirical **cdf** using both regime hypotheses allows for an adaptive algorithm to be used for selecting n.

Similarly, there is a tradeoff in the choice of the **CI**. Choosing a larger **CI** in general increases the number of samples necessary to discriminate between the two regimes, and thus the time to detect track-loss will be longer. But choosing a smaller **CI** in general increases the chances that at any given timestep there will be a threshold for n, above which both regime hypotheses are rejected and below which both regime hypotheses are accepted. This is because the "granularity" between the critical values, $d_{n+1}(\alpha) - d_n(\alpha)$, is larger for smaller **CI**s.

An elementary set of rules for an adaptive algorithm is as follows. A desired **CI** and initial value of n are selected. Then if the **TRD**: i) assigns the sample to both regimes, increase n, ii) assigns the sample to exactly one regime, retain n, iii) assigns the sample to neither regime, decrease n. If this strategy fails to determine a value of n which uniquely assigns the empirical **cdf** to a regime, the **CI** should be increased.

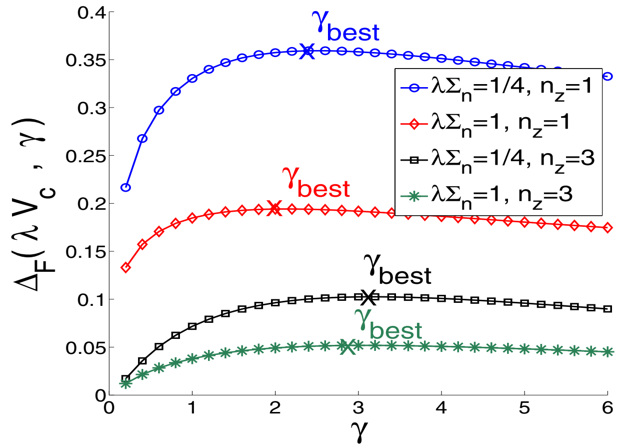

Fig. 2. Curves of $\Delta_F(\lambda V_c, \gamma)$ vs. γ for various n_z and $\lambda \mathbf{\Sigma}_n$.

VI. Simulation Results

Under the assumptions of Gaussian distributed truth measurements in the tracking regime and uniformly distributed clutter measurements in both regimes, the **TRD** will work with any data association method provided that there is an estimate of the innovations covariance. Thus the following PDAF example makes an excellent stand-in for KF-based data association methods in general.

A. Illustrative PDAF Example

The kinematic model system (with timestep δ)

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \delta^2/2 \\ \delta \end{bmatrix} \mathbf{w}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

is the standard zero-order hold discrete approximation to a continuous double-integrator system. This model is implemented in a PDAF with

$$\delta = 0.1, \qquad \lambda = 0.2,$$
$$\mathbf{Q} = q = 500, \qquad \mathbf{R} = r = 0.5\,.$$

A four standard deviation target tracking gate and a unity probability of truth measurement detection:

$$\gamma = 16 \quad \Rightarrow \quad P_G = 0.99994$$
$$P_D \quad = \quad 1$$

yields

$$V_k \quad = \quad c_{n_z}\, \gamma^{n_z/2}\, |\mathbf{S}(k)|^{1/2} \quad = \quad 8\, |\mathbf{S}(k)|^{1/2}\,.$$

This system gives an experimentally measured average gating volume (averaged over many tracks) of

$$V_{\mathbf{pdaf-T}} = 6.2195$$

for the tracking regime, and

$$V_{\mathbf{pdaf-L}} = 478.3304$$

for the track-lost regime. The data for individual tracks was generated by running the PDAF filter for 1000 timesteps without track-loss (as defined by the truth measurement never

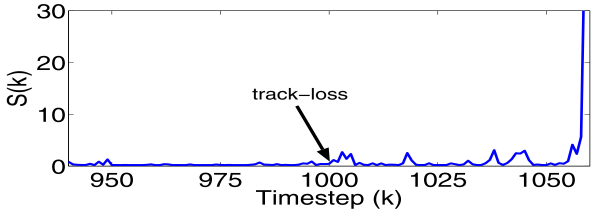

Fig. 3. Innovations variance $\mathbf{S}(k)$ before and after track-loss.

failing to gate for more than 5 timesteps), then forcing track-loss by making $P_D = 0$ at timestep $k = 1001$. The filter was then allowed to run for 1000 timesteps in the track-lost regime. For the example system, this is a tracking problem of "moderate" difficulty using the PDAF [11], [14].

Figure 3 is a plot of the innovations variance $\mathbf{S}(k)$ of an example target track, over approximately 50 timesteps before and after track-loss is forced at timestep 1001. During the first timesteps after track-loss, $\mathbf{S}(k)$, and thus the target tracking gate, spike repeatedly and then explode to orders of magnitude larger than in the tracking regime. This is typical behavior for a KF-based data association filter after track-loss. Thus even over short horizons, the distribution of $|\mathbf{S}(k)|^{1/2}$ in the track-lost regime typically has a large skewness and variance, and a constant volume gate is necessary to test the track-lost regime hypothesis.

Figures 4 and 5 show the results of two different **TRD**s using $\mathbf{CI} = 95\%$ with fixed $n = 5$ and $n = 50$, respectively, on an example target track; again track-loss is forced at timestep 1001. Looking at Figure 4, before track-loss the **TRD** for $n = 5$ does not consistently assign the track to exactly one regime, and thus n is too small. Looking at Figure 5, the **TRD** for $n = 50$ consistently assigns the track solely to the tracking regime, and 40 timesteps after track-loss it begins to consistently assign the track solely to the track-lost regime. The 22 timesteps after track-loss when the **TRD** rejects both regime hypotheses is a good indicator of possible track-loss. Using the same data, the adaptive **TRD** described earlier "converges" near $n = 50$, but with track-loss detected more quickly than for the fixed n **TRD** of Figure 5 because n is reduced when both regime hypotheses are rejected, thus considering less information from near the track-loss transition.

B. Robustness of the Track Regime Detector

The tracking regime (**NG**) portion of the **TRD** is sensitive to modeling error introduced by the steady-state innovations covariance assumption. Noting that the expected number of clutter measurements in the gate is λV_k [1], the modeling errors occurring from large-value outliers in $|\mathbf{S}(k)|$ (and thus V_k) tend to dominate all other sources of error. Large outliers occur when the track is "almost" lost and then recovered, such as around timestep 950 in Figure 3.

Another view of this can be seen in Figure 6, which shows results of both regime tests on tracking regime data of a target track from the PDAF example system. The regime **cdfs** are offset slightly horizontally to improve legibility. The tests were performed using $\mathbf{CI} = 95\%$ on $n = 500$ measurement

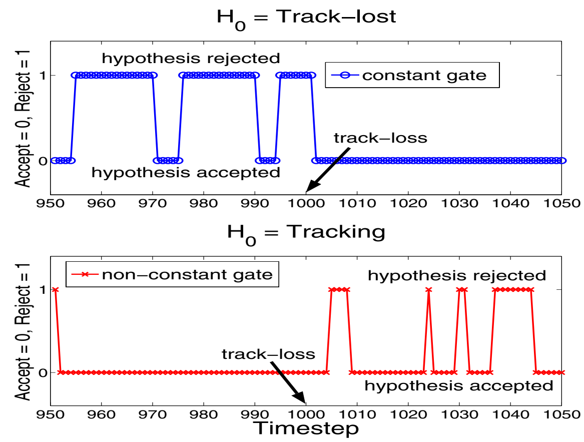

Fig. 4. Results of **TRD** for $\mathbf{CI} = 95\%$, $n = 5$.

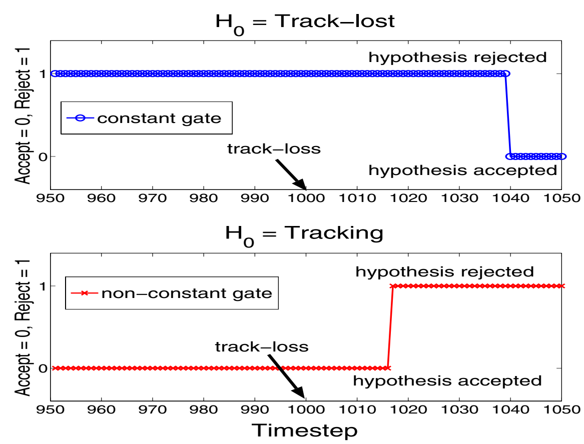

Fig. 5. Results of **TRD** for $\mathbf{CI} = 95\%$, $n = 50$.

count samples from the tracking regime ($d_n(\alpha) = 0.0604$ using the continuous recursion method), with $\mathbf{\Sigma}_n = 0.8875$, resulting in $\gamma_{\text{best}} = 2.6$ and $V_c = 2.8620$. The average data association gating volume over the 500 timesteps was 7.0997, 14% larger than $V_{\mathbf{pdaf-T}}$, indicating the track was "almost" lost at least once over this interval. The resulting outliers in $|\mathbf{S}(k)|^{1/2}$ bias V_c to be larger, effectively shifting the theoretical tracking regime **cdf**s F_{T} and F_{L} to the right toward larger m_k. The result is that both of the regime hypotheses are rejected, similar to the interval from timestep 1018 to 1039 in Figure 5. However, adapting n as described in Section V.F handles this problem, as when n is reduced to $n = 200$ (and thus $d_n(\alpha)$ increased to 0.0952), then the tracking regime hypothesis is uniquely accepted.

The track-lost regime (**CG**) portion of the **TRD** is sensitive only to modeling errors in λ. The sensitivity to errors caused by using a theoretical clutter density, λ_{theory}, which is different than the actual clutter density, λ_{actual}, is dependent on the gating volumes of the two regimes, and clearly track-lost regime data begins to resemble the tracking regime **cdf** as $\lambda_{\text{actual}} V_c \rightarrow \lambda_{\text{theory}} V_k + P_D P_G$. Since the impact of errors in λ_{theory} scales with V_c and the trend for KF-based data association algorithms is for V_c to grow larger after track-loss, it is preferable to err on the high side when choosing λ_{theory}, again effectively shifting the theoretical tracking regime **cdf**s F_{T} and F_{L} to the right toward larger

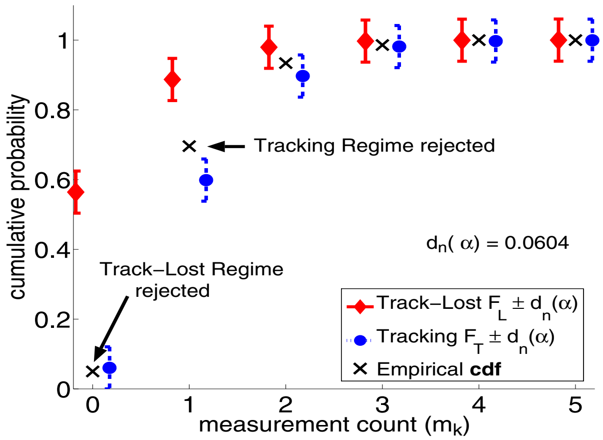

Fig. 6. Tracking regime data: $\mathbf{CI} = 95\%$, $n = 500$, $\gamma_{\text{best}} = 2.6$.

m_k and thus biasing the **TRD** in favor of accepting the track-lost regime hypothesis.

Figure 7 shows results of both the tracking and track-lost regime tests, this time for track-lost regime data from the PDAF example system. However, the theoretical regime **cdf**s were calculated for both regimes using $\lambda_{\text{theory}} = 0.85\,\lambda_{\text{actual}}$, which biases the **TRD** toward accepting the tracking regime hypothesis. The tests were performed using $\mathbf{CI} = 95\%$ on $n = 50$ measurement count samples from the track-lost regime ($d_n(\alpha) = 0.1884$ using the continuous recursion method), with $\boldsymbol{\Sigma}_n = 9.7145$, resulting in $\gamma_{\text{best}} = 2.2$ and $V_c = 28.8177$. Even this -15% error in λ_{theory} does not result in falsely rejecting the track-lost regime hypothesis in this example.

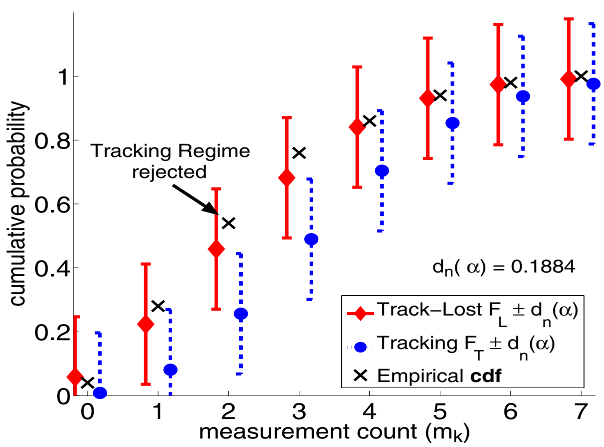

Fig. 7. Track-lost regime data: $\mathbf{CI} = 95\%$, $n = 50$, $\gamma_{\text{best}} = 2.2$.

VII. Conclusions

A track regime detector for Kalman filter based data association algorithms was developed that determines the filter operating regime (tracking or track-lost) based on the statistics of the number of measurements present in two track regime detection gates, assuming uniform clutter measurement density. Approximations for the tracking regime of the measurement count distribution inside a data-dependent gating volume as well as an exact formulation for the track-lost measurement count distribution inside a constant gating volume were derived. Once the measurement count distributions of the two regimes are available, Kolmogorov-Smirnov tests can be performed on empirical **cdf**s constructed from data sampled by these two gating strategies over consecutive timesteps to determine whether the filter is operating in a given regime with a given degree of confidence. Additionally, the gating parameter for the most powerful test in an asymptotic sense can be determined. Information from the resulting track regime detector can be used to adaptively determine the number of consecutive timesteps to sample from, in order to make the test both indicate track-loss as quickly as possible and also uniquely assign the track (where possible) to either the tracking or track-lost regime. Good performance of the track-loss detector was demonstrated for a PDAF example. The robustness of the track regime detector to variations in clutter measurement density and gating volumes has been initially addressed through the adaptive nature of the detector.

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